Errata for: High-Dimensional Probability, by Roman Vershynin

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Lemma 1.2.1: remove the second X from the first sentence.

Exercise 2.2.8: replace "probability $1-\varepsilon$" by "probability at least $1-\varepsilon$".

Section 2.5, the line below the first displayed formula: replace "term $a_i X_i$" by "term X_i".

Proof of Proposition 2.5.2, p. 23, replace line 3 by

$$\leq 3p(p/2)^{p/2} \quad (\text{since } \Gamma(x) \leq 3x^x \text{ for all } x \geq 1/2).$$

In the next line, replace $K_2 \leq 2$ with $K_2 \leq 3$.

Proof of Proposition 2.5.2, p. 23, l.15: replace $K_3 = 1/2\sqrt{e}$ by $K_3 = 2\sqrt{e}$.

Remark 2.5.3: Replace "by other absolute constants" with "by any other absolute constant that is larger than 1".

Proof of Proposition 2.7.1, p. 30, l.12: replace $K_5 = 1/2e$ by $K_5 = 2e$.

Exercise 2.8.5: replace "a random variable" by "a mean-zero random variable".

Exercise 3.2.2 (b): replace "a random vector with invertible covariance matrix" by "a random vector with mean μ and invertible covariance matrix".

Section 3.3.5, the sentence below the firt displayed formula: replace "Denote the covariance matrix" by "Assume that $\mathbb{E}\, X = 0$ (translate K appropriately to achieve this) and denote covariance matrix".

Section 3.4.3, first paragraph: replace "which we discussed in Section 3.4.3" by "which we discussed in Section 3.3.1".

Exercise 3.4.9: Replace part (b) with the following: "Show that the subgaussian norm of this distribution is *not* bounded by an absolute constant as the dimension n grows."

Section 3.6.2, bottom of p. 61: replace "four black vertices" by "three black vertices". Replace "seven white vertices" by "four white vertices".

Section 3.6.3, Equation (3.25): add a comma between X_i and X_j. Also, right after the colon, add "$X_i \in \mathbb{R}^n$, " Section 3.6.3, third paragraph, second sentence: after "Choose a random hyperplane in $\mathbb{R}^n$" add "passing through the origin".

Exercise 3.7.5 part (b), after "polynomial $f : \mathbb{R} \to \mathbb{R}$", add "with non-negative coefficients,"

Section 4.1.5, first sentence: replace $s_r n(A)$ with $s_n(A)$.

Section 4.2, Figure 4.1(b), caption: replace $\mathcal{P}(K, \varepsilon) \leq 10$ by $\mathcal{P}(K, \varepsilon) \geq 10$.

Remark 4.2.3: Replace "metric space" with "complete metric space".

Definition 4.2.4, first line: replace (T, d, ε) with (T, d).

Lemma 4.2.8, proof, last sentence: replace "the upper bound in the lemma is proved" by "the lower bound in the lemma is proved".

Section 4.3.1, proof of Proposition 4.3.1, second sentence: replace "into bit string" by "into bit strings".

Section 4.3.1, proof of Proposition 4.3.1, in the displayed formula, replace $d(x, x_0) + d(y, y_0)$ by $d(x, x_0) + d(y, x_0)$.

Theorem 4.7.5, last sentence: replace "misclassified vertices" by "misclassified points".

Exercise 5.1.2, part 1: replace the right hand side of the bound by $\sup_{x \in \mathbb{R}^n} \|\nabla f(x)\|_2$.

Lemma 5.1.7: at the end of the footnote, replace ε by t.

Exercise 5.1.9 part (b), replace the displayed formula by

$$\sigma(A_{2t}) \geq 1 - 2\exp(-ct^2).$$

Exercise 5.1.14, footnote 6 on p. 104: replace ε by t.

Exercise 5.2.4: replace $p > 0$ by $p \geq 1$.

Section 5.2.6, p. 108, l.6: Replace "The probability P" by "The probability $\mathbb{P}$".

Remark 5.2.8: Replace "we can consider the singular value decomposition" by "we can make it from an $n \times n$ Gaussian random matrix G with $N(0, 1)$ independent entries. Indeed, consider the singular value decomposition"

Section 5.3, p. 113, l. 5, replace $1 - 3\exp(-c\varepsilon^2 m/2)$ by $1 - 2\exp(-c\varepsilon^2 m/2)$.

Exercise 5.3.3: replace $Q = (1/\sqrt{m})A$ by $Q = (1/\sqrt{n})A$.

Definition 5.4.4: replace "if all eigenvalues" by "if X is symmetric and all eigenvalues".

Exercise 5.4.5 should be replaced by the following expanded version:

Exercise 5.4.5. Prove the following properties.

(a) $\|X\| \le t$ if and only if $-tI \preceq X \preceq tI$.
(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. If $f(x) \le g(x)$ for all $x \in \mathbb{R}$ satisfying $|x| \le K$, then $f(X) \preceq g(X)$ for all X satisfying $\|X\| \le K$.
(c) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function and X, Y are commuting matrices. Then $X \preceq Y$ implies $f(X) \preceq f(Y)$.
(d) Give an example showing that property (c) may fail for non-commuting matrices.

Hint: Find 2×2 matrices such that $0 \preceq X \preceq Y$ but $X^2 \not\preceq Y^2$.

(e) In the following parts of the exercise, we develop weaker versions of property (c) that hold for arbitrary, not necessarily commuting, matrices. First, show that $X \preceq Y$ always implies $\operatorname{tr} f(X) \le \operatorname{tr} f(Y)$ for any increasing function $f : \mathbb{R} \to \mathbb{R}$.

Hint: Using Courant-Fisher's min-max principle (**??**), show that $\lambda_i(X) \le \lambda_i(Y)$ for all i.

(f) Show that $0 \preceq X \preceq Y$ implies $X^{-1} \succeq Y^{-1}$ if X is invertible.

Hint: First consider the case where one of the matrices is the identity. Next, multiply the inequality $X \preceq Y$ by $Y^{-1/2}$ on the left and on the right.

(g) Show that $0 \preceq X \preceq Y$ implies $\log X \preceq \log Y$.

Hint: Check and use the identity $\log x = \int_0^\infty (\frac{1}{1+t} - \frac{1}{x+t})\, dt$ and property (f).

Section 5.4.3, in Step 1, the line below displayed formula (5.13): replace $\lambda_{\max(S)}$ with $\lambda_{\max}(S)$.

Section 5.4.3, p. 118, l.-9, replace the sentence in parentheses by the following one: "(Here we used Exercise 5.4.5 again: part (g) to take logarithms on both sides, and then part (e) to take traces of the exponential of both sides.)"

Proof of Lemma 5.4.10, p. 119, l.2: replace $3/K'$ by $3/K$.

Exercise 5.4.11, displayed formula: change $\log n$ to $1+\log n$ in both occurrences.

Section 5.6, Formula (5.18): add $\mathbb{E}$ right after $\frac{1}{m}$ in the middle expression.

Exercise 5.6.6, footnote 7 on p. 124: put the subscript 2 in $\|u_j\|_2$.

Theorem 5.6.1: add "$, n \ge 2$" at the end of the first sentence.

Proof of Theorem 5.6.1, p. 123, l.7 (the first displayed formula after 5.19): change the first $\preceq$ to $=$.

Proof of Theorem 5.6.1: $\operatorname{tr}(\Sigma)$ was typeset in the book as $\operatorname{tr} \Sigma$, i.e. without the parentheses. Put back the parentheses throughout this proof. For example, $\operatorname{tr} \Sigma\, \Sigma$ should be replaced with $\operatorname{tr}(\Sigma)\Sigma$.

Proof of Theorem 6.1.1, p. 129, middle of the page: replace "Using Jensen's and Fubini inequalities" by "Using Jensen's inequality and Fubini theorem".

Exercise 6.1.5, replace "convex function" by "convex and increasing function".

Section 6.2, second line: replace "Berstein's" by "Bernstein's".

Proof of Theorem 6.2.1, the line above (6.8): replace “Chebyshev’s” by “Markov’s”.

The line above Lemma 6.2.2: replace “general distributions” by “general subgaussian distributions”.

Proof of Lemma 6.2.2, p. 131, line -4: remove the 4 from the left hand side.

The line above Lemma 6.2.3: replace “general distributions” by “general subgaussian distributions”.

Proof of Lemma 6.2.3:
(a) Remove K^2 from Equation (6.7)
(b) On the next line, replace $\mu = \sqrt{2}C\lambda$ by $\mu = \sqrt{2C}K\lambda$
(c) In the rest of the proof of this lemma, replace $\sqrt{2}C$ by $\sqrt{2C}K$ in both occurrences.

Exercise 6.2.6: replace B^T by B in both parts (a) and (b).

Exercise 6.2.6: In the first displayed formula, replace $|\lambda| \le \frac{c}{\|B\|}$ by $|\lambda| \le \frac{c}{K\|B\|}$.

Proof of Theorem 6.3.2:
(a) Second displayed formula: add $\|A\|_F =$ immediately before $\|B^\mathsf{T}B\|_F$
(b) Replace “deduce a concentration inequality for $\|X\|_2$” by “deduce a concentration inequality for $\|BX\|_2$”

Exercise 6.4.6: replace “independent random variables” by “independent, mean zero random variables”.

Proof of Theorem 6.5.1, p.140, first paragraph:
(a) replace the first two sentences by the following:
“(Check the first identity carefully!) In other words, $\sum_{i\le j} Z_{ij}^2$ is a diagonal matrix, and its diagonal entries equal $\|A_i\|_2^2$.”
(b) Remove the factor 2 from the right hand side of the displayed formula.

Proof of Lemma 6.7.4:
(a) in the proof of the lower bound, replace the second equation sign with $\le$. (Thus, it should read $\le \mathbb{E}_g \mathbb{E}_X \cdots$.)
(b) in the last sentence of the proof, remove “of the variables”.

In the following places, replace $\|X_t - X_s\|_2$ by $\|X_t - X_s\|_{L^2}$, and replace $\|Y_t - Y_s\|_2$ by $\|Y_t - Y_s\|_{L^2}$: Section 7.1.1 the line above Example 7.1.6, (7.13), Example 8.1.2, Theorem 8.6.1, Corollary 8.6.2, proof of Corollary 8.6.2.

Section 7.2.1, paragraph after Exercise 7.2.2: in the displayed equation for $f(x)$, replace u with τ.

Proof of Lemma 7.2.7, footnote 4 on p. 153: replace $g_i : \mathbb{R} \to \mathbb{R}^n$ by $g_i : \mathbb{R} \to \mathbb{R}$.

Paragraph above Theorem 7.4.1, footnote 6 on p. 170: change $N(t, d, \varepsilon) = \infty$ to $N(T, d, \varepsilon) = \infty$

Section 7.4.1, first sentence: remove “geometric”.

Exercise 7.4.5: replace $\log N$ by $C \log N$.

Proof of Theorem 7.7.1, Step 1: in the sentence “Similarly to our older arguments” and the two next sentences, replace S^{n-1} by S^{m-1}. There are a total of three occurrences.

Replace $N(...)$ by $\mathcal{N}(...)$ in the following places: Theorem 8.1.10, Example 8.1.11 (twice), Exercise 8.1.14, Proof of Theorem 8.3.18 on p. 195 (in the first displayed formula, and only in the right hand side of it).

Section 8.2.3, below the third displayed equation: replace ”the deviation of the sample expectation” by ”the deviation of the population expectation”.

Exercise 8.3.8: Replace $[a, b] \times [a, b]$ by $[a, a + d] \times [b, b + d]$.

The line above Lemma 8.3.13: replace “there are as many shattered subsets” by “there are at least as many shattered subsets”

Theorem 8.3.26, footnote 11 on p. 198: replace “sure” by “almost sure”.

Section 8.4.2, after formula (8.32), add the following sentence: “Here X denotes a random variable with distribution $\mathbb{P}$, i.e. with the same distribution as the sample points $X_1, \dots, X_n \in \Omega$.”

Exercise 8.5.2: change $\log n$ to $\log k$ in the denominator.

Proof of Theorem 8.5.3: remove the black square from p. 208. (This is not the end of the proof yet.)

Below the proof of Theorem 8.5.3, last line on p. 208: change $\le 2^{2^{k+1}}$ to $= 2^{2^{k+1}}$.

Exercise 8.5.6: in the second line, replace $u + 2^k$ with $u + 2^{k/2}$.

Corollary 8.6.2: replace “be a Gaussian process” by “be a mean zero Gaussian process”.

Exercise 8.6.4:
(a) Modify the second sentence as follows: “Assume that[1] $X_0 = 0$, and for all $x, y \in T \cup \{0\}$ we have...”
(b) Replace the hint to this exercise (included in the end of the book) by the following: “Use Remark 8.5.4and the majorizing measure theorem to get a bound in terms of the Gaussian width $w(T \cup \{0\})$, then pass to Gaussian complexity using Exercise 7.6.9.”
(c) Put two coffee cups instead of three for this exercise.

Exercises 8.6.5 and 8.6.6: replace “in the setting of Corollary 8.6.3” by “in the setting of Exercise 8.6.4”.

[1] If $0 \notin T$, then simply define $X_0 := 0$.

Exercise 8.6.5, hint: add the first sentence to the hint: “Argue like in Exercise 8.6.4.”

Figure 8.9: the picture redrawn by CUP has two typos:
(a) Below the horizontal axis, the label should read $X_1 X_2 X_3 X X_n$. Thus, we need to replace the second occurrence of X_1 by X.
(b) Above the curve, replace the rightmost label $T(X)_n$ by $T(X_n)$.

Section 8.7, third paragraph: replace “our bound on (6.2)” by “our bound on (8.46)”.

Proof of Theorem 8.7.1, p. 213, l.7: replace $\|u - w\|$ by $\|u - w\|_2$.

Proof of Theorem 8.7.1, p. 213, l.12, displayed formula for Y_{uv}: switch rad(S) and rad(T). The correct formula should read

$$\|Y_{uv} - Y_{wz}\|_2^2 = \|u - w\|_2^2 \operatorname{rad}(S)^2 + \|v - z\|_2^2 \operatorname{rad}(T)^2.$$

Proof of Theorem 9.2.4, in the last displayed formula replace Σ_N by Σ_m.

Section 10.2, Equation (10.4): replace $x \in T$ by $x' \in T$.

Exercise 10.3.1: change the inequality to $m \geq 2\|x\|_0$.

Exercise 10.3.2(c): change $\|x\|_p$ to $\|x\|_p^p$.

Exercise 10.3.8: put two coffee cups.

Exercise 10.5.9: replace $s_n(A_I)$ by $s_s(A_I)$.

Proof of Lemma 10.5.3, first line: replace “Hölder’s” by “Cauchy–Schwarz”.

Section 10.6.2, middle of page 251: replace “This completes the proof of Lemma 10.26.” by “This completes the proof of lemma 10.6.7.”

Section 11, p. 254, l.5: replace ”whether k is larger” by ”whether m is larger”.

Exercise 11.2.2: Replace ”the ℓ_1 and ℓ_∞ norms” by ”the ℓ_1 norm.”

Exercise 11.2.3: Change the wording to ”matrix $Q := C(\log m)^{-1/2} A$, for some appropriate constant C,”

Exercise 11.3.2: change ”$V = B_2^m$” to ”$\operatorname{conv}(V) = B_2^m$”.

Exercise 11.3.4: replace Tx_0 by x_0.

Exercise 11.3.7: change $\sim \log n$ to $\sim \sqrt{\log n}$.

Exercise 11.3.8: change $m \lesssim r(S)$ to $m \gtrsim r(S)$.

Exercise 11.3.9: change conv(AT) to conv(PT).

Section 11.5, last displayed formula: change $\sqrt{\frac{m}{n}}$ to $\sqrt{\frac{m}{n}} \operatorname{diam}(T)$.