Symplectic Geometry: Reduction, Convexity, and Unimodularity

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Contents

Current Section

Symplectic Geometry

- M a smooth manifold. $E \to M$ a vector bundle. A **geometric structure** on M involves a smooth section T of a tensor bundle, $T \in \Gamma\left(E^{\otimes_{\mathbb{F}}^{k}} \otimes_{\mathbb{F}} (E^*)^{\otimes_{\mathbb{F}}^{j}}\right)$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.
- Examples of geometric structures:
  - **Riemannian geometry**: Inner product at each point.
  - **Symplectic geometry**: closed nondegenerate skew-symmetric bilinear form ω, i.e., $\ker \widetilde{\omega_p} = 0$ and $d\omega = 0$.
  - **Kähler geometry**: Compatible symplectic, complex, and Riemannian structures.
- Examples of symplectic manifolds:
  - $(\mathbb{C}^n, \omega_0) = (\mathbb{R}^{2n}, \omega_0)$.
  - Cotangent bundles T^*M.
  - Flag $\mathfrak{fl}(n) = \mathrm{GL}(n, \mathbb{C})/B = \mathrm{U}(n)/\mathbb{T}^n$.

Symplectic Geometry

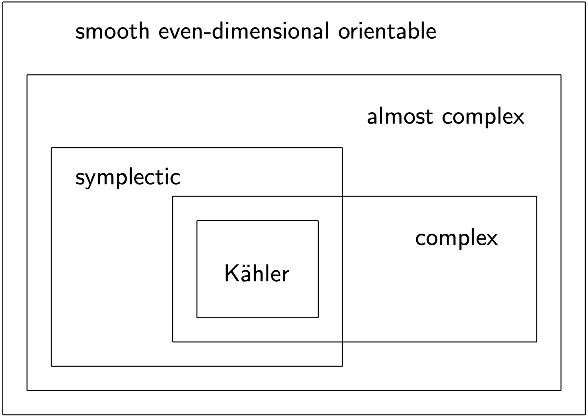


Figure 1: Geometries (Venn diagram from [1]).

Hamiltonian Actions

A diffeomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ is called a **symplectomorphism** if

$$\varphi^* \omega_2 = \omega_1.$$

Suppose Lie group G acts on (M, ω) via symplectomorphisms:

$$\Psi : G \to \mathrm{Sympl}(M, \omega)$$

(M, ω) is called a **Hamiltonian G-space** if it has a **moment map** $\mu : M \to \mathfrak{g}^*$ defined below.

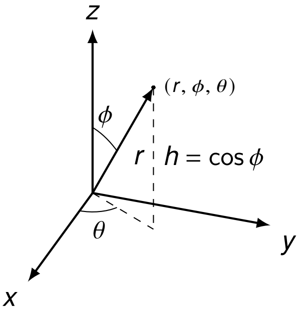
- Moment Map Characterization:
  - Hamiltonian condition: For any $X \in \mathfrak{g}$, $d\mu^X = \iota_{X^\#}\omega$, where $\mu^X(p) = \langle \mu(p), X \rangle$ and $X^\#$ = the vector field generated by one-paramater subgroup $\exp tX$.
  - Equivariance condition: $\mu \circ \Psi_g = \mathrm{Ad}^*_g \circ \mu$.
- Comoment Map Characterization (for connected Lie groups):
  - Hamiltonian condition: $\mu^*(X) := \mu^X$ is Hamiltonian for $X^\#$.
  - Equivariance condition: μ^* is a Lie algebra homomorphism, i.e., $\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)\}$. where $\{\cdot, \cdot\}$ is the Poisson bracket.

Example 1 (Sphere)

Consider the sphere $\mathbb{S}^2$. A point P on it can be written as

$$(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

so it has "height" $\cos\phi$. We thus define the **height function** as $H(\theta, h) = h$ on the sphere with symplectic form $\omega = d\theta \wedge dh$, the standard form for chart $(U, (\theta, h))$.



Consider the circle action on the sphere below. What's its moment map μ? Hint: height function, but why?

$$\Psi: \quad \mathbb{S}^1 \longrightarrow \mathrm{Sympl}\left(\mathbb{S}^2, \omega\right)$$
$$e^{i\theta} \longmapsto \text{ rotation by angle } \theta \text{ around } z\text{-axis}$$

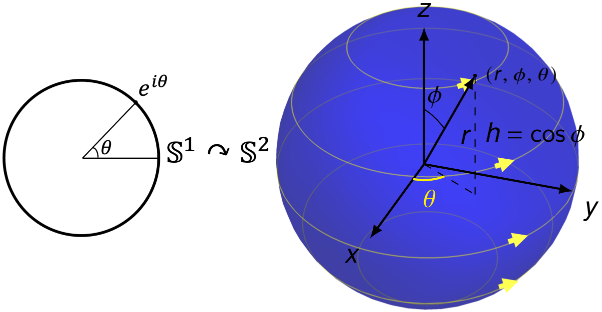


Figure 2: Circle action on a sphere and spherical coordinates.

Three Theorems: Reduction, Convexity, and Unimodularity

- Reduction Theorem (Meyer [2], Marsden-Weinstein [3])
  - Conserved quantities reduce phase space.
  - Applications: classical mechanics [4, 5].
- Convexity Theorem (Atiyah [6], Guillemin-Sternberg [7])
  - Image of the moment maps are convex polytopes.
  - Schur-Horn theorem and Horn's conjecture on Hermitian spectra.
  - Generalization to semisimple Lie group actions (Weinstein [8]).
- Unimodularity Theorem (Delzant [9])
  - $\{\text{Symplectic toric manifolds}\}/\sim \longleftrightarrow \{\text{Unimodular polytopes}\}/\sim$.
  - Non-compact symplectic toric manifolds (Karshon-Lerman [10]).

One Motivation: Counting Integer-points in Polytopes

Pukhlikov-Khovanskii ([11]): Let $\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \geq \lambda_i, i = 1, \cdots, m\}$ and $\Delta_h = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \geq \lambda_i + h_i, i = 1, \cdots, m\}$. Then

$$\#(\Delta \cap \mathbb{Z}^n) = \text{Todd}_h \left(\text{vol}(\Delta_h) \right)\big|_{h=0}$$

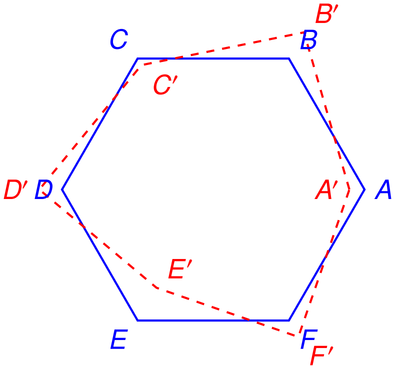


Figure 3: Perturbation Δ_h of a polytope Δ.

Current Section

Marsden-Weinstein-Meyer Theorem

Theorem 2 (Meyer [2], Marsden-Weinstein [3])

Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Suppose G acts freely on $\mu^{-1}(0)$. Then:

- *The quotient space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a smooth manifold.*
- *The projection $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ defines a principal G-bundle.*
- *There exists a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{\mathrm{red}}$.*

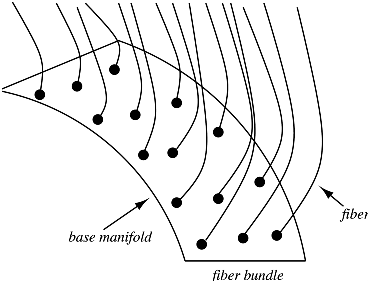


Figure 4: A line bundle. Pic. taken from Wolfram.

More Hamiltonian G-spaces

Theorem 3 (Commuting Actions)

Let (M, ω, G, μ) be a Hamiltonian G-space and $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be the symplectic reduction. Suppose that another Lie group H acts on (M, ω) in a Hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. If H-action commutes with the G-action and ϕ is G-invariant, then the action of H on M_{red} admits a Hamiltonian action of H with moment map ϕ_{red} .

Theorem 4 (Lie Subgroup Actions)

Let G be any Lie group and H a closed subgroup of G, with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. The projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Suppose that (M, ω, G, ϕ) is a Hamiltonian G-space. The restriction of the G-action to H is Hamiltonian with moment map

$$i^* \circ \phi : M \longrightarrow \mathfrak{h}^*$$

Complex Projective Space

- The complex projective space $\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*$ is obtained from $\mathbb{C}^{n+1} \backslash \{0\}$ by making the identifications $(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \backslash \{0\}$; $[z_0 : \cdots : z_n]$ is the equivalence class of $(z_0, \cdots, z_n)$. For $j = 0, 1, \cdots, n$, let

$$\mathcal{U}_j = \left\{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\right\}$$

$$\varphi_j : \mathcal{U}_j \to \mathbb{C}^n \quad \varphi_j\left([z_0 : \cdots : z_n]\right) = \left(\frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j}\right).$$

  This gives a complex atlas for $\mathbb{CP}^n$. It can be shown that $\omega = \frac{i}{2}\partial\bar{\partial} \log\left(\|z\|^2 + 1\right)$ is Kähler and thus symplectic on $\mathbb{C}^n$ and that $\omega_k := \varphi_k^* \omega$ agrees with ω_l on $\mathcal{U}_k \cap \mathcal{U}_l$. Thus, these ω_j's glue together to define a symplectic form ω_{FS}, called **Fubini-Study form**, on $\mathbb{CP}^n$.

Complex Projective Space

- $G = \mathbb{S}^1$**-Action** Φ_G **on** $\mathbb{C}^{n+1}$:

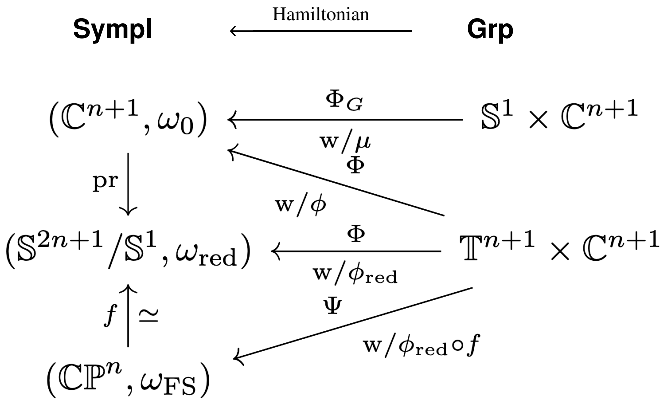
$$e^{i\theta} \cdot (z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n),$$
$$\mu(z) = -\frac{1}{2}\|z\|^2 + \frac{1}{2}.$$

- $H = \mathbb{T}^{n+1}$**-Action** Φ_H **on** $\mathbb{C}^{n+1}$:

$$(t_0, \ldots, t_n) \cdot (z_0, \ldots, z_n) = (t_0 z_0, \ldots, t_n z_n),$$
$$\phi(z) = -\frac{1}{2}\left(|z_0|^2, \ldots, |z_n|^2\right) + (\text{ constant })$$

- $(\mathbb{C}^{n+1}, \omega_0, \mathbb{S}^1, \mu)$ reduces to $(\mathbb{S}^{2n+1}/\mathbb{S}^1, \omega_{\mathrm{red}})$, which is symplectomorphic to $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ via an f. The other Hamiltonian group action over $\mathbb{C}^{n+1}$ descends naturally to the reduced space. The symplectomorphism f then transfers these data to the complex projective space $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$. It is now a Hamiltonian $\mathbb{T}^{n+1}$-space.

Complex Projective Space



Current Section

Atiyah-Guillemin-Sternberg Theorem

Any compact connected abelian Lie group must be a torus $G = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.

Theorem 5 (Atiyah [6], Guillemin-Sternberg [7])

Let (M, ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m torus. Suppose that $\psi : \mathbb{T}^m \to \mathsf{Sympl}(M, \omega)$ is a Hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:

(1) *the levels of μ are connected;*

(2) *the image of μ is convex;*

(3) *the image of μ is the convex hull of the images of the fixed points of the action.*

*The image $\mu(M)$ of the moment map is hence called the **moment polytope**.*

Atiyah-Guillemin-Sternberg Theorem

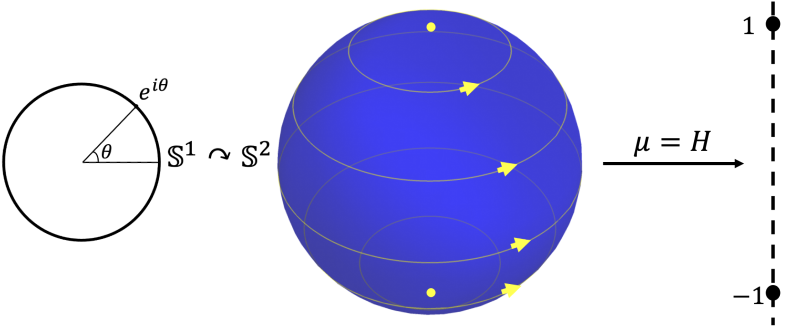


Figure 5: Circle action $\mathbb{S}^1$ over sphere $(\mathbb{S}^2, \omega = d\theta \wedge dh)$ by rotations with height function as the moment map $\mu(\theta, h) = H(\theta, h) = h$. The image polytope is $\mathrm{Im}(\mu) = [-1, 1]$.

Atiyah-Guillemin-Sternberg Theorem

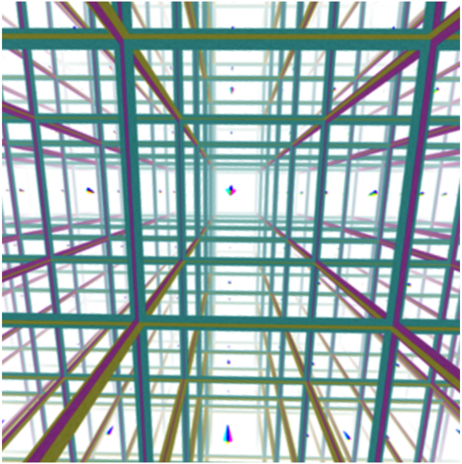


Figure 6: A three-torus, where an observer sees the back of his own head. Pic. taken from Wikipedia.

Atiyah-Guillemin-Sternberg Theorem

proof sketch.
Atiyah's proof of theorem 5 uses induction over $m = \dim \mathbb{T}^m$. Consider the statements:

- A_m: “the levels of μ are connected, for any $\mathbb{T}^m$-action;"
- B_m: “the image of μ is convex, for any $\mathbb{T}^m$-action."

Then

- The connectedness statement (1) $\Longleftrightarrow \mathrm{A}_m$ holds for all m,
- The convexity statement (2) $\Longleftrightarrow \mathrm{B}_m$ holds for all m.

The base case A_1 uses the fact that μ^X is a Morse-Bott function. For the induction $\mathrm{A}_{m-1} \implies \mathrm{A}_m$, see [12].
We show B_1, the induction $\mathrm{B}_{m-1} \implies \mathrm{B}_m$, and verify that the vertices supporting image polytope are the fixed points of the action.

Convexity

- Base case B_1: For $m = 1$, $\mathbb{T}^m = \mathbb{S}^1$ and $\mathfrak{g}^* = \mathbb{R}$. Since M is connected, $\mu(M)$ is also connected. In $\mathbb{R}$, connectedness implies convexity.
- Induction $\mathrm{B}_{m-1} \implies \mathrm{B}_m$: Denote $H = \mathbb{T}^{m-1}$ and $G = \mathbb{T}^m$, so $\mathrm{Lie}(H) = \mathfrak{h}^*$ and $\mathrm{Lie}(G) = \mathfrak{g}^*$. Choose an injective matrix $A \in \mathbb{Z}^{m\times(m-1)}$, so it can be either seen as a map $A : \mathbb{R}^{m-1} \cong \mathfrak{h} \to \mathfrak{g} \cong \mathbb{R}^m$ (so $A^t : \mathbb{R}^m \cong \mathfrak{g}^* \to \mathfrak{h}^* \cong \mathbb{R}^{m-1}$) or as a map

$$A : \mathbb{T}^{m-1} \longrightarrow \mathbb{T}^m$$
$$\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_{m-1}}\right) \longmapsto \left(e^{2\pi i \sum_{j=1}^{m-1} a_{1j}\theta_j}, \cdots, e^{2\pi i \sum_{j=1}^{m-1} a_{mj}\theta_j}\right).$$

  Consider the action of an $(m-1)$-subtorus

$$\psi_A : \mathbb{T}^{m-1} \longrightarrow \mathrm{Sympl}(M, \omega)$$
$$\theta \longmapsto \psi_{A\theta}$$

- Reduced Hamiltonian action: The $(m-1)$-torus action ψ_A on M has a moment map $\mu_A = A^t\mu$.

Convexity

- Connected level set: Fix $p_0 \in \mu_A^{-1}(\xi)$. The level set

$$\mu_A^{-1}(\xi) = \left\{p \in M \mid \mu(p) - \mu\,(p_0) \in \ker A^t\right\}$$

  is connected and $\ker A^t$ is 1-dimensional. This will force the convexity.
- Rational approximation: $p, q \in M$ arbitrary. Use compactness of $\mu(M)$ to choose sequences approaching them. Then use rationality to choose A.

Examples

Example 6

$(\mathbb{CP}^n, \omega_{\text{FS}}, \mathbb{T}^{n+1}, \phi_{\text{red}} \circ f)$ has action

$$(t_0, \cdots, t_n) \cdot [z_0 : \cdots : z_n] \mapsto [t_0 z_0 : \cdots : t_n z_n]$$

with moment map $\phi_{\text{red}} \circ f([z_0 : \cdots : z_n]) = -\frac{1}{2\|z\|^2}\left(|z_0|^2, \cdots, |z_n|^2\right)$.

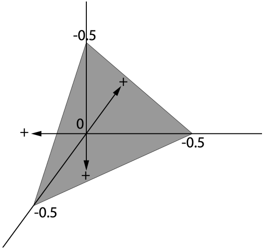


Figure 7: Moment polytope $\text{Im}(\phi_{\text{red}} \circ f)$ when $n = 2$.

Examples

Example 7

$(\mathbb{CP}^n, \omega_{\mathrm{FS}}, \mathbb{T}^n, \nu)$

$$\left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right) \cdot [z_0 : z_1 : \cdots : z_n] = \left[z_0 : e^{i\theta_1} z_1 : \cdots : e^{i\theta_n} z_n\right]$$

with moment map $\nu([z_0 : z_1 : \cdots : z_n]) = -\frac{1}{2\|z\|^2}\left(|z_1|^2, \cdots, |z_n|^2\right) = -\frac{1}{2}(x_1, \cdots, x_n)$

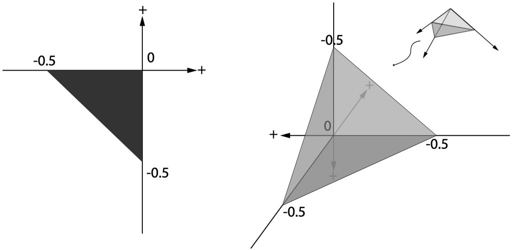


Figure 8: Moment polytopes $\mathrm{Im}(\nu)$ when $n = 2, 3$.

Current Section

Symplectic Toric Manifolds

An action of a group G on a manifold M is called **effective** if each group element $g \neq e$ moves at least one $p \in M$, that is,

$$\bigcap_{p \in M} G_p = \{e\}$$

where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

Theorem 8

Let $(M, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then $\dim M \geq 2m$.

Symplectic Toric Manifolds

A **(symplectic) toric manifold** is a compact connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold:

$$\dim \mathbb{T} = \frac{1}{2} \dim M$$

and with a choice of a corresponding moment map μ.

Example 9

Complex projective space $\mathbb{CP}^n$ is a symplectic toric manifold through the action $\mathbb{T}^n$ but not $\mathbb{T}^{n+1}$.

Delzant's Classification Theorem

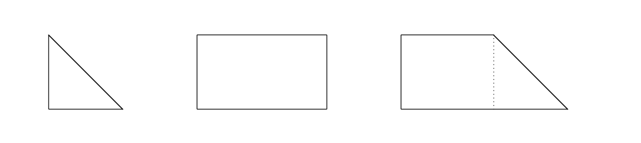
Definition 10 (Unimodular Polytope)

A convex polytope $\Delta \subset \mathbb{R}^n$ is called **Delzant**, or **unimodular** if it satisfies

- (Simplicity) there are n edges meeting at each vertex,
- (Rationality) the edges meeting at the vertex p are rational in the sense that every edge E_k is of the form $p + tu_k$ where $t \in [0, T]$ and $u_k \in \mathbb{Z}^n$,
- (Smoothness) for each vertex with edges $E_1, \ldots, E_n$ the corresponding vectors $u_1, \ldots, u_n$ spanning the edges can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Example 11 (Examples of Unimodular Polytopes)

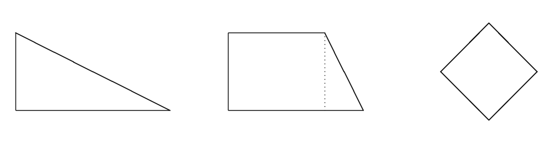
The following examples of unimodular polytopes are from [1].



Delzant's Classification Theorem

Example 12 (Non-examples of Unimodular Polytopes)

The following non-examples of unimodular polytopes are from [1].



Definition 13 (Symplectic Toric Isomorphisms)

Two symplectic toric manifolds, $(M_k, \omega_k, \mathbb{T}^n, \mu_k)$, $k = 1, 2$, are **isomorphic** if there exists an equivariant symplectomorphism $\varphi : M_1 \to M_2$, i.e., a symplectomorphism φ such that $\varphi([\theta] \cdot p) = [\theta] \cdot \varphi(p)$.

Delzant's Classification Theorem

Theorem 14 (Delzant, [9])

Symplectic toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:

$$\frac{\{\textit{symplectic toric manifolds}\}}{\{\textit{isomorphisms}\}} \longleftrightarrow \frac{\{\textit{Delzant polytopes}\}}{\{\textit{translations}\}}$$

$$\left(M^{2n}, \omega, \mathbb{T}^n, \mu\right) \longmapsto \mu\,(M)\,.$$

Steps of the proof.

1. The map is well-defined: M is toric $\Longrightarrow \mu(M)$ is Delzant. This is a consequence of the equivariant Darboux theorem (see [13] for example).

Delzant's Classification Theorem

2. The map is surjective: let $M \mapsto \mu(M)$ be denoted by f and define $g : \Delta \to M_\Delta$ where Δ is Delzant and M_Δ is toric with $\omega_\Delta, \mathbb{T}^n, \mu_\Delta$. The constant involved in μ_Δ can be chosen such that $\mu(M_\Delta) = \Delta$, i.e., $f \circ g = \mathrm{id}$. This will prove the surjectivity of f. This part follows from Delzant's construction of M_Δ.
3. The map is injective: we also need to show $g \circ f = \mathrm{id}$. [13, Sections 2.4 and 2.5] show how Lerman did a different construction, i.e., a symplectic toric manifold E^Δ from a given Δ such that $\mu(E^\Delta) = \Delta$ (so surjectivity is fulfilled); but also that if we start with M and let $\Delta = \mu(M)$, then E^Δ is isomorphic to M (this shows injectivity). We will not include Lerman's construction here.

Current Section

Hermitian Spectra

Theorem 15 (Schur-Horn Theorem, [14])

Let $d_1, \dots, d_n$ and $\lambda_1, \dots, \lambda_n$ be real numbers. There is an $n \times n$ Hermitian matrix with diagonal entries $d_1, \dots, d_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if the vector $(d_1, \dots, d_n)$ lies in the convex hull of the set of vectors whose coordinates are all possible permutations of $(\lambda_1, \dots, \lambda_n)$.

Theorem 16 (Birkhoff-von Neumann Theorem)

A bistochastic matrix = a convex combination of permutation matrices.

Consider $T : \mathcal{H} \to \mathfrak{u}(n)^*; \xi \mapsto \mathrm{tr}(i\xi\,\cdot\,)$. Then T is an intertwining operator or $\mathrm{U}(n)$-equivariant isomorphism for conjugation representation and coadjoint representation of unitary group, i.e., $\forall A \in \mathrm{U}(n)$, $\mathrm{Ad}^*(A) \circ T = T \circ \Psi(A)$. Lie group theory gives orbits $\mathcal{H}_\lambda$'s C^∞-structures and KKS-form gives it symplectic structures. $\mathcal{H}_\lambda$ is a Hamiltonian $\mathrm{U}(n)$-space. Now apply AGS theorem.

Hermitian Spectra

More general Hermitian spectra problems like $\{(\lambda, \mu, \nu) | \mathcal{H}_\lambda + \mathcal{H}_\mu + \mathcal{H}_\nu = 0\}$; see Knutson and Tao's work [15, 16].

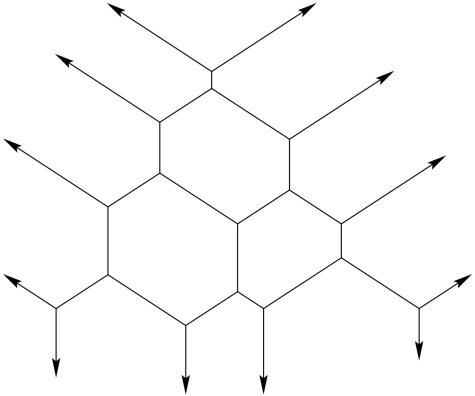


Figure 9: A honeycomb. Pic. taken from [16].

Semisimple Lie Group Actions

Knutson used in his paper [15] the following Kirwan's generalization of AGS theorem for nonabelian Lie groups.

Theorem 17 (Kirwan, [17])

Let (M, ω, G, ϕ) be a compact Hamiltonian G-manifold with G a compact Lie group. Then the intersection of the image $\phi(M)$ with the positive Weyl chamber $\mathfrak{t}^*_+$ let $\mathfrak{t}^*_+$ be a positive Weyl chamber for a maximal compact subgroup K of G is a convex polytope.

Weinstein generalized that to noncompact cases.

Theorem 18 (Weinstein, [8])

Let G be a semisimple Lie group, let $\mathfrak{t}^*_+$ be a positive Weyl chamber for a maximal compact subgroup K of G, and let $\mathcal{U}$ be a coadjoint-invariant open subset of the set $\mathcal{D} \subset \mathfrak{g}^*$ such that $\mathcal{U} \cap \mathfrak{t}^*_+$ is convex. If (M, μ) is a connected, proper, Hamiltonian $(G, \mathcal{U})$-space, then $\mu(M) \cap \mathfrak{t}^*_+$ is a closed, convex, locally polyhedral subset of $\mathfrak{t}^*_+ \cap \mathcal{U}$, and $\mu^{-1}(\xi)$ is connected for each $\xi \in \mathcal{U}$.

Geometric Representation Theory

Kirillov-Kostant Souriau form on coadjoint orbit.

Symplectic toric manifolds.

(M, ω) is **prequantizable** if there is a Hermitian line bundle $\mathbb{L} \to M$ and a connection ∇ on $\mathbb{L}$ whose curvature form is ω. Geometric quantization associates M with a prequantum Hilbert space $Q(M, \omega)$ that satisfies several axioms:

(1) Multiplicity: $Q(M_1 \times M_2) = Q(M_1) \otimes Q(M_2)$;

(2) Duality: $Q(M, -\omega) = Q^*(M, \omega)$;

(3) Finiteness: M compact $\implies \dim Q(M) < \infty$;

(4) Functoriality: G compact Lie group. To every Hamiltonian action $\Phi : G \to \mathrm{Sympl}(M, \omega)$ corresponds a unitary representation ρ of G on $Q(M)$. This is illustrated by the following Kostant's formulation of Bott-Borel-Weil theorem.

Geometric Representation Theory

Theorem 19 (Bott-Borel-Weil Theorem; Kostant's Formulation)

For a compact connected Lie group, there is a one-to-one correspondence between irreducible unitary representations of G and prequantizable coadjoint orbits.

There is another axiom:

(5) Reduction: (M, ω, G, μ) Hamiltonian. $X = \mu^{-1}(0)/G$ reduced space. ρ be the corresponded BBW representation. Then let $Q(M)_G$ be the invariant subspace $\{v \in Q(M) | \rho_g(v) = v, \forall g\}$. Then the axiom is that $Q(X) = Q(M)_G$.

Geometric Representation Theory

Guillemin and Sternberg [18] proposed a counting formula on corresponded unitary irreducible representations. Before that, we recall the familiar version of Schur's lemma in finite dimensional vector space over $\mathbb{C}$ that the subspace of all intertwining operators between irreducible representations φ and ρ is

$$\operatorname{Hom}_G(\varphi,\rho) = \begin{cases} 0, & \varphi \nsim \rho \quad (1) \\ \mathbb{C}\,\mathrm{id}, & \varphi = \rho \quad (2) \end{cases}$$

Note that (2) requires $\mathbb{F} = \mathbb{C}$ and finite dimension while (1) do not need that. (2) says that $\dim_{\mathbb{C}} \operatorname{Hom}_G(\varphi,\varphi) = 1$.

Lemma 20 (Schur's lemma for a Hilbert space representation; see here)

Let φ be an irreducible representation of group G on a Hilbert space $\mathcal{H}$. The subspace of all bounded operators in $\operatorname{Hom}_G(\varphi,\varphi)$ is $\mathbb{C}\,\mathrm{id}$.

Geometric Representation Theory

- **Multiplicity conjecture:** For compact connected Lie group G and Hamiltonian G-space (M, ω, G, μ) and M_O = reduced space of $M \times O^-$,

$$\text{Multiplicity of } \rho_O \text{ in } Q(M) = \text{Riemann-Roch number of } M_O$$

- **Toric case:** $M = \mathbb{C}^d$, G abelian, i.e., $= \mathbb{T}^n$, coadjoint orbits O are just constants ξ in $(\mathbb{R}^n)^*$. Let Δ be a Delzant polytope by eqs. $\langle x, u_i \rangle \geq \lambda_i$, $i = 1, \cdots, d$ and $\pi : \mathbb{R}^d \to \mathbb{R}^n$; $e_i \mapsto u_i$. There induce a map $\pi : \mathbb{T}^d \to \mathbb{T}^n$ with kernel N. $(\mathbb{C}^d, \omega_0, \mathbb{T}^d, \mu)$ restricts to $(\mathbb{C}^d, \omega_0, N, i^* \circ \mu)$. Let $\lambda^0 = i^*(\lambda)$. Reduce $(\mathbb{C}^d, \omega_0, N, i^* \circ \mu)$ at level $-\lambda^0$ will get us Delzant's construction (M_Δ, μ_Δ). This is the symplectic manifold such that $\mu_\Delta(M_\Delta) = \Delta$. In previous context, $M_O = M_\Delta$.
- **RHS** is defined using Chern class from which Todd class is extracted, and this is equal to $\mathrm{Todd}_h(\mathrm{vol}(\Delta_h))|_{h=0}$.
- **LHS** is $\#\rho_{\lambda^0}$ in $Q(\mathbb{C}^d) = \#(\Delta \cap (\mathbb{Z}^n)^*) = \#(\Delta \cap \mathbb{Z}^n)$.

Geometric Representation Theory

- Pukhlikov-Khovanskii [11] equated these two:

$$\#(\Delta \cap \mathbb{Z}^n) = \mathrm{Todd}_h(\mathrm{vol}(\Delta_h))|_{h=0}$$

- So the Multiplicity conjecture is true for symplectic toric case.
- This was conjectured in 1980s by Guillemin and Sternberg [18] and was proven in 1990s by Eckhard Meinrenken as well as Youliang Tian and Weiping Zhang. It is now known as quantization commutes with reduction.

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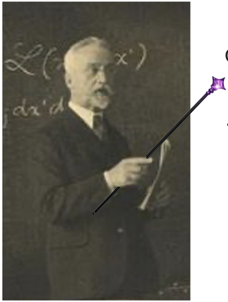
Questions?



Cartan's magic formula

$$\mathcal{L}_X = \mathrm{d}\iota_X + \iota_X\,\mathrm{d}$$

$$\begin{aligned}\mathcal{L}_{X_f} g &= \iota_{X_f} dg \\ &= \iota_{X_f}\iota_{X_g}\omega \\ &= \omega\left(X_g, X_f\right) \\ &= -\{f, g\}\end{aligned}$$

Schur–Horn theorem

$$\begin{aligned} d_1 &\le \lambda_1 \\ d_2 + d_1 &\le \lambda_1 + \lambda_2 \\ \vdots &\le \vdots \\ d_{N-1} + \cdots + d_2 + d_1 &\le \lambda_1 + \lambda_2 + \cdots + \lambda_{N-1} \\ d_N + d_{N-1} + \cdots + d_2 + d_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_{N-1} + \lambda_N \end{aligned}$$