Lecture Notes on Polytopes

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Contents

Chapter 1

Introduction to Polytopes

1.1 Why study polytopes?

- Classical: Euclid's Elements presents the platonic solids as a crowning achievement of Greek mathematics.
- Useful: Linear optimization is equivalent to finding points in polytopes.
- Interdisciplinary: Provide combinatorial tools to other areas of mathematics (e.g. symplectic geometry, algebraic geometry, number theory, etc.)
- Fun for some.

1.2 What is a polytope?

1.2.1 Affine subspaces

The nonempty **affine subspaces**, or **flats**, are the translates of linear subspaces (the vector subspaces of $\mathbb{R}^d$ containing the origin $0 \in \mathbb{R}^d$). The **dimension of an affine subspace** is the dimension of the corresponding linear vector space. Affine subspaces of dimensions $0, 1, 2$, and $d-1$ in $\mathbb{R}^d$ are called points, lines, planes, and hyperplanes, respectively. We take for granted the fact that affine subspaces can be described by as the affine image of some real vector space $\boldsymbol{a} + V$ (where V is a linear subspace) or as the set of all affine combinations of a finite set of points,

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} = \lambda_0 \boldsymbol{x}_0 + \ldots + \lambda_n \boldsymbol{x}_n \text{ for } \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

That is, every affine subspace can be described both as an intersection of affine hyperplanes, and as the **affine hull** of a finite point set (i.e., as the intersection of all affine flats that contain the set). A set of $n \geqslant 0$ points is **affinely independent** if its affine hull has dimension $n-1$, that is, if every proper subset has a smaller affine hull.

Proposition 1.2.1. The two definitions of affine subspace $\boldsymbol{a} + V$ and $\{\sum \lambda_i \boldsymbol{x}_i \mid \sum \lambda_i = 1\}$ are equivalent.

Proof.
From $\boldsymbol{a} + V$ to Affine Combinations:
Given $\boldsymbol{a} + V$, where $\boldsymbol{a}$ is a particular point and V is a vector space, any point in $\boldsymbol{a} + V$ can be written as $\boldsymbol{a} + \boldsymbol{v}$, where $\boldsymbol{v} \in V$. If we choose a basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ for V, then any $\boldsymbol{v} \in V$ can be expressed as a

linear combination $\boldsymbol{v}=\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$, where λ_i are scalars. Therefore, any point in $\boldsymbol{a}+V$ can be written as $\boldsymbol{a}+\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$. If we set $\lambda_0=1-\sum_{i=1}^{k} \lambda_i$, then we can write this as $\lambda_0 \boldsymbol{a}+\sum_{i=1}^{k} \lambda_i\left(\boldsymbol{a}+\boldsymbol{x}_i\right)$, ensuring that $\sum_{i=0}^{k} \lambda_i=1$. This shows that every point in $\boldsymbol{a}+V$ can be seen as an affine combination of points in the subspace.

From Affine Combinations to $\boldsymbol{a}+V$:
Conversely, consider a set defined by affine combinations $\{\sum_{i=0}^{n} \lambda_i \boldsymbol{x}_i \mid \sum \lambda_i=1\}$. Let's choose one of these points, say $\boldsymbol{x}_0$, to play the role of $\boldsymbol{a}$ in the $\boldsymbol{a}+V$ definition. We can then view the differences $\boldsymbol{x}_i-\boldsymbol{x}_0$ as elements of a vector space V, since they represent directions (or displacements) from $\boldsymbol{x}_0$ to other points in the set. This shows that the set of affine combinations can be expressed as $\boldsymbol{a}+V$, where $\boldsymbol{a}=\boldsymbol{x}_0$ and V is the span of $\{\boldsymbol{x}_i-\boldsymbol{x}_0\}$. □

1.2.2 Polytopes

A point set $K \subseteq \mathbb{R}^d$ is **convex** if with any two points $\boldsymbol{x}, \boldsymbol{y} \in K$ it also contains the straight line segment $[\boldsymbol{x}, \boldsymbol{y}]=\{\lambda x+(1-\lambda) y: 0 \leqslant \lambda \leqslant 1\}$.

Clearly, every intersection of convex sets is convex, and $\mathbb{R}^d$ itself is convex. Thus for any $K \subseteq \mathbb{R}^d$, the "smallest" convex set containing K, called the **convex hull** of K, can be constructed as the intersection of all convex sets that contain K :

$$\operatorname{conv}(K):=\bigcap\left\{K^{\prime} \subseteq \mathbb{R}^d: K \subseteq K^{\prime}, K^{\prime} \text{ convex }\right\}$$

For any finite set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq K$ and parameters $\lambda_1, \ldots, \lambda_k \geqslant 0$ with $\lambda_1+\ldots+\lambda_k=1$, the convex hull $\operatorname{conv}(K)$ must contain the point $\lambda_1 \boldsymbol{x}_1+\ldots+\lambda_k \boldsymbol{x}_k$: this can be seen by induction on k, using

$$\lambda_1 \boldsymbol{x}_1+\ldots+\lambda_k \boldsymbol{x}_k=\left(1-\lambda_k\right)\left(\frac{\lambda_1}{1-\lambda_k} \boldsymbol{x}_1+\ldots+\frac{\lambda_{k-1}}{1-\lambda_k} \boldsymbol{x}_{k-1}\right)+\lambda_k \boldsymbol{x}_k$$

for $\lambda_k<1$. When $k=1$, the convex hull of a single point is itself. When $k=2$, every convex set containing $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must contain $[\boldsymbol{x}_1, \boldsymbol{x}_2]$, so their intersection has to contain $[\boldsymbol{x}_1, \boldsymbol{x}_2]$. Then do the induction on k, the size of finite subset in K, by above formula. This will show the $\supseteq$ direction of the following relationship:

$$\operatorname{conv}(K)=\left\{\lambda_1 \boldsymbol{x}_1+\ldots+\lambda_k \boldsymbol{x}_k \,\middle|\, \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq K, \lambda_i \geqslant 0, \sum_{i=1}^k \lambda_i=1\right\}$$

But the right-hand side of this equation is easily seen to be convex, which proves the equality.

Now if $K=\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^d$ is itself finite, then we get the definition of a polytope.

Definition 1.2.2. A **polytope**, or a $\mathcal{V}$**-polytope**, is the convex hull of a finite set of points in some $\mathbb{R}^d$.

$$\operatorname{conv}(K)=\left\{\lambda_1 \boldsymbol{x}_1+\ldots+\lambda_n \boldsymbol{x}_n: n \geqslant 1, \lambda_i \geqslant 0, \sum_{i=1}^n \lambda_i=1\right\} .$$

We consider a generalization.

Definition 1.2.3. A **cone** is a nonempty set of vectors $C \subseteq \mathbb{R}^d$ that with any finite set of vectors also contains all their linear combinations with nonnegative coefficients. In particular, every cone contains 0 . For an arbitrary subset $Y \subseteq \mathbb{R}^d$, we define its conical hull (or positive hull) cone (Y) as the intersection of all cones in $\mathbb{R}^d$ that contain Y. Clearly $C:=\operatorname{cone}(Y)$ is a cone for every Y. Similar to the situation for convex hulls (Lecture 0), one can easily see that

$$\operatorname{cone}(Y)=\{\lambda_{\mathbf{1}} \boldsymbol{y}_1+\ldots+\lambda_k \boldsymbol{y}_k:\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\} \subseteq Y, \lambda_i \geqslant 0\}$$

In the case where $Y = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} \subseteq \mathbb{R}^d$ is a finite set - this is the only case we will need here - this reduces to

$$\operatorname{cone}(Y) := \{t_1\boldsymbol{y}_1 + \ldots + t_n\boldsymbol{y}_n : t_i \geqslant 0\} = \{Y\boldsymbol{t} : \boldsymbol{t} \geqslant 0\}$$

We define that $\operatorname{cone}(Y) = \{0\}$ if Y is the empty set, i.e., if $n = 0$. The **vector sum** (or **Minkowski sum**) of two sets $P, Q \subseteq \mathbb{R}^d$ is defined to be

$$P + Q := \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in P, \boldsymbol{y} \in Q\}$$

Definition 1.2.4. A $\mathcal{V}$**-polyhedron** is any finitely generated convexconical combination: a set $P \subseteq \mathbb{R}^d$ that is given in the form

$$P = \operatorname{conv}(V) + \operatorname{cone}(Y) \quad \text{for some } V \in \mathbb{R}^{d\times n}, Y \in \mathbb{R}^{d\times n'},$$

as the Minkowski sum of a convex hull of a finite point set and the cone generated by a finite set of vectors.

Thus, comparing this to definition of a polytope we get that a $\mathcal{V}$-polytope is a $\mathcal{V}$ polyhedron that is bounded, that is, contains no ray $\{\boldsymbol{u} + \boldsymbol{t}\boldsymbol{v} : t \geqslant 0\}$ with $v \neq 0$. For this we only need to observe that $\operatorname{conv}(V)$ is always bounded. This follows from a trivial computation: if $\boldsymbol{x} \in \operatorname{conv}(V)$, then

$$\min\{v_{ik} : 1 \leqslant i \leqslant n\} \leqslant x_k \leqslant \max\{v_{ik} : 1 \leqslant i \leqslant n\},$$

which encloses $\operatorname{conv}(V)$ in a bounded box.

The **dimension** of a polytope is the dimension of its affine hull. A d**-polytope** is a polytope of dimension d in some $\mathbb{R}^e(e \geqslant d)$. Two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are **affinely isomorphic**, denoted by $P \cong Q$, if there is an affine map $f : \mathbb{R}^d \longrightarrow \mathbb{R}^e$ that is a bijection between the points of the two polytopes. (Note that such a map need not be injective or surjective on the "ambient spaces.")

Example 1.2.5.
The standard d-simplex is $\Delta_d := \operatorname{conv}\{e_1, \ldots, e_{d+1}\} \subseteq \mathbb{R}^{d+1}$
The d-cube is $C_d := \operatorname{conv}\{0,1\}^d = [0,1]^d \subseteq \mathbb{R}^d$. In fact, $C_d = \{\boldsymbol{x} \in \mathbb{R}^d \mid 0 \leqslant x_i \leqslant 1\}$.
The d-cross polytope is $\diamond_d := \operatorname{conv}\{\pm e_1, \ldots, \pm e_d\} \subseteq \mathbb{R}^d$.
Two-dimensional polytopes are called polygons.

We consider another approach to define polyhedron and polytope.

Definition 1.2.6. An $\mathcal{H}$**-polyhedron** is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$. An $\mathcal{H}$**-polytope** is an $\mathcal{H}$-polyhedron that is bounded in the sense that it does not contain a ray $\{\boldsymbol{x} + t\boldsymbol{y} : t \geqslant 0\}$ for any $\boldsymbol{y} \neq 0$. An $\mathcal{H}$-polyhedron can be represented by

$$P = P(A, \boldsymbol{z}) = \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} \leqslant \boldsymbol{z}\} \quad \text{for some } A \in \mathbb{R}^{m\times d}, \boldsymbol{z} \in \mathbb{R}^m.$$

(Here " $A\boldsymbol{x} \leqslant \boldsymbol{z}$ " is the usual shorthand for a system of inequalities, namely $\boldsymbol{a}_1\boldsymbol{x} \leqslant z_1, \ldots, \boldsymbol{a}_m\boldsymbol{x} \leqslant z_m$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are the rows of A, and $z_1, \ldots, z_m$ are the components of $\boldsymbol{z}$.)

We now show that the two definitions are equivalent.

Theorem 1.2.7 (Main theorem for polytopes)**.**

$$\{\mathcal{H}\text{-polytope}\} = \{\mathcal{V}\text{-polytope}\}.$$

A subset $P \subseteq \mathbb{R}^d$ is the convex hull of a finite point set (a $\mathcal{V}$-polytope)

$$P = \operatorname{conv}(V) \quad \text{for some } V \in \mathbb{R}^{d\times n}$$

if and only if it is a bounded intersection of halfspaces (an $\mathcal{H}$-polytope)

$$P = P(A, z) \quad \text{for some } A \in \mathbb{R}^{m\times d}, z \in \mathbb{R}^m$$

This result contains two implications, which are equally "geometrically clear" and nontrivial to prove, and which in a certain sense are equivalent.

This theorem provides two independent characterizations of polytopes that are of different power, depending on the problem we are studying. For example, consider the following four statements.
- Every intersection of a polytope with an affine subspace is a polytope.
- Every intersection of a polytope with a polyhedron is a polytope.
- The Minkowski sum of two polytopes is a polytope.
- Every projection of a polytope is a polytope.

The first two statements are trivial for a polytope presented in the form $P = P(A, \boldsymbol{z})$ (where the first is a special case of the second), but both are nontrivial for the convex hull of a finite set of points. Similarly the last two statements are easy to see for the convex hull of a finite point set, but are nontrivial for bounded intersections of halfspaces.

Theorem 1.2.7 is the version we really need, a very basic statement about polytopes; however, it is not the most straightforward version to prove. Therefore we generalize it to a theorem about polyhedra, due to Motzkin.

Theorem 1.2.8 (Main theorem for polyhedra)**.**

$$\{\mathcal{H}\text{-polyhedron} = \mathcal{V}\text{-polyhedron}\}.$$

A subset $P \subseteq \mathbb{R}^d$ is a sum of a convex hull of a finite set of points plus a conical combination of vectors (a $\mathcal{V}$-polyhedron)

$$P = \operatorname{conv}(V) + \operatorname{cone}(Y) \quad \text{for some } V \in \mathbb{R}^{d\times n}, Y \in \mathbb{R}^{d\times n'}$$

if and only if is an intersection of closed halfspaces (an $\mathcal{H}$-polyhedron)

$$P = P(A, z) \quad \text{for some } A \in \mathbb{R}^{m\times d}, z \in \mathbb{R}^m.$$

First note that Theorem 1.2.7 follows from Theorem 1.2.8: we have already seen that polytopes are bounded polyhedra, in both the $\mathcal{V}$ and the $\mathcal{H}$ versions.

Proof. Sketch of proof of $\supseteq$. Let $P = \operatorname{conv}(V) + \operatorname{cone}(Y)$ and identify V with the $d \times n$ with columns the elements of V and similarly Y with an $d \times m$ matrix. Note that

$$P = \left\{x \in \mathbb{R}^d \,|\exists \lambda \in \mathbb{R}^n, \mu \in \mathbb{R}^m |\, x = V\lambda + Y\mu, \sum \lambda_i = 1, \lambda_i \geqslant 0, \mu_i \geqslant 0\right\}.$$

Let

$$Q = \left\{ \begin{bmatrix} x \\ \lambda \\ \mu \end{bmatrix} \in \mathbb{R}^{d+n+m} \,\middle|\, x \in P \right\}.$$

Note that Q is given by the half-spaces $x - V\lambda - Y\mu \geqslant 0, x - V\lambda - Y\mu \leqslant 0, \sum \lambda_i = 1$, $\lambda_i \geqslant 0, \mu_i \geqslant 0$. Moreover, P is a projection of Q. Thus this direction relies on showing that the projection of an $\mathcal{H}$-polyhedron is an $\mathcal{H}$-polyhedron. This is done using Fourier-Motzkin elimination. □

Example 1.2.9. Suppose Q is the polyhedron given by

$$x_1 - x_2 \leqslant -1, \quad x_1 + x_2 \leqslant 5, \quad -x_1 + x_2 \leqslant 3, \quad -x_1 \leqslant 0$$

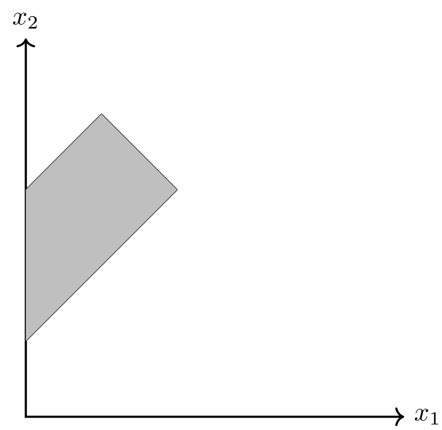
and we wish to project onto the x_1-axis. To do so we should eliminate the x_2-variable. Note,

$$x_1 + 1 \leqslant x_2 \leqslant -x_1 + 5, x_1 + 3.$$

Thus, the projection is given by

$$-x_1 \leqslant 0, x_1 + 1 \leqslant -x_1 + 5, x_1 + 1 \leqslant x_1 + 3$$

which becomes $0 \leqslant x_1 \leqslant 2$. Fourier-Motzkin elimination generalizes this.



Proof. Sketch of proof of $\subseteq$. Let $P = P(A, z)$. Consider $Q = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{d+n} \middle| Ax \leqslant y \right\}$. We will show that Q is a V-polyhedron. Note that $P = Q \cap \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{d+n} \middle| y = z \right\}$, where the latter is an affine hyperplane. We will also show that the intersection of a V-polyhedron with an affine hyperplane is a V-polyhedron.
(1) Q is a $\mathcal{V}$-polyhedron. Note that

$$\begin{aligned} Q &= \left\{ \begin{bmatrix} x \\ Ax + w \end{bmatrix} \middle| x \in \mathbb{R}^d, w \in R^n_{\geqslant 0} \right\} \\ &= \text{cone} \left\{ \begin{bmatrix} \pm e_1 \\ \pm Ae_1 \end{bmatrix}, \ldots, \begin{bmatrix} \pm e_d \\ \pm Ae_d \end{bmatrix}, \begin{bmatrix} 0 \\ f_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ f_n \end{bmatrix} \right\} \end{aligned}$$

where $e_1, \ldots, e_d \in \mathbb{R}^d \times 0$ are the standard basis vectors and $f_1, \ldots, f_n \in 0 \times \mathbb{R}^n$ are the standard basis vectors.
(2) The intersection of a $\mathcal{V}$-polyhedron with an affine hyperplane is a $\mathcal{V}$-polyhedron. We are skipping the proof. □

1.3 Farkas Lemma

The following version of Farkas lemma yields a characterization for the solvability of a system of inequalities (we are using [12]'s numbering).

Proposition 1.3.1 (Farkas lemma I). Let $A \in \mathbb{R}^{m \times d}$ and $z \in \mathbb{R}^m$. Either
(i) there exists a point $\boldsymbol{x} \in \mathbb{R}^d$ with $A\boldsymbol{x} \leqslant \boldsymbol{z}$, or
(ii) there exists a row vector $\boldsymbol{c} \in (\mathbb{R}^m)^*$ with $\boldsymbol{c} \geqslant \mathbb{O}, \boldsymbol{c}A = \mathbb{O}$ and $\boldsymbol{cz} < 0$,
but not both.

The next version of Farkas Lemma states that either a system of equations has a positive solution or a vector that certifies that such a solution does not exist.

Proposition 1.3.2 (Farkas lemma II). Let $A \in \mathbb{R}^{m \times d}$ and $\boldsymbol{z} \in \mathbb{R}^m$. Either
(i) there exists a point $\boldsymbol{x} \in \mathbb{R}^d$ with $A\boldsymbol{x} = \boldsymbol{z}, \boldsymbol{x} \geqslant 0$, or
(ii) there exists a row vector $\boldsymbol{c} \in (\mathbb{R}^m)^*$ with $\boldsymbol{c}A \geqslant \mathbb{O}$ and $\boldsymbol{cz} < 0$,
but not both.

Proof. We have the following equivalences:

$$\begin{aligned}
&\exists \boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{z}, \boldsymbol{x} \geqslant \mathbf{0} \\
&\Longleftrightarrow \exists \boldsymbol{x} : A\boldsymbol{x} \leqslant \boldsymbol{z}, (-A)\boldsymbol{x} \leqslant -\boldsymbol{z}, -\boldsymbol{x} \leqslant \mathbf{0} \\
&\Longleftrightarrow \exists \boldsymbol{x} : \begin{pmatrix} A \\ -A \\ -I_d \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} \boldsymbol{z} \\ -\boldsymbol{z} \\ \mathbf{0} \end{pmatrix} \\
&\overset{\text{FL I}}{\Longleftrightarrow} \nexists c_1 \geqslant \mathbb{0}, c_2 \geqslant \mathbb{0}, b \geqslant 0 : \\
&\qquad (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{b}) \begin{pmatrix} A \\ -A \\ -I_d \end{pmatrix} = \mathbb{0}, (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{b}) \begin{pmatrix} \boldsymbol{z} \\ -\boldsymbol{z} \\ \mathbf{0} \end{pmatrix} < 0 \\
&\Longleftrightarrow \nexists \boldsymbol{c}_1 \geqslant \mathbb{0}, \boldsymbol{c}_2 \geqslant \mathbb{0}, \boldsymbol{b} \geqslant \mathbb{0} : (\boldsymbol{c}_1 - \boldsymbol{c}_2) A - \boldsymbol{b} = \mathbb{0}, (\boldsymbol{c}_1 - \boldsymbol{c}_2) \boldsymbol{z} < 0 \\
&\Longleftrightarrow \nexists \boldsymbol{c} = \boldsymbol{c}_1 - \boldsymbol{c}_2, \boldsymbol{b} \geqslant \mathbb{0} : \boldsymbol{c}A - \boldsymbol{b} = \mathbb{0}, \boldsymbol{c}\boldsymbol{z} < 0 \\
&\Longleftrightarrow \nexists \boldsymbol{c} : \boldsymbol{c}A \geqslant \mathbb{0}, \boldsymbol{c}\boldsymbol{z} < 0.
\end{aligned}$$

□

Proposition 1.3.3 (Farkas lemma IV)**.** Let $V \in \mathbb{R}^{d \times n}, Y \in \mathbb{R}^{d \times n'}$, and $\boldsymbol{x} \in \mathbb{R}^d$. Either
(i) there exist $\boldsymbol{t}, \boldsymbol{u} \geqslant \mathbf{0}$ with $\mathbb{1}\boldsymbol{t} = \mathbf{1}$ and $\boldsymbol{x} = V\boldsymbol{t} + \boldsymbol{Y}\boldsymbol{u}$, or
(ii) there exists a row vector $(\alpha, \boldsymbol{a}) \in \left(\mathbb{R}^{d+1}\right)^*$ with $\boldsymbol{a}\boldsymbol{v}_i \leqslant \alpha$ for all $i \leqslant n$, $\boldsymbol{a}\boldsymbol{y}_j \leqslant 0$ for all $j \leqslant n'$, while $\boldsymbol{a}\boldsymbol{x} > \alpha$, but not both.

Proof. The "either" condition can be stated as

$$\exists \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{u} \end{pmatrix} \geqslant \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} : \quad \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ V & Y \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix}$$

which by version II of the Farkas lemma is equivalent to

$$\begin{aligned}
&\overset{\text{FL II}}{\Longleftrightarrow} \nexists (\alpha, -\boldsymbol{a}) \in \left(\mathbb{R}^{d+1}\right)^* : \quad (\alpha, -\boldsymbol{a}) \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ V & Y \end{pmatrix} \geqslant (\mathbb{0}, \mathbb{0}),\ (\alpha, -a) \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} < 0 \\
&\Longleftrightarrow \nexists (\alpha, -\boldsymbol{a}) \in \left(\mathbb{R}^{d+\mathbf{1}}\right)^* : \quad \alpha\mathbb{1} - \boldsymbol{a}V \geqslant \mathbb{0}, \boldsymbol{a}Y \leqslant \mathbb{0}, \boldsymbol{a}\boldsymbol{x} > \alpha,
\end{aligned}$$

which is equivalent to the negation of the "or" condition. □

1.4 Faces of polytopes

Definition 1.4.1. Let $P \subseteq \mathbb{R}^d$ be a convex polytope. Let $\boldsymbol{c}$ be a row vector. A linear inequality $\boldsymbol{c}\boldsymbol{x} \leqslant c_0$ is **valid** for P if it is satisfied for all points $\boldsymbol{x} \in P$. A **face** of P is any set of the form

$$F = P \cap \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{c}\boldsymbol{x} = c_0\right\}$$

where $\boldsymbol{c}\boldsymbol{x} \leqslant c_0$ is a valid inequality for P. Thus, equivalently, if $\boldsymbol{c}$ is a column vector, a **face** of P can also be written as

$$F = \{\boldsymbol{x} \in P : \forall \boldsymbol{y} \in P, \langle \boldsymbol{y}, \boldsymbol{c} \rangle \leqslant \langle \boldsymbol{x}, \boldsymbol{c} \rangle\}$$

The dimension of a face is the **dimension** of its affine hull: $\dim(F) := \dim(\operatorname{aff}(F))$.

For the valid inequality $\mathbb{0}\mathbf{x} \leqslant 0$, we get that P itself is a face of P. All other faces of P, satisfying $F \subset P$, are called **proper faces**. For the inequality $\mathbb{0}\mathbf{x} \leqslant 1$, we see that $\varnothing$ is always a face of P. The faces of dimensions $0, 1, \dim(P) - 2$, and $\dim(P) - 1$ are called **vertices**, **edges**, **ridges**, and **facets**, respectively. Thus, in particular, the vertices are the minimal nonempty faces, and the facets are the maximal proper faces. The set of all vertices of P, the vertex set, will be denoted by vert (P).

Example 1.4.2. Let $P = \operatorname{conv}(0, e_1, e_2)$.
- If $m = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, then $P_m = \{0\}$.
- If $m = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $P_m = \operatorname{conv}(e_1, e_2)$.
- If $m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $P_m = \{e_2\}$.
- If $m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $P_m = P$.

Proposition 1.4.3 (Ziegler proposition 2.2)**.** Let $P \subseteq \mathbb{R}^d$ be a polytope.
(i) Every polytope is the convex hull of its vertices: $P = \operatorname{conv}(\operatorname{vert}(P))$.
(ii) If a polytope can be written as the convex hull of a finite point set, then the set contains all the vertices of the polytope: $P = \operatorname{conv}(V)$ implies that $\operatorname{vert}(P) \subseteq V$.

Proof. Write $P = \operatorname{conv}(V)$ with V finite. If any $v \in V$ can be written as a convex combination of elements in $V' := V - \{v\}$ then $P = \operatorname{conv}(V')$. Repeat until no longer possible until we get $P = \operatorname{conv}(W)$. We claim that $W = \operatorname{verts}(P)$.

$\supseteq$: Let $v = \lambda_1 w_1 + \cdots + \lambda_n w_n \in \operatorname{verts}(P)$ with $w_1, \ldots, w_n \in W, \sum \lambda_i = 1$, and $\lambda_i \geqslant 0$. Let c be such that $P_c = \{v\}$ and note that for all $i, \langle c, w_i \rangle \langle \langle c, v \rangle$. It follows that

$$\langle c, v \rangle = \sum \lambda_i \langle c, w_i \rangle < \langle c, v \rangle$$

a contradiction.

$\subseteq$: Let $w \in W$ and consider $W' = W - \{w\}$. Since $w \notin \operatorname{conv}(W')$ there does not exist $t \geqslant 0$ such that $w = W't$ and $\mathbb{1}t = 1$. Equivalently, there does not exist $t \geqslant 0$ such that

$$\begin{bmatrix} \mathbb{1} \\ W' \end{bmatrix} t = \begin{bmatrix} 1 \\ w \end{bmatrix}.$$

By Farkas Lemma II, there exists c such that $c \begin{bmatrix} \mathbb{1} \\ W' \end{bmatrix} \geqslant 0$ and $c \begin{bmatrix} 1 \\ w \end{bmatrix} < 0$. Writing $c = (\beta, -b)$, then $\beta\mathbb{1} - bW' \geqslant 0$ and $\beta - bw < 0$. It follows that $bW' \leqslant (\beta, \ldots, \beta)$ and $bw > \beta$, i.e. $P_b = \{w\}$. □

Proposition 1.4.4 (Ziegler proposition 2.3)**.** Let $P \subseteq \mathbb{R}^d$ be a polytope, and $V := \operatorname{vert}(P)$. Let F be a face of P.
(i) The face F is a polytope, with $\operatorname{vert}(F) = F \cap V$.
(ii) Every intersection of faces of P is a face of P.
(iii) The faces of F are exactly the faces of P that are contained in F.
(iv) $F = P \cap \operatorname{aff}(F)$.

We will need another construction: the **vertex figure** obtained by cutting a polytope by a hyperplane that cuts off a single vertex.

For this, we consider a polytope P with $V = \operatorname{vert}(P)$, and a vertex $v \in V$. Let $\boldsymbol{c}\boldsymbol{x} \leqslant c_0$ be a valid inequality with

$$\{\boldsymbol{v}\} = P \cap \{\boldsymbol{x} : \boldsymbol{c}\boldsymbol{x} = c_0\}.$$

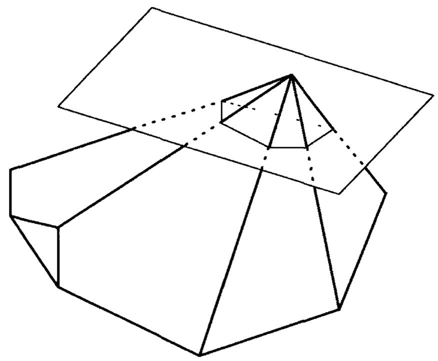


Figure 1.1: Vertex figure.

Furthermore, we choose some $c_1 < c_0$ with $\boldsymbol{cv}' < c_1$ for all $\boldsymbol{v}' \in \mathrm{vert}(P)\backslash\boldsymbol{v}$. Then we define a vertex figure of P at v as the polytope

$$P/\boldsymbol{v} := P \cap \{\boldsymbol{x} : \boldsymbol{cx} = c_1\}.$$

Note that the construction of P/v depends on the choice of c_1 and of the inequality $\boldsymbol{cx} \leqslant c_0$; however, the following result shows that the combinatorial type of P/v is independent of this.

Proposition 1.4.5 (Ziegler Proposition 2.4)**.** There is a bijection between the k-dimensional faces of P that contain v, and the $(k-1)$-dimensional faces of P/v, given by

$$\pi : \quad F \longmapsto F \cap \{\boldsymbol{x} : \boldsymbol{cx} = c_1\},$$
$$\sigma : P \cap \mathrm{aff}\left(\{v\} \cup F'\right) \longleftarrow F'.$$

1.4.1 Face lattices

A **partial ordering** on a (nonempty) set S is a binary relation on S, denoted $\leqslant$, which satisfies the following properties:

- reflexive: for all $s \in S, s \leqslant s$,
- antisymmetric: if $s \leqslant s'$ and $s' \leqslant s$ then $s = s'$,
- transitive: if $s \leqslant s'$ and $s' \leqslant s''$ then $s \leqslant s''$.

When we fix a partial ordering $\leqslant$ on S, we refer to S (or, more precisely, to the pair $(S, \leqslant)$) as a **partially ordered set**, also abbreviated as **poset**.

It is important to notice that we do not assume all pairs of elements in S are **comparable** under $\leqslant$: for some s and s' we may have neither $s \leqslant s'$ nor $s' \leqslant s$. If all pairs of elements can be compared (that is, for all s and s' in S either $s \leqslant s'$ or $s' \leqslant s$) then we say S is **totally ordered** with respect to $\leqslant$.

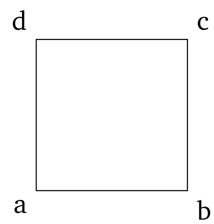
A **chain** in S is a totally ordered subset of S; its **length** is its number of elements minus 1.

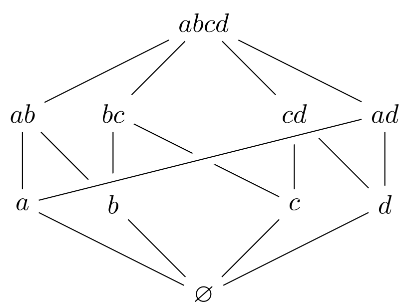
Example 1.4.6. The usual ordering relation $\leqslant$ on $\mathbb{R}$ or on $\mathbb{Z}^+$ is a partial ordering of these sets. In fact it is a total ordering on either set. This ordering on $\mathbb{Z}^+$ is the basis for proofs by induction.

Example 1.4.7. On $\mathbb{Z}^+$, declare $a \leqslant b$ if $a \mid b$. This partial ordering on $\mathbb{Z}^+$ is different from the one in previous example and is called ordering by divisibility. It is one of the central relations in number theory. (Proofs about $\mathbb{Z}^+$ in number theory sometimes work not by induction, but by starting on primes, then extending to prime powers, and then extending to all positive integers using prime factorization. Such proofs view $\mathbb{Z}^+$ through the divisibility relation rather than through the usual ordering relation.) Unlike the ordering on $\mathbb{Z}^+$ in previous example, $\mathbb{Z}^+$ is not totally ordered by divisibility: most pairs of integers are not comparable under the divisibility relation. For instance, 3 doesn't divide 5 and 5 doesn't divide 3 . The subset $\{1, 2, 4, 8, 16, \ldots\}$ of powers of 2 is totally ordered under divisibility.

Definition 1.4.8. The **face lattice** of a convex polytope P is the poset $L := L(P)$ of all faces of P, partially ordered by inclusion.

Example 1.4.9.
(1) The **Boolean lattice** is the poset given by $(2^{[d]}, \subseteq)$, where we use $[d]$ to denote $\{1, \cdots, d\}$ and 2^X is the power set of X.
(2) Face lattice $L(C_2)$ of cycle C_2 using **Hasse's diagram** of poset (the element in the poset that is higher contains those that are lower). The top is the whole polytope, and the bottom is the empty face. The second line has the edges, and the third line has the vertices.





(3) Exercise: show that $L(\triangle_d)$ is the Boolean lattice.

For elements $x, y \in S$ with $x \leqslant y$, we denote by

$$[x, y] := \{w \in S : x \leqslant w \leqslant y\}$$

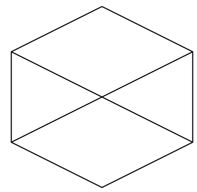
the **interval** between x and y. An interval in S is **boolean** if it is isomorphic to the poset $B_k = \left(2^{[k]}, \subseteq\right)$ of all subsets of a k-element set, for some k.

A poset is **bounded** if it has a unique minimal element, denoted $\hat{0}$, and a unique maximal element, denoted $\hat{1}$. The **proper part** of a bounded poset S is $\bar{S} := S \backslash \{\hat{0}, \hat{1}\}$.

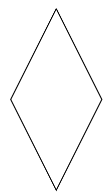
A poset is **graded** if it is bounded, and every maximal chain has the same length. In this case the length of a maximal chain in the interval $[\hat{0}, x]$ is the **rank** of x, denoted by $r(x)$. The rank $r(S) := r(\hat{1})$ is also called the **length** of S. For example, every chain is a graded poset, with $r(C) = |C| - 1$, and the boolean posets B_k are graded of length $r\left(B_k\right) = k$, for all $k \geqslant -1$.

A poset is a **lattice** if it is bounded, and every two elements $x, y \in S$ have a unique minimal upper bound in S, called the **join** $x \vee y$, and every two elements $x, y \in S$ have a unique maximal lower bound in S, called the **meet** $x \wedge y$. (In fact, any two of these three conditions imply the third; also, if every pair of elements has a join respectively meet, then also every finite subset has a join respectively meet.)

Example 1.4.10. The following poset is not a lattice.



Theorem 1.4.11 (Ziegler Theorem 2.7.). Let P be a convex polytope.
(i) For every polytope P the face poset $L(P)$ is a graded lattice of length $\dim(P)+1$, with rank function $r(F)=\dim(F)+1$.
(ii) Every interval $[G,F]$ of $L(P)$ is the face lattice of a convex polytope of dimension $r(F)-r(G)-1$.
(iii) ("Diamond property") Every interval of length 2 has exactly four elements. That is, if $G \subseteq F$ with $r(F)-r(G)=2$, then there are exactly two faces H with $G \subset H \subset F$, and the interval $[G,F]$ looks like



Proof. To see that $L(P)$ is a lattice it suffices to see that it has a unique maximal element $\hat{1}=P$ and a unique minimal element $\hat{0}=\varnothing$, and that meets exist, with $F \wedge G=F \cap G$; this is true because $F \cap G$ is a face of F and of G, and thus of P, by Proposition 1.4.4(ii). And clearly every face of P that is contained in F and in G must be contained in $F \cap G$.

We continue with part (ii). For this we can assume that $F=P$, by Proposition 1.4.4(iii). Now if $G=\varnothing$, then everything is clear. If $G \neq \varnothing$, then it has a vertex $v \in G$ by Proposition 1.4.3(i), which is a vertex of P by Proposition 1.4.4(iii). Now the face lattice of P/v is isomorphic to the interval $[\{v\}, P]$ of the face lattice $L(P)$, by Proposition 2.4. Thus we are done by induction on $\dim(G)$.

For part (i) it remains to see that the lattice $L(P)$ is graded. If $G \subset F$ are faces of P, then from $G = P \cap \operatorname{aff}(G) \subseteq P \cap \operatorname{aff}(F)=F$, which holds by Proposition 1.4.4(iv), we can conclude that $\operatorname{aff}(G) \subset \operatorname{aff}(F)$, and thus that $\dim(G)<\dim(F)$. So it suffices to show that if $\dim(F)-\dim(G) \geqslant 2$, then there is a face $H \in L(P)$ with $G \subset H \subset F$. But by part (ii) the interval $[G,F]$ is the face lattice of a polytope of dimension at least 1 , so it has a vertex, which yields the desired H. Part (iii) is a special case of (ii): the "diamond" is the face lattice of a 1-dimensional polytope. □

Definition 1.4.12. Two polytopes P, Q are **combinatorially equivalent** if $L(P) \simeq L(Q)$.

Example 1.4.13. Up to combinatorial equivalence, for each n there is exactly one polygon with n vertices.

Recall that last time we also defined the f-vector of P to be $f(P):=(f_{-1}, f_0, \ldots, f_d)$, where f_i is the number of faces of dimension i, and the f-polynomial of P to be $f_P(t):=\sum_{i=0}^d f_i t^i$

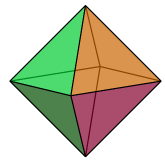
Exercise 1.4.14. Do there exist two non-combinatorially equivalent 3 -dimensional polytopes with the same f-vector?

1.4.2 f-vectors

Definition 1.4.15. The f**-vector** of P is $f(P):=(f_{-1}, f_0, \ldots, f_d)$, where f_i is the number of faces of dimension i. The f**-polynomial** of P is $f_P(t):=\sum_{i=0}^d f_i t^i$.

Example 1.4.16. Let $P=\operatorname{conv}(0, e_1, e_2)$. Then $f(P)=(1,3,3,1)$ and $f_P(t)=3+3t+t^2$.

Example 1.4.17. Let P be an octahedron. Then $f(P)=(1,6,12,8,1)$ and $f_P(t)=6+12t+8t^2+t^3$.



Example 1.4.18. Consider the d-cube $C_d = \left\{x \in \mathbb{R}^d \mid \forall i, -1 \leqslant x_i \leqslant 1\right\}$. Given $v \in \mathbb{R}^d$, we have that

$$(C_d)_v = \{x \in C_d \mid v_1 x_1 + \cdots v_d x_d \max\}.$$

Note that - If $v_i > 0$ then $x_i = 1$ maximizes $v_i x_i$.
- If $v_i < 0$ then $x_i = -1$ maximizes $v_i x_i$.
- If $v_i = 0$ then x_i can be anything.

For example, if $v = (+, -, 0, 0, -, +, 0, +, 0)$, then

$$(C_d)_v = \{(1, -1, a, b, -1, 1, c, 1, d) \mid a, b, c, d \in [-1, 1]\} \simeq C_4.$$

It follows that the faces of C_d are in one-to-one correspondence with d-tuples in $\{\pm 1, 0\}^d$. Moreover, the dimension of the face corresponding to a tuple is the number of 0 s. Therefore, $f_k = \left(\begin{array}{c} d \\ k \end{array}\right) 2^{d-k}$ and $f_{C_d}(t) = \sum_0^d \left(\begin{array}{c} d \\ k \end{array}\right) 2^{d-k} t^k = (2+t)^d$.

Exercise 1.4.19. Compute f_{Δ_d}.

A key question in combinatorics asks the following:

Q: What is the structure of the collection of f-vectors of d-dimensional polytopes? (It is also interesting for other manifolds.)

Example 1.4.20. For 2-dimensional polytopes, aka polygons, the answer is simple. The f-vector is $(1, n, n, 1)$ for some n. For 3-dimensional polytopes, the answer is more complicated, but settled.

Theorem 1.4.21 (Euler)**.** Let P be a 3-dimensional polytope. Then

$$f_0 - f_1 + f_2 = 2.$$

Theorem 1.4.22 (Steinitz)**.**

$$\begin{aligned} & \left\{f \in \mathbb{Z}^5 \mid \exists P, f(P) = f\right\} \\ = & \left\{f \in \mathbb{Z}^5 \mid f_{-1} = f_3 = 1, f_0 - f_1 + f_2 = 2, f_2 \leqslant 2f_0 - 4, f_0 \leqslant 2f_2 - 4\right\}. \end{aligned}$$

In arbitrary dimension, much less is known.

Theorem 1.4.23 (Euler-Poincaré equation)**.** Let P be a d-dimensional polytope. Then $-f_{-1} + f_0 + \cdots + (-1)^d f_d = 0$.

Definition 1.4.24. A polytope is **simplicial** if all of its faces are combinatorially equivalent to standard simplices.

Remark 1.4.25. Billera-Lee and Stanley proved the g-Theorem which gives a characterization for the f-vector of simplicial polytopes. This could be a good topic for the long presentation. There are still contributions being done.

Theorem 1.4.26 ((Xue, 20+))**.** Let P be a d-dimensional polytope with $d+s$ vertices, where $s \geqslant 2$ and $d \geqslant s$. Then for every k, $f_k(P) \geqslant \left(\begin{array}{c} d+1 \\ k+1 \end{array}\right) + \left(\begin{array}{c} d \\ k+1 \end{array}\right) - \left(\begin{array}{c} d+1-s \\ k+1 \end{array}\right)$.

Remark 1.4.27 ((Kalais' 3^d conjecture, '89))**.** If P is centrally symmetric (i.e. v is a vertex if and only if $-v$ is a vertex), then P has at least 3^d nonempty faces, where $d = \dim(P)$.

Remark 1.4.28 (Open question)**.** Is $\left(1, 10^3, 10^5, 10^5, 10^3, 1\right)$ the f-vector of a 4-dimensional polytope?

1.5 Simplicial, Cyclic, and Simple Polytopes

We say that $d+1$ vectors are **affinely independent** if the smallest affine space containing them has dimension d. If P is the convex hull of $d+1$ affinely independent vectors, then P is a d-dimensional polytope and all these vectors are vertices. A d-simplex is the convex hull of $d+1$ affinely independent vectors.

Definition 1.5.1. A d-dimensional polytope is **simplicial** if all of its facets are $(d-1)$ simplices. One can recognize affinely independent vectors by looking at determinants.

Lemma 1.5.2. Let $a_0, \ldots, a_d \in \mathbb{R}^d$. Then $a_0, \ldots, a_d$ are affinely independent if and only if

$$\det \left[\begin{array}{ccc} 1 & \cdots & 1 \\ a_0 & \cdots & a_d \end{array}\right] \neq 0.$$

Definition 1.5.3. Let $d \in \mathbb{N}$. The **moment curve** in $\mathbb{R}^d$ is

$$\mu_d : \mathbb{R} \to \mathbb{R}^d, \quad t \mapsto \left[t, t^2, \ldots, t^d\right].$$

The **cyclic polytope** $C_d\left(t_1, \ldots, t_n\right) = \operatorname{conv}\left\{\mu_d\left(t_1\right), \ldots, \mu_d\left(t_n\right)\right\}$, where $t_1 < \cdots < t_n$ and $n > d$.

The next theorem says that cyclic polytopes have the largest possible number of faces among all convex polytopes with a given dimension and number of vertices.

Theorem 1.5.4 (Upper bound Theorem - McMullen)**.** For any polytope P of dimension d and n verices we have that $f_k(P) \leqslant f_k\left(C_d\left(t_1, \ldots, t_n\right)\right)$ for any k.

Theorem 1.5.5. Let $d \geqslant 2$.
(1) The cyclic polytope $C_d\left(t_1, \ldots, t_n\right)$ is simplicial.
(2) For $S \subseteq [n]$ with $|S| = d$ we have that $\left\{\mu_d\left(t_s\right) \mid s \in S\right\}$ forms a facet if and only if for all $i < j$ not in S, $|\{k \mid k \in S, i < k < j\}|$ is even.

Lemma 1.5.6 (Vandermonde determinant)**.** Let $a_0, \ldots, a_d \in \mathbb{R}$. Then

$$\det \left[\begin{array}{ccc} 1 & \cdots & 1 \\ a_0 & \cdots & a_d \\ \vdots & & \vdots \\ a_0^d & \cdots & a_d^d \end{array}\right] = \prod_{0 \leqslant i < j \leqslant d} \left(a_j - a_i\right).$$

proof of the theorem. (1) By the lemma, any $d+1$ points $\mu_d\left(t_{i_0}\right), \ldots, \mu_d\left(t_{i_d}\right)$ are affinely independent. It follows that all the $\mu_d\left(t_i\right)$ are vertices and that all the facets are simplices.

(2) Let $S \subseteq [n]$ with $S = \left\{s_1, \ldots, s_d\right\}$. Let H_S be the hyperplane through $\mu_d\left(t_{s_1}\right), \ldots, \mu_d\left(t_{s_d}\right)$. Observe that

$$H_S = \left\{x \in \mathbb{R}^d \,\middle|\, \det \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ x & \mu_d\left(t_{s_1}\right) & \cdots & \mu_d\left(t_{s_d}\right) \end{array}\right] = 0\right\}.$$

Let F_S be the defining equation of H_S. As we can see in Figure 1.2, H_S is a facet if and only if $F_S\left(\mu_d\left(t_i\right)\right)$ has the same sign for all $i \notin S$. The sign of $F_S\left(\mu_d(t)\right)$ changes sign as it passes through its zeroes, which are precisely the $\mu_d\left(t_{s_i}\right)$ (since F_S is a polynomial of degree d). It follows that for $i < j, F_S\left(\mu_d\left(t_i\right)\right)$ has the same sign as $F_S\left(\mu_d\left(t_{s_j}\right)\right)$ if and only if there is an even number of sign changes between them.

□

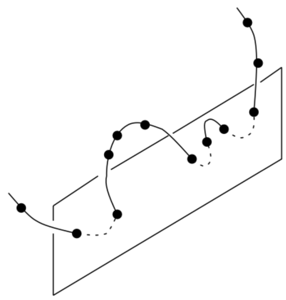


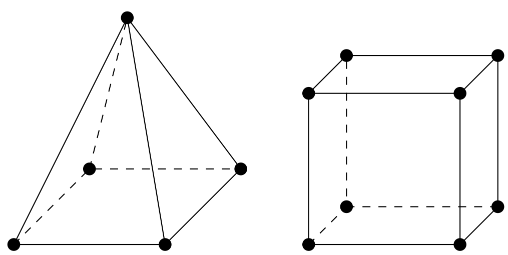
Figure 1.2: H_S with moment curve.

Corollary 1.5.7. The combinatorial type of $C_d(t_1, \ldots, t_n)$ only depends on d, n.

Sketch of proof. Fix d, n. The V-description of the facets of any cyclic polytope is the same. Since the faces are the intersections of the facets, the structure of the face lattice is the same.

Definition 1.5.8. A polytope of dimension d is **simple** if each vertex is contained in exactly d facets.

Example 1.5.9.
(1) The d-cube is simple.
(2) The pyramid with square base is not simple.



Exercise 1.5.10. Show that if P is simple, then every interval $[F, G]$ of $L(P)$ with $f \neq \varnothing$ is a Boolean lattice.

For these polytopes there is a more compact vector encoding the f-vector.

Definition 1.5.11. Let $f(P) = (f_{-1}, f_0, \ldots, f_d)$. The h-polynomial of P is $h_P(t) = f_P(t-1)$. The h-vector of P is $h(P) = (h_0, \ldots, h_d)$ consisting of the coefficients of h_P.

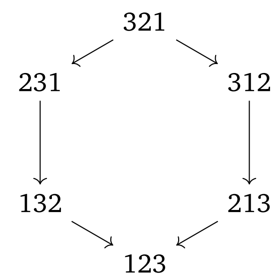
Example 1.5.12.
(1) Let P be the octahedron. We saw that $f_P(t) = 6 + 12t + 8t^2 + t^3$. It follows that $h_P(t) = 1 - t + 5t^2 + t^3$.
(2) Let P be the 3-cube. Then $f_P(t) = 8 + 12t + 6t^2 + t^3$ and $h_P(t) = 1 + 3t + 3t^2 + t^3$.
(3) We saw $f_{C_d}(t) = (2+t)^d$. It follows that $h_{C_d}(t) = (1+t)^d$.

Theorem 1.5.13 (Dehn-Sommerville equations)**.** Let P be a simple d-polytope. Then $h_i = h_{d-i}$ for all i, i.e. $h_P(t)$ is palindromic.

Reason: if P is simple, then the h-polynomial is the Poincare polynomial of a smooth toric variety. This equations reflect Poincaré duality. There is a method to compute the h-vector of a simple polytope.
(1) Find $\lambda : \mathbb{R}^n \to \mathbb{R}$ linear such that $\lambda(u) \neq \lambda(v)$ for all edges $[u, v]$ of P.
(2) For each vertex u, define $\beta(u) := | \{v \in \mathrm{verts}(P) \mid [u, v] \text{ is an edge}, \lambda(v) > \lambda(u)\} |$.

Theorem 1.5.14. Let P be a simple polytope. Then $h_P(t) = \sum_{u \in \mathrm{verts}(P)} t^{\beta(u)}$.

Example 1.5.15. Let $P = \operatorname{conv}\{(w_1, w_2, w_3) \mid w \text{ is a permutation of } [3]\}$. Let $\lambda(x, y, z) = x + 10^2 y + 10^5 z$. The following figure shows the orientation on the edges of P given by $u \to v$ if $\lambda(v) > \lambda(u)$. It follows that $h_P(t) = 1 + 4t + t^2$.



1.6 Permutohedron

A **permutahedron** is defined as

$$P = \operatorname{conv}\{(w_1, w_2, \cdots, w_n) \mid w \text{ is a permutation of } [n]\}$$

where we mean a sequence of numbers $1, \cdots, n$ as a permutation instead of an element of the symmetric group S_n in above definition, and $(w_1, \cdots, w_n)$ is a point in $\mathbb{R}^n$, with components the 1-st, 2-nd, $\cdots$, n-th number of the sequence w.

Alternatively, we can say $w \in S_n$ but mean the second row of its matrix notation instead of meaning its cycle notation. For example, The cycle notation $(1\ 2) \in S_3$ is represented in matrix as

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and is then abbreviated as $w = (2\ 1\ 3) \in S_3$ with $w_1 = 2$, $w_2 = 1$, and $w_3 = 3$.

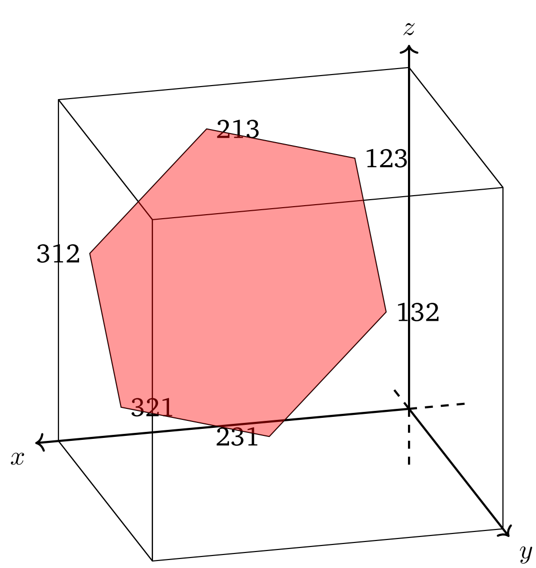


Figure 1.3: conv(S_3)

Just as conv(S_3) can be represented in two dimensions, Figure 1.6 gives the three-dimensional representation of conv(S_4) in four dimensions.

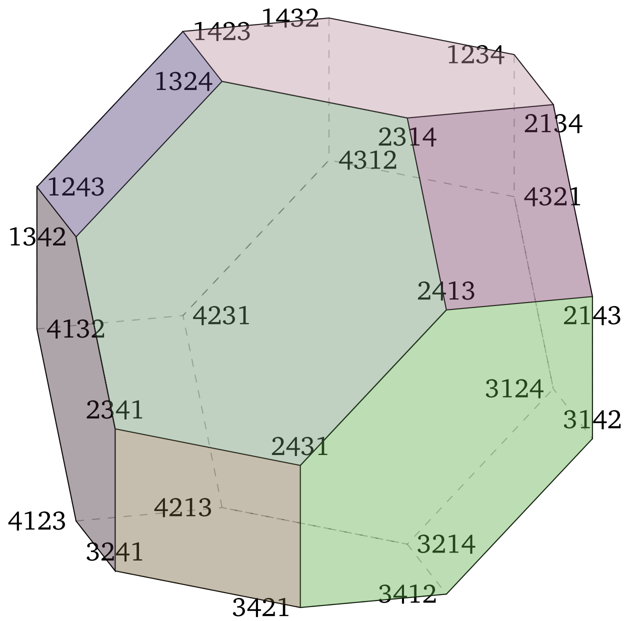


Figure 1.4: conv(S_4)

Recall: There is a method to compute the h-vector of a simple polytope.
(1) Find $\lambda : \mathbb{R}^n \to \mathbb{R}$ linear such that $\lambda(u) \neq \lambda(v)$ for all edges $[u, v]$ of P.
(2) For each vertex u, define $\beta(u) :=| \{v \in \text{verts}(P) \mid [u, v] \text{ is an edge}, \lambda(v) > \lambda(u)\} |$.

Theorem 1.6.1. Let P be a simple polytope. Then $h_P(t) = \sum_{u \in \text{verts}(P)} t^{\beta(u)}$.

Let S_n denote the set of permutations of $[n]$. A **descent** of $w \in S_n$ is an $i \in [n-1]$ such that $w(i) > w(i+1)$. The **Eulerian number** $A(d, i) :=| \{w \in S_n \mid w \text{ has exactly } i \text{ descents}\} |$.

Proposition 1.6.2. Let $\Pi_d := \text{conv}(S_n)$ be the permutohedron. The h-polynomial is $h_{\Pi_d}(t) = \sum_{i=0}^{d-1} A(d, i)t^i$.

Proof. Let us choose a linear function $\lambda(x) = \lambda_1 x_1 + \cdots + \lambda_n x_n$ with $\lambda_1 << \cdots << \lambda_n$. Then λ satisfies condition (1). Let $v = (i, i+1)w$ be adjacent to w. Then, $\lambda(v) > \lambda(w) \Leftrightarrow \lambda_{w^{-1}(i)} > \lambda_{w^{-1}(i+1)} \Leftrightarrow w^{-1}(i) > w^{-1}(i+1) \Leftrightarrow i$ is a descent of w^{-1}. Thus, $\beta(w) = \#\{i \mid i \text{ is a descent of } w^{-1}\}$ and the result follows. □

Exercise 1.6.3. Prove that verts $(\Pi_d) = S_n$ and the edges of Π_d are given by $[w, (i, i+1)w]$ for some i.

1.7 Dual/Polar Polytopes

This is based on [12] section 2.3.

Definition 1.7.1. The polar of $P \subseteq \mathbb{R}^d$ is the set

$$P^\Delta = \left\{c \in \mathbb{R}^d \mid \forall x \in P, \langle x, c\rangle \leqslant 1\right\}$$

Notation: given $m \in \mathbb{R}^d$ and $b \in \mathbb{R}$, let $H_{m,b} = \left\{x \in \mathbb{R}^d \mid \langle m, x\rangle \leqslant b\right\}$.

Example 1.7.2. Let $P = \operatorname{conv}\{(0,2),(-1,1),(-1,0),(0,-1),(1,-1)\}$. Then $P^\Delta \subseteq H_{m,1}$ for all $m \in P$. In fact, taking the vertices is enough so that

$$\begin{aligned} P^\Delta &= \left\{(x,y) \in \mathbb{R}^d \mid 2y, -x+y, -x, -y, x-y \leqslant 1\right\} \\ &= \operatorname{conv}\{(-.5,.5),(-1,0),(-1,-1),(0,-1),(1.5,.5)\} \end{aligned}$$

The following figure shows P and P^Δ.

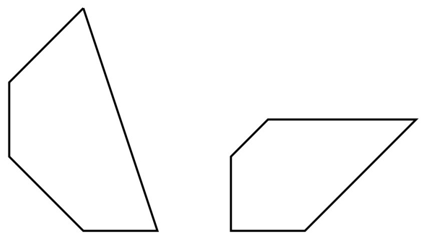


Figure 1.5: P and P^Δ.

Example 1.7.3. Let $P = \operatorname{conv}\{(0,1),(1,1),(1,0)\}$. Then $P^\Delta = \left\{c \in \mathbb{R}^2 \mid c_1 + c_2 \leqslant 1, c_1 \leqslant 1, c_2 \leqslant 1\right\}$. The following figure shows P and P^Δ.

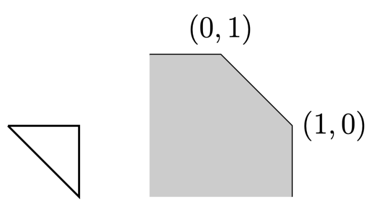


Figure 1.6: P and P^Δ.

Theorem 1.7.4 (Ziegler Theorem 2.11)**.**
(i) $P \subseteq Q$ implies $Q^\Delta \subseteq P^\Delta$.
(ii) $P \subseteq P^{\Delta\Delta}$.
(v) If $0 \in P$, then $P = P^{\Delta\Delta}$.
(vi) If $0 \in \operatorname{int}(P)$ and $P = \operatorname{conv}(V)$, then $P^\Delta = \bigcap_{v \in V} H_{v,1}$.
(vii) If $P = \{x \mid Ax \leqslant \mathbb{1}\}$, then P^Δ is the convex hull of the rows of A.

Example 1.7.5. The cube is the polar to the octahedron. To observe (vi) and (vii), note that the vertices of the octahedron are $(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)$. This gives us the inequalities of the cube.

Example 1.7.6. The cube is the polar to the octahedron. To observe (vi) and (vii), note that the vertices of the octahedron are $(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)$. This gives us the inequalities of the cube.

Remark 1.7.7 (For those taking toric variety)**.** A different way to define the polar is as

$$P^* = \left\{c \in \mathbb{R}^d \mid \forall x \in P, \langle x, c\rangle \geqslant -1\right\}.$$

This is prefered in toric geometry. The polytopes P^Δ and P^* are related by the linear isomorphism $\varphi(x) = -x$. This is because P^Δ is the convex hull of the outer normals of the facets of P whereas P^* is the convex hull of the inner normals.

proof of the theorem. (i)-(ii) Exercises.

(v) By (ii) we only need to show that $P^{\Delta\Delta} \subseteq P$. Suppose that $q \in P^{\Delta\Delta}$ but $q \notin P$. Let $c \in P^{\Delta}$. By definition, $\langle c, q\rangle \leqslant 1$. Since $q \notin P$, there exists a hyperplane separating q from P. Suppose that $H_{m,b}$ is such that $q \notin H_{m,b}$ and $P \subseteq H_{m,b}$. Further assume that the boundary of $H_{m,b}$ is disjoint from P. Since $0 \in P$, we have that $b > 0$. It follows that $\langle m/b, x\rangle < b/b = 1$ for all $x \in P$, i.e., $m/b \in P^{\Delta}$. Since $q \in P^{\Delta\Delta}$, we have that $\langle m/b, q\rangle \leqslant 1$. However, this contradicts that $q \notin H_{m,b}$.

(vi) Since $V \subseteq P$, it follows that $P^{\Delta} \subseteq \bigcap_{v\in V} H_{v,1}$. For the opposite containment, let a be such that $\langle a, v\rangle \leqslant 1$ for all $v \in V$. Suppose that $\langle a, x\rangle > 1$ for some $x \in P$. Since the linear functional $\langle a, -\rangle$ is maximized at the face P_a we can take a vertex of P_a which also has to be in V. Now, $\langle a, v\rangle \geqslant \langle a, x\rangle > 1$, contradicting $\langle a, v\rangle \leqslant 1$. □

Next we want to compare the face lattices of P and P^{Δ}.

Example 1.7.8. Let P be the cube and P^{Δ} be the octahedron. We can see that the face lattices are opposites.

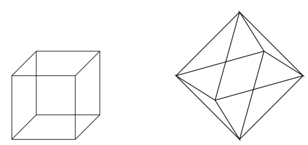


Figure 1.7: P and P^{Δ}.

Definition 1.7.9. Let $(S, \leqslant)$ be a poset. The **opposite poset** $(S, \preceq)$ is defined by $x \preceq y$ if and only if $y \leqslant x$.

Proposition 1.7.10. The face lattice of P^{Δ} is the opposite of the face lattice of P.

This corollary is a consequence of the following theorem.

Theorem 1.7.11 (Ziegler, Theorem 2.12)**.** Let $P = \operatorname{conv}(V) = \{x \mid Ax \leqslant \mathbb{1}\}$ and consider a face $F = \operatorname{conv}(V') = \{x \mid A''x \leqslant \mathbb{1}, A'x = \mathbb{1}\}$, where A', A'' together form the rows of A. Then P^{Δ} has a dual face $F^{\Delta} = \operatorname{conv}(\text{ rows of } A') = \{a \mid aV'' \leqslant 1, aV' = 1\}$. Moreover, every face of P^{Δ} is of this form.

We wish to prove:

Theorem 1.7.12 (Ziegler, Theorem 2.11)**.** (vii) If $P = \{x \mid Ax \leqslant \mathbb{1}\}$, then P^{Δ} is the convex hull of the rows of A.

First, we need to introduce a version of the Farkas Lemma.

Lemma 1.7.13 (Farkas Lemma III)**.** . Let $A \in \mathbb{R}^{m\times d}, z \in \mathbb{R}^m, a \in \mathbb{R}^d$, and $z_0 \in \mathbb{R}$. The polyhedron $P = \left\{x \in \mathbb{R}^d \mid Ax \leqslant z\right\}$ is nonempty if and only if
(1) there exists a vector $c \geqslant 0$ such that $cA = a$ and $\langle c, z\rangle \leqslant z_0$, or
(2) there exists a vector $c \geqslant 0$ such that $cA = 0$ and $\langle c, z\rangle < 0$, or both.

Proof of (vii). The containment $\supseteq$ is straightforward since every row of A has to be in P^{Δ}. For the opposite containment, let $a \in P^{\Delta}$. Since $P \neq \varnothing$ and condition (2) cannot hold, Farkas Lemma III implies that there exists $c \geqslant 0$ such that $cA = a$ and $\langle c, \mathbb{1}\rangle \leqslant 1$. This is close, but what we really need is $c' \geqslant 0$ such that $c'A = a$ and $\langle c', \mathbb{1}\rangle = 1$. To do so, we will find $d = c' - c \geqslant 0$ with $dA = 0$ and $\langle d, \mathbb{1}\rangle = 1 - c > 0$ which leads us to the desired c' by scaling. Farkas Lemma II says that we can find d unless there exist x, y such that

$$Ax = \mathbb{1}y \quad \text{and} \quad y < 0.$$

However, this would imply that for all $\lambda > 0, \lambda x \in P$ contradicting that P is bounded.

Chapter 2

Graph of Polytopes

2.1 $G(P)$ and linear programming

Let P be a convex polytope. The vertices and the edges of P form an abstract, finite, undirected, simple graph, called the **graph of** P and denoted by $G(P)$.

For every face $F \in L(P)$, we denote by $G(F)$ the **induced subgraph of** $G(P)$ on the subset $\mathrm{vert}(F) \subseteq \mathrm{vert}(P)$ of the vertices of $G(P)$, that is, the graph of all vertices in F, and all edges of P between them. This coincides with the graph of F, if F is itself considered as a polytope.

Definition 2.1.1. A linear function $\lambda : \mathbb{R}^d \to \mathbb{R}$ is in **general position** with respect to a polytope P if for all $u, v \in \text{Vert}(P)$, $\lambda(u) \neq \lambda(v)$.

Definition 2.1.2. We will consider **orientations** of $G(P)$, which assign a direction to every edge. An orientation is **acyclic** if there is no directed cycle in it. This implies (because all our graphs are finite) that there is a sink: a vertex that does not have an edge directed away from it. (Proof: Start at any vertex, and keep on walking along directed edges until you close a directed cycle or get stuck in a sink.)

If λ is in general position with repsect to P, then it induces an orientation of $G(P)$. Concretely, $u \to v$ if $\lambda(u) > \lambda(v)$.

Proposition 2.1.3. The orientation of $G(P)$ induced by λ in general position is acyclic and has a unique sink. Moreover, λ is maximized over P at the sink.

Proof. If there was a cycle $v_1 \to \cdots \to v_k \to v_1$, then $\lambda(v_1) > \cdots > \lambda(v_k) > \lambda(v_1)$, a contradiction. Every acyclic graph has a sink. Suppose t is a sink. Let $N(t) = \{v \in \mathrm{verts}(P) \mid [t, v] \text{ edge } \}$ be the neighbors of t. Recall that the vertex figure P/t is obtained by cutting P by a hyperplane that separates v from the other vertices of P. The vertices of P/t are in 1-1 correspondence with the elements of $N(t)$. It follows that $P \subseteq t + \mathrm{cone}(v - t \mid v \in N(t))$. Now, since $\lambda(v) \leqslant \lambda(t)$ for all $v \in N(t)$ we have that given $p \in P$,

$$\lambda(p) = \lambda(t) + \sum_{v \in N(t)} \lambda(v - t) \leqslant \lambda(t),$$

since $\lambda(v - t) < 0$ for all $v \in N(t)$. It follows that t is the unique sink and that it maximizes λ. $\square$

This proposition gives us a method to maximize λ over P.

Proposition 2.1.4 (Dantzig's simplex algorithm).
(1) Start at a vertex v.

(2) If v is a sink, then stop.
(3) Otherwise, move to a neighbor w of v such that $\lambda(w) > \lambda(v)$.

.

This hides the fact that finding a vertex of P is nontrivial. The full simplex algorithm takes care of this.

2.2 The diameter of a polytope.

Definition 2.2.1. The **diameter** $\delta(P)$ of P is the the smallest δ such that any two vertices in P can be connected by a path with at most δ edges.

The diameter of a polytope is a measure of how hard it is to optimize a linear function over it using the simplex algorithm. Concretely, it gives a lower bound on the number of iterations needed.

Let $\Delta(d, n)$ be the maximum diameter of a d-dimensional polytope with at most n facets.

Example 2.2.2. $\Delta(2, n) = \lfloor n/2 \rfloor$.

Question: What is the behavior of $\Delta(d, n)$?

Conjecture(Hirsch Conjecture, '57). $\Delta(d, n) \leqslant n - d$.

This was disproven:

Theorem 2.2.3 (Santos, '10)**.** The Hirsch Conjecture is false. Namely, there exists a 43-dimensional polytope with 86 facets such that $\delta(P) \geqslant 44$.

Conjecture(Polynomial Hirsch Conjecture). There is a polynomial $f(n)$ such that the diameter of every polytope with n facets is bounded above by $f(n)$.

2.3 Simple polytope and graph

Theorem 2.3.1 (Perles '70 - conjecture, Blind-Mani '87)**.** If P is simple then $G(P)$ determines P up to combinatorial equivalence.

In other words, if two simple polytopes have isomorphic graphs, then their face lattices are isomorphic as well.

We will discuss Kalai's simple proof of this result. A key observation will be that if P is simple, then for any vertex v any k edges incident to v determine a face.

proof of the theorem. Let $\mathcal{O}(P)$ be the set of all orientations of $G(P)$. Let us say that $\mathcal{O} \in \mathcal{O}(P)$ is **good** if for all faces F of P, $\mathcal{O}|_{\text{verts}(F)}$ has a unique sink. (Otherwise, we say that $\mathcal{O}$ is **bad**.) We say that a graph is k**-regular** if all its vertices have degree k. The result follows from the following two claims.

Claim 1. Let $U \subseteq \text{verts}(P)$. Then U is the vertex set of a k-dimensional face of P if and only if $G(P)|_U$ is k-regular and there exists a good orientation $\mathcal{O}$ for which U is a downset. Here the last condition means that if $v \in U$ and $u \to v$ is an edge of $\mathcal{O}$, then $u \in U$.

Let $f^{\mathcal{O}} = \sum_{v \in \text{verts}(P)} 2^{\text{indeg(v)}}$.

Claim 2:

$$\{ \text{ good orientations } \} = \left\{ \text{ orientations with } \max f^{\mathcal{O}} \right\}.$$

Given $\mathcal{O} \in \mathcal{O}(P)$, note that

$$\begin{aligned} |\{\text{ nonempty faces of } P\}| \leqslant &|\{(F, v) \mid F \text{ face of } P \text{ and } v \text{ is a sink vertex of } F\}| \\ &= \sum_{v \in \operatorname{verts}(P)} |\{F \mid v \text{ is a sink vertex of } F\}| \\ &= \sum_{v \in \operatorname{verts}(P)} 2^{\operatorname{indeg}(v)} \end{aligned}$$

where the last equality follows from the fact that P is simple. More concretely, for any vertex v any k edges incident to v determine a face. It follows that if v has indegree k, then there are 2^k faces of P with v as a sink. Moreover, we have equality if and only if $\mathcal{O}$ is good. This gives us the most recent claim.

We are now ready to prove the first claim.

$\Rightarrow$ Suppose that U are the vertices of a face F. Since P is simple, it is immediate that $\left.G(P)\right|_U$ is k-regular. Let λ be such that F minimizes λ over P. This λ may not be in general position, but we can perturb by a small amount to be so. The resulting orientation is good and U is a *downset*.

$\Leftarrow$ Suppose that U and $\mathcal{O}$ are as desired. Since $\mathcal{O}_U$ is acyclic, let x be a sink and note that it has indegree k. Let F be the k-dimensional face determined by these k edges. Since $\mathcal{O}$ is good, x is the unique sink of F. Since $u \rightarrow x$ for all vertices along these k edges and U is a downset, then $\operatorname{verts}(F) \subseteq U$. Now, we must have that verts $(F)=U$ since both $\left.G\right|_U$ and $\left.G\right|_{\operatorname{verts}(F)}$ are connected and k-regular. ☐

Theorem 2.3.2 (Balinksi's Theorem)**.** A graph is connected if there is a path between any two vertices.

Definition 2.3.3. A graph is d**-connected** if it stays connected after removing any $\leqslant d-1$ vertices (and their incident edges).

Theorem 2.3.4. If P is d-dimensional, then $G(P)$ is d-connected.

In particular, this says that the degree of any vertex is $\geqslant d$.

Proof. To ease the proof, suppose that $P \subseteq \mathbb{R}^d$. Let $V=\operatorname{verts}(P)$ and $S \subseteq V$ be such that $|S| \leqslant d-1$. To show that $G(P)-S$ is connected we use induction on d. The base case $d=1$ is immediate. For the inductive case, let $L=\operatorname{span}(S)$ and consider two cases.

(1) Suppose L doesn't intersect the interior of P. Then S are the vertices of a face $P_c \subsetneq P$. Consider the face P_{-c} and the orientation of $G(P)$ given by the function $\lambda(x)=\langle-c, x\rangle$. By the argument we used in the proof of Proposition 2.1.3, we have have that every vertex is either in P_{-c},or it has a neighbor $x \notin S$ whose $\langle c, x\rangle$-value is smaller. Thus, there is a c decreasing path from any vertex in $V \backslash S$ to a vertex in P_{-c}. By induction, $G\left(P_{-c}\right)$ is connected and we are done.

(2) Suppose L intersects the interior of P. Let $H=\left\{x \mid\langle c, x\rangle=c_0\right\}$ be a hyperplane containing S and at least one $v \in V \backslash S$. Note that since $L \subseteq H$, then H also intersects the interior of P. This is possible because every set of d points is contained in a hyperplane Consider the faces P_c and P_{-c}. Let $P^{+}=\left\{x \in P \mid\langle c, x\rangle \geqslant c_0\right\}$ and $P^{-}=\left\{x \in P \mid\langle c, x\rangle \leqslant c_0\right\}$. Note that every vertex of P^{+}has a c-increasing path to P_c. Since $G\left(P_c\right)$ is connected, by induction, it follows that $G\left(P^{+}\right) \backslash S$ is connected. Similarly, $G\left(P^{-}\right) \backslash S$ is connected. Since v is in both graphs, we conclude that $G(P) \backslash S$ is connected. ☐

Question: Can we characterize the graphs of polytopes?

Theorem 2.3.5 (Steinitz' Theorem)**.** G is the graph of a 3-dimensional polytope if and only if it is simple, planar, and 3-connected (Simple means that it has no loops or multiple edges.)

Proof. Proof of $\Rightarrow$. Let G be the graph of a 3-dimensional polytope P. It is immediate that it is simple. Also, Balinksi's Theorem implies it is 3-connected. Last, it is planar by blowing up a facet.

Proof of $\Leftarrow$. See [2, Chapter 4]. $\square$

Remark 2.3.6. No similar theorem is known, and it seems that no similarly effective theorem is possible, in higher dimensions.

Chapter 3

The Ehrhart Theory

We shift to [4] for the main reference. The main theme is to count the number of integer points inside a polytope. We begin with some examples.

A convex polytope $\mathcal{P}$ is called **integral** if all of its vertices have integer coordinates, and $\mathcal{P}$ is called **rational** if all of its vertices have rational coordinates. A **unit d-cube**

$$\begin{aligned}\square &:= [0,1]^d = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : \text{ all } x_k = 0 \text{ or } 1\} \\ &= \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \leqslant x_k \leqslant 1 \text{ for all } k = 1, 2, \ldots, d\right\}\end{aligned}$$

We now compute the discrete volume of an integer dilate of $\square$. That is, we seek the number of integer points $t \square \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{>0}$. Here $t\mathcal{P}$ denotes the dilated polytope $\{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\}$ for a polytope $\mathcal{P}$. What is the discrete volume of $\mathcal{P} = \square$? We dilate by the positive integer t, as depicted in Figure 3.1, and count:

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) = \#\left(t \square \cap \mathbb{Z}^d\right) = \#\left([0,t]^d \cap \mathbb{Z}^d\right) = (t+1)^d,$$

a polynomial in the integer variable t. Notice that the coefficients of this polynomial are the binomial coefficients. The number of interior integer points in $t\square^\circ$ is $L_{\square^\circ}(t) = \#\left(t \square^\circ \cap \mathbb{Z}^d\right) = \#\left((0,t)^d \cap \mathbb{Z}^d\right) = (t-1)^d$. Notice that this polynomial equals $(-1)^d L_{\square}(-t)$, the evaluation of the polynomial $L_{\square}(t)$ at negative integers, up to a sign.

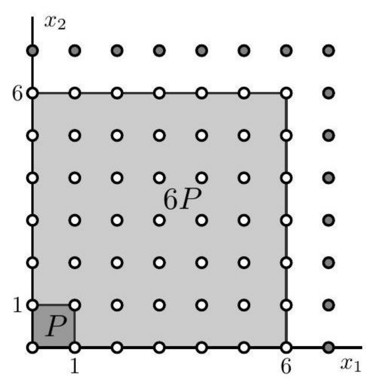


Figure 3.1: The 6^{th} dilate of $\square$ in dimension 2.

The generating function of $L_{\mathcal{P}}$ is called **Ehrhart series** of $\mathcal{P}$:

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t) z^t$$

The Ehrhart series of $\mathcal{P} = \square$ takes on a special form.

$$\begin{aligned}\mathrm{Ehr}_{\square}(z) &= 1 + \sum_{t \geqslant 1}(t+1)^d z^t = \sum_{t \geqslant 0}(t+1)^d z^t = \frac{1}{z}\sum_{t \geqslant 1} t^d z^t \\ &= \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}.\end{aligned}$$

where $A(d,k)$ is the **Eulerian number**, which counts the number of permutations with exactly k descents.

The standard simplex Δ in dimension d is

$$\mathrm{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\} = \left\{(x_1, x_2 \ldots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leqslant 1 \text{ and all } x_k \geqslant 0\right\}$$

(we now include zero as a vertex now). The dilate $t\Delta$ is

$$t\Delta = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leqslant t \text{ and all } x_k \geqslant 0\right\}$$

The lattice-point enumerator of Δ is the polynomial $L_{\Delta}(t) = \left(\begin{array}{c} d+t \\ d \end{array}\right)$. Its evaluation at negative integers yields $(-1)^d L_{\Delta}(-t) = L_{\Delta^{\circ}}(t)$. The Ehrhart series of Δ is $\mathrm{Ehr}_{\Delta}(z) = \frac{1}{(1-z)^{d+1}}$.

3.1 Triangulations

Because most of the proofs that follow work like a charm for a simplex, we first dissect a polytope into simplices. This dissection is captured by the following definition.

A triangulation of a convex d-polytope $\mathcal{P}$ is a finite collection T of d-simplices with the following properties:

- $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$.
- For every $\Delta_1, \Delta_2 \in T, \Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2.

Figure 3.2 exhibits two triangulations of the 3-cube. We say that $\mathcal{P}$ can be triangulated using no new vertices if there exists a triangulation T such that the vertices of every $\Delta \in T$ are vertices of $\mathcal{P}$.

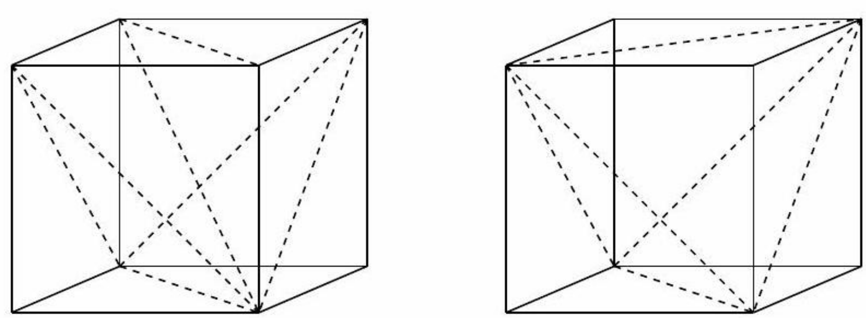


Figure 3.2: Two (very different) triangulations of the 3-cube.

Theorem 3.1.1 (Existence of triangulations). Every convex polytope can be triangulated using no new vertices.

This theorem seems intuitively obvious, but it is not entirely trivial to prove.

In order to prove this, let us consider regular subdivisions.

Let $P=\operatorname{conv}(V) \subseteq \mathbb{R}^d$. Choose $h: V \rightarrow \mathbb{R}$ and let $P^{\prime}=\operatorname{conv}\left\{\left[\begin{array}{c} v \\ h(v) \end{array}\right] \middle| \, v \in V\right\}$. We say that a face F of $P^{\prime}$ is lower if $F=P_c^{\prime}$ for some $c \in \mathbb{R}^{d+1}$ with $c_{d+1}<0$.

Let $\pi: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be the projection onto the first d coordinates.

Proposition 3.1.2. The set $\{\pi(F) \mid F$ lower face of $P^{\prime}\}$ is a subdivision of P. If h is generic, then this subdivision is a triangulation.

Proof. We will only provethe second claim. Suppose $P \subseteq \mathbb{R}^d$ is d-dimensional. First, we will show that each lower facet of $P^{\prime}$ is a simplex, i.e., the convex hull of $d+1$ affinely independent vectors. Suppose we have $d+1$ affinely independent vertices of $P, v_1, \ldots, v_{d+1}$. Let $H \subseteq \mathbb{R}^{d+1}$ be the hyperplane given by the equation

$$\left[\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 1 \\ v_1 & v_2 & \cdots & v_{d+1} & \boldsymbol{x} \\ h\left(v_1\right) & h\left(v_2\right) & \cdots & h\left(v_{d+1}\right) & x_{d+1} \end{array}\right]$$

Note that for all $i,\left[\begin{array}{c} v_i \\ h\left(v_i\right) \end{array}\right] \in H$. Also, note that if we fix $\boldsymbol{x}$, then there is a unique x_{d+1} that makes this equation hold. Thus, if $v \neq v_i$ for all i, then since we chose the $h(v)$ at random, then $\left[\begin{array}{c} v \\ h(v) \end{array}\right]$ is not in H. This proves that $P^{\prime}$ is simplicial, so all of its faces are simplices.

Next, we will show that $\bigcup_{\Delta \in T} \Delta=P$. It suffices to show that $\operatorname{int}(P) \subseteq \bigcup_{\Delta \in T} \Delta$. Let $x \in \operatorname{int}(P)$ and consider the vertical line $\mathcal{L} \subseteq \mathbb{R}^{d+1}$ through x. Since $\mathcal{L} \cap \operatorname{int}\left(P^{\prime}\right) \neq \varnothing$, then $\mathcal{L} \cap P^{\prime}$ is a line segment with endpoints (x, y) and $(x, z), y<z$. Since $(x, y) \in \partial P^{\prime}$, then $(x, y) \in P_{\left(c, c_{d+1}\right)}^{\prime}$ for some $\left(c, c_{d+1}\right)$. Note then that $\left\langle\left(c, c_{d+1}\right),(x, y)\right\rangle>\left\langle\left(c, c_{d+1}\right),(x, z)\right\rangle$ and since $y<z$ this can only hold if $c_{d+1}<0$. It follows that $P_c^{\prime}$ is a lower face and $x \in \pi\left(P_c^{\prime}\right)$.

The last property is left as an exercise

□

3.2 The Ehrhart Series of an Rational Polytope

By now, we have computed several instances of counting functions by setting up a generating function that fits the particular problem in which we are interested. In this subsection, we set up such a generating function for the latticepoint enumerator of an arbitrary rational polytope. Such a polytope is given by its hyperplane description as an intersection of half-spaces and hyperplanes. The half-spaces are algebraically given by linear inequalities, the hyperplanes by linear equations. If the polytope is rational, we can choose the coefficients of these inequalities and equations to be integers (Exercise). To unify both descriptions, we can introduce slack variables to turn the half-space inequalities into equalities. Furthermore, by translating our polytope into the nonnegative orthant (we can always shift a polytope by an integer vector without changing the lattice-point count), we may assume that all points in the polytope have nonnegative coordinates. In summary, after a harmless integer translation, we can describe every rational polytope $\mathcal{P}$ as

$$\mathcal{P}=\left\{\mathbf{x} \in \mathbb{R}_{\geqslant 0}^d: \mathbf{A} \mathbf{x}=\mathbf{b}\right\}$$

for some integral matrix $\mathbf{A} \in \mathbb{Z}^{m\times d}$ and some integer vector $\mathbf{b} \in \mathbb{Z}^m$. (Note that d is not necessarily the dimension of $\mathcal{P}$.) To describe the t^{th} dilate of $\mathcal{P}$, we simply scale a point $\mathbf{x} \in \mathcal{P}$ by $\frac{1}{t}$, or alternatively, multiply $\mathbf{b}$ by t :

$$t\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d_{\geqslant 0} : \mathbf{A}\frac{\mathbf{x}}{t} = \mathbf{b}\right\} = \left\{\mathbf{x} \in \mathbb{R}^d_{\geqslant 0} : \mathbf{A}\mathbf{x} = t\mathbf{b}\right\}$$

Hence the lattice-point enumerator of $\mathcal{P}$ is the counting function

$$L_{\mathcal{P}}(t) = \#\left\{\mathbf{x} \in \mathbb{Z}^d_{\geqslant 0} : \mathbf{A}\mathbf{x} = t\mathbf{b}\right\}$$

Consider the polytope $\mathcal{P}$ given by (3.2), we denote the columns of $\mathbf{A}$ by $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ and expand the function

$$\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)\mathbf{z}^{t\mathbf{b}}}$$

in terms of geometric series:

$$\left(\sum_{n_1\geqslant 0} \overbrace{\mathbf{z}^{n_1\mathbf{c}_1}}^{=(\mathbf{z}^{\mathbf{c}_1})^{n_1}}\right)\left(\sum_{n_2\geqslant 0} \overbrace{\mathbf{z}^{n_2\mathbf{c}_2}}^{=(\mathbf{z}^{\mathbf{c}_2})^{n_2}}\right)\cdots\left(\sum_{n_d\geqslant 0} \overbrace{\mathbf{z}^{n_d\mathbf{c}_d}}^{=(\mathbf{z}^{\mathbf{c}_d})^{n_d}}\right)\frac{1}{\mathbf{z}^{t\mathbf{b}}}.$$

Here we use the abbreviating notation $\mathbf{z}^{\mathbf{a}} := z_1^{a_1} z_2^{a_2}\cdots z_m^{a_m}$ for the vectors $\mathbf{z} = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$. In multiplying out everything, a typical exponent will look like

$$n_1\mathbf{c}_1 + n_2\mathbf{c}_2 + \cdots + n_d\mathbf{c}_d - t\mathbf{b} = \mathbf{A}\mathbf{n} - t\mathbf{b}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$. That is, if we take the constant term of our generating function (3.2), we are counting integer vectors $\mathbf{n} \in \mathbb{Z}^d_{\geqslant 0}$ satisfying

$$\mathbf{A}\mathbf{n} - t\mathbf{b} = \mathbf{0}, \quad \text{that is,} \quad \mathbf{A}\mathbf{n} = t\mathbf{b}.$$

So this constant term will pick up exactly the number of lattice points $\mathbf{n} \in \mathbb{Z}^d_{\geqslant 0}$ in $t\mathcal{P}$:

Theorem 3.2.1 (Euler's generating function)**.** Suppose the rational polytope $\mathcal{P}$ is given by (3.2). Then the lattice-point enumerator of $\mathcal{P}$ can be computed as follows:

$$L_{\mathcal{P}}(t) = \operatorname{const}\left(\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)\mathbf{z}^{t\mathbf{b}}}\right)$$

We finish this section with rephrasing this constant-term identity in terms of Ehrhart series.

Corollary 3.2.2. Suppose the rational polytope $\mathcal{P}$ is given by (3.2). Then the Ehrhart series of $\mathcal{P}$ can be computed as

$$\operatorname{Ehr}_{\mathcal{P}}(x) = \operatorname{const}\left(\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)\left(1-\frac{x}{\mathbf{z}^{\mathbf{b}}}\right)}\right)$$

Proof. By above theorem,

$$\begin{aligned}\operatorname{Ehr}_{\mathcal{P}}(x) &= \sum_{t\geqslant 0}\operatorname{const}\left(\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)\mathbf{z}^{t\mathbf{b}}}\right)x^t \\ &= \operatorname{const}\left(\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)}\sum_{t\geqslant 0}\frac{x^t}{\mathbf{z}^{t\mathbf{b}}}\right) \\ &= \operatorname{const}\left(\frac{1}{\left(1-\mathbf{z}^{\mathbf{c}_1}\right)\left(1-\mathbf{z}^{\mathbf{c}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{c}_d}\right)}\frac{1}{1-\frac{x}{\mathbf{z}^{\mathbf{b}}}}\right).\end{aligned}$$

□

3.3 Discrete Volume of Cones

3.3.1 Cones

A pointed cone $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_m \mathbf{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geqslant 0\}$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ are such that there exists a hyperplane H for which $H \cap \mathcal{K} = \{\mathbf{v}\}$; that is, $\mathcal{K} \backslash \{\mathbf{v}\}$ lies strictly on one side of H. The vector $\mathbf{v}$ is called the apex of $\mathcal{K}$, and the $\mathbf{w}_k$ are the generators of $\mathcal{K}$. The cone is rational if $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Q}^d$, in which case we may choose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Z}^d$ by clearing denominators. The dimension of $\mathcal{K}$ is the dimension of the affine space spanned by $\mathcal{K}$; if $\mathcal{K}$ is of dimension d, we call it a d-cone. The d-cone $\mathcal{K}$ is simplicial if $\mathcal{K}$ has precisely d linearly independent generators.

Just as polytopes have a description as an intersection of half-spaces, so do pointed cones: a rational pointed d-cone is the d-dimensional intersection of finitely many half-spaces of the form

$$\left\{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \cdots + a_d x_d \leqslant b\right\}$$

with integral parameters $a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$ such that the corresponding hyperplanes of the form

$$\left\{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \cdots + a_d x_d = b\right\}$$

meet in exactly one point.

Cones are important for many reasons. The most practical for us is a process called coning over a polytope. Given a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we lift these vertices into $\mathbb{R}^{d+1}$ by adding a 1 as their last coordinate. So, let

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_n = (\mathbf{v}_n, 1)$$

Now we define the cone over $\mathcal{P}$ as

$$\operatorname{cone}(\mathcal{P}) = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_n \mathbf{w}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0\} \subset \mathbb{R}^{d+1}$$

This pointed cone has the origin as apex, and we can recover our original polytope $\mathcal{P}$ (strictly speaking, the translated set $\{(\mathbf{x}, 1) : \mathbf{x} \in \mathcal{P}\}$) by cutting cone $(\mathcal{P})$ with the hyperplane $x_{d+1} = 1$, as shown in Figure 3.3.

By analogy with the language of polytopes, we say that the hyperplane $H = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\right\}$ is a supporting hyperplane of the pointed d-cone $\mathcal{K}$ if $\mathcal{K}$ lies entirely on one side of H, that is,

$$\mathcal{K} \subseteq \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leqslant b\right\} \quad \text{or} \quad \mathcal{K} \subseteq \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geqslant b\right\}$$

A face of $\mathcal{K}$ is a set of the form $\mathcal{K} \cap H$, where H is a supporting hyperplane of $\mathcal{K}$. The $(d-1)$-dimensional faces are called facets, and the 1-dimensional faces edges, of $\mathcal{K}$. The apex of $\mathcal{K}$ is its unique 0 -dimensional face.

Just as polytopes can be triangulated into simplices, pointed cones can be triangulated into simplicial cones. So, a collection T of simplicial d-cones is a triangulation of the d-cone $\mathcal{K}$ if it satisfies the following:

- $\mathcal{K} = \bigcup_{\mathcal{S} \in T} \mathcal{S}$.

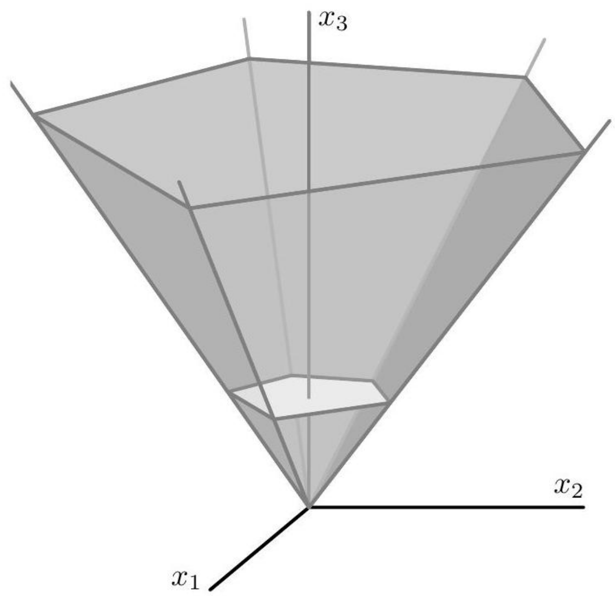


Figure 3.3: Coning over polytope

- For every $\mathcal{S}_1, \mathcal{S}_2 \in T, \mathcal{S}_1 \cap \mathcal{S}_2$ is a face common to both $\mathcal{S}_1$ and $\mathcal{S}_2$.

We say that $\mathcal{K}$ can be triangulated using no new generators if there exists a triangulation T such that the generators of every $\mathcal{S} \in T$ are generators of $\mathcal{K}$.

Theorem 3.3.1. Every pointed cone can be triangulated into simplicial cones using no new generators.

Proof. Given a pointed d-cone $\mathcal{K}$ with apex $\mathbf{v}$, there exists a hyperplane H that intersects $\mathcal{K}$ only at $\mathbf{v}$. Choose $\mathbf{w} \in \mathcal{K}^{\circ}$; then

$$\mathcal{P} := (\mathbf{w} - \mathbf{v} + H) \cap \mathcal{K}$$

is a $(d-1)$-polytope whose vertices are determined by the generators of $\mathcal{K}$. (This construction yields a variant of Figure 3.4.) Now triangulate $\mathcal{P}$ using no new vertices. Each simplex Δ_j in this triangulation gives naturally rise to a simplicial cone

$$\mathcal{S}_j := \{\mathbf{v} + \lambda \mathbf{x} : \lambda \geqslant 0, \mathbf{x} \in \Delta_j\},$$

and these simplicial cones, by construction, triangulate $\mathcal{K}$. □

3.3.2 Integer-Point Transforms for Rational Cones

We want to encode the information contained by the lattice points in a set $S \subset \mathbb{R}^d$. It turns out that the following multivariate generating function allows us to do this in an efficient way if S is a rational cone or polytope:

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, z_d) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$$

The generating function σ_S simply lists all integer points in S in a special form: not as a list of vectors, but as a formal sum of Laurent monomials. We call σ_S the **integer-point transform** of S; the function σ_S also goes by the name moment generating function or simply generating function of S. The integer-point transform σ_S opens the door to both algebraic and analytic techniques.

Example 3.3.2. As a warmup example, consider the 1-dimensional cone $\mathcal{K} = [0, \infty)$. Its integer-point transform is our old friend

$$\sigma_{\mathcal{K}}(z) = \sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \sum_{m \geqslant 0} z^m = \frac{1}{1-z}$$

Example 3.3.3. Now we consider the 2-dimensional cone

$$\mathcal{K} := \{\lambda_1(1,1) + \lambda_2(-2,3) : \lambda_1, \lambda_2 \geqslant 0\} \subset \mathbb{R}^2$$

depicted in Figure 3.4. To obtain the integer-point transform $\sigma_{\mathcal{K}}$, we tile $\mathcal{K}$ by copies of the fundamental parallelogram

$$\Pi := \{\lambda_1(1,1) + \lambda_2(-2,3) : 0 \leqslant \lambda_1, \lambda_2 < 1\} \subset \mathbb{R}^2$$

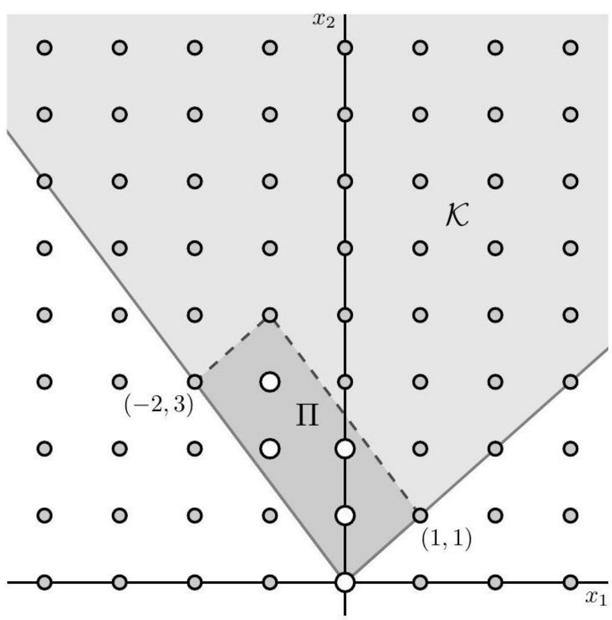


Figure 3.4: The cone $\mathcal{K}$ and its fundamental parallelogram.

More precisely, we translate Π by nonnegative integer linear combinations of the generators $(1,1)$ and $(-2,3)$, and these translates will exactly cover $\mathcal{K}$. How can we list the integer points in $\mathcal{K}$ as Laurent monomials? Let's first list all vertices of the translates of Π. These are nonnegative integer combinations of the generators $(1,1)$ and $(-2,3)$, so we can list them using geometric series:

$$\sum_{\substack{\mathbf{m}=j(1,1)+k(-2,3) \\ j,k \geqslant 0}} \mathbf{z}^{\mathbf{m}} = \sum_{j \geqslant 0} \sum_{k \geqslant 0} \mathbf{z}^{j(1,1)+k(-2,3)} = \frac{1}{(1 - z_1 z_2)\left(1 - z_1^{-2} z_2^3\right)}$$

We now use the integer points $(m,n) \in \Pi$ to generate a subset of $\mathbb{Z}^2$ by adding to (m,n) nonnegative linear integer combinations of the generators $(1,1)$ and $(-2,3)$. Namely, we let

$$\mathcal{L}_{(m,n)} := \{(m,n) + j(1,1) + k(-2,3) : j,k \in \mathbb{Z}_{\geqslant 0}\}.$$

It is immediate that $\mathcal{K} \cap \mathbb{Z}^2$ is the disjoint union of the subsets $\mathcal{L}_{(m,n)}$ as (m,n) ranges over $\Pi \cap \mathbb{Z}^2 =$

$\{(0,0),(0,1),(0,2),(-1,2),(-1,3)\}$. Hence

$$\begin{aligned}
\sigma_{\mathcal{K}}(\mathbf{z}) &= \sum_{(m,n)\in\{(0,0),\cdots,(-1,3)\}} \sum_{j,k\geqslant 0} \mathbf{z}^{(m,n)+j(1,1)+k(-2,3)} \\
&= \sum_{(m,n)\in\{(0,0),\cdots,(-1,3)\}} z_1^m z_2^n \sum_{j,k\geqslant 0} \mathbf{z}^{j(1,1)+k(-2,3)} \\
&= \left(1+z_2+z_2^2+z_1^{-1}z_2^2+z_1^{-1}z_2^3\right) \sum_{\substack{\mathbf{m}=j(1,1)+k(-2,3)\\ j,k\geqslant 0}} \mathbf{z}^{\mathbf{m}} \\
&= \frac{1+z_2+z_2^2+z_1^{-1}z_2^2+z_1^{-1}z_2^3}{(1-z_1z_2)\left(1-z_1^{-2}z_2^3\right)}
\end{aligned}$$

Similar geometric series suffice to describe integer-point transforms for rational simplicial d-cones. The following result utilizes the geometric series in several directions simultaneously. We recall that a d-dimensional cone is **simplicial** if it has d linearly independent generators.

Theorem 3.3.4. Suppose

$$\mathcal{K} := \{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d : \lambda_1,\lambda_2,\ldots,\lambda_d \geqslant 0\}$$

is a simplicial d-cone, where $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_d \in \mathbb{Z}^d$. Then for $\mathbf{v}\in\mathbb{R}^d$, the integer-point transform $\sigma_{\mathbf{v}+\mathcal{K}}$ of the shifted cone $\mathbf{v}+\mathcal{K}$ is the rational function

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})}$$

where Π is the half-open parallelepiped

$$\Pi := \{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d : 0\leqslant \lambda_1,\lambda_2,\ldots,\lambda_d < 1\}$$

The half-open parallelepiped Π is called the fundamental parallelepiped of $\mathcal{K}$.

Proof. In $\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \sum_{\mathbf{m}\in(\mathbf{v}+\mathcal{K})\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$, we list each integer point $\mathbf{m}$ in $\mathbf{v}+\mathcal{K}$ as the Laurent monomial $\mathbf{z}^{\mathbf{m}}$. Such a lattice point can, by definition, be written as

$$\mathbf{m} = \mathbf{v}+\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d$$

for some numbers $\lambda_1,\lambda_2,\ldots,\lambda_d\geqslant 0$. Because the $\mathbf{w}_k$ form a basis of $\mathbb{R}^d$, this representation is unique. Let's write each λ_k in terms of its integer and
fractional parts: $\lambda_k = \lfloor\lambda_k\rfloor + \{\lambda_k\}$. So $\mathbf{m} = \mathbf{v} + (\{\lambda_1\}\mathbf{w}_1+\{\lambda_2\}\mathbf{w}_2+\cdots+\{\lambda_d\}\mathbf{w}_d) + \lfloor\lambda_1\rfloor\mathbf{w}_1+\lfloor\lambda_2\rfloor\mathbf{w}_2+\cdots+\lfloor\lambda_d\rfloor\mathbf{w}_d$, and we should note that since $0\leqslant\{\lambda_k\}<1$, the vector

$$\mathbf{p} := \mathbf{v}+\{\lambda_1\}\mathbf{w}_1+\{\lambda_2\}\mathbf{w}_2+\cdots+\{\lambda_d\}\mathbf{w}_d$$

is in $\mathbf{v}+\Pi$. In fact, $\mathbf{p}\in\mathbb{Z}^d$, since $\mathbf{m}$ and $\lfloor\lambda_k\rfloor\mathbf{w}_k$ are all integer vectors. Again, the representation of $\mathbf{p}$ in terms of the $\mathbf{w}_k$ is unique. In summary, we have proved that every $\mathbf{m}\in\mathbf{v}+\mathcal{K}\cap\mathbb{Z}^d$ can be uniquely written as

$$\mathbf{m} = \mathbf{p}+k_1\mathbf{w}_1+k_2\mathbf{w}_2+\cdots+k_d\mathbf{w}_d \tag{3.4}$$

for some $\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d$ and some integers $k_1,k_2,\ldots,k_d\geqslant 0$. On the other hand, let's write the rational function on the right-hand side of the theorem as a product of series:

$$\frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} = \left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geqslant 0}\mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geqslant 0}\mathbf{z}^{k_d\mathbf{w}_d}\right).$$

If we multiply everything out, a typical exponent will look exactly like (3.4). □

Our proof contains a crucial geometric idea. Namely, we tile the cone $\mathbf{v}+\mathcal{K}$ with translates of $\mathbf{v}+\Pi$ by nonnegative integral combinations of the $\mathbf{w}_k$. It is this tiling that gives rise to the nice integer-point transform in theorem 3.3.4. Computationally, we therefore favor cones over polytopes due to our ability to tile a simplicial cone with copies of the fundamental domain, as above.

theorem 3.3.4 shows that the real complexity of computing the integerpoint transform $\sigma_{\mathbf{v}+\mathcal{K}}$ is embedded in the location of the lattice points in the parallelepiped $\mathbf{v}+\Pi$.

By mildly strengthening the hypothesis of theorem 3.3.4, we obtain a slightly easier generating function.

Corollary 3.3.5. Suppose

$$\mathcal{K}:=\{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d:\lambda_1,\lambda_2,\ldots,\lambda_d\geqslant 0\}$$

is a simplicial d-cone, where $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_d\in\mathbb{Z}^d$, and $\mathbf{v}\in\mathbb{R}^d$, such that the boundary of $\mathbf{v}+\mathcal{K}$ contains no integer point. Then

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z})=\frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{\left(1-\mathbf{z}^{\mathbf{w}_1}\right)\left(1-\mathbf{z}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{z}^{\mathbf{w}_d}\right)},$$

where Π is the open parallelepiped

$$\Pi:=\{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d:0<\lambda_1,\lambda_2,\ldots,\lambda_d<1\}$$

Proof. The proof of theorem 3.3.4 goes through almost verbatim, except that $\mathbf{v}+\Pi$ now has no boundary lattice points, so that there is no harm in choosing Π to be open. □

Since a general pointed cone can always be triangulated into simplicial cones, the integer-point transforms add up in an inclusion-exclusion manner (note that the intersection of simplicial cones in a triangulation is again a simplicial cone). Hence we have the following corollary.

Corollary 3.3.6. For a pointed cone

$$\mathcal{K}=\{\mathbf{v}+\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_m\mathbf{w}_m:\lambda_1,\lambda_2,\ldots,\lambda_m\geqslant 0\}$$

with $\mathbf{v}\in\mathbb{R}^d,\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m\in\mathbb{Z}^d$, the integer-point transform $\sigma_{\mathcal{K}}(\mathbf{z})$ evaluates to a rational function in the coordinates of $\mathbf{z}$.

Philosophizing some more, one can show that the original infinite sum $\sigma_{\mathcal{K}}(\mathbf{z})$ converges only for $\mathbf{z}$ in a subset of $\mathbb{C}^d$, whereas the rational function that represents $\sigma_{\mathcal{K}}$ gives us its meromorphic continuation.

Example 3.3.7. Consider $K=\operatorname{cone}\{-1,1\}=\mathbb{R}$, which is not pointed. Then,

$$\sigma_K(z)=\sum_{t\geqslant 0}z^t+\sum_{t\geqslant 0}z^{-t}-1=0.$$

In general, $\sigma_K(z)=0$ for any non-pointed cone.

3.4 From cones to polytopes

summary

- Given $S\subseteq\mathbb{R}^d$, let $\sigma_S(\mathbf{z})=\sum_{\mathbf{m}\in S\cap\mathbb{Z}^d}\mathbf{z}^{\mathbf{m}}$.

- $L_P(t)=\left|tP\cap\mathbb{Z}^d\right|$.

- $\operatorname{Ehr}_P(z)=\sum_{t\geqslant 0}L_P(t)z^t$.

Let $P = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^d$, where $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^d$; such polytopes are called **integral polytopes**. Note that $L_P(t) = \sigma_{tP}(1)$. The cone over this polytope is

$$C(P) := \text{ cone } \left\{ \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix} \right\}$$

(you can think about it in this way: placing the vertices of the polytopes on the plane and then leveling them up by 1.) Note that $tP \cong C(P) \cap \left\{x \in \mathbb{R}^{d+1} \mid x_{d+1} = t\right\}$. By cutting the cone with the hyperplane $x_{d+1} = 2$, we obtain a copy of $\mathcal{P}$ dilated by a factor of 2. More generally, we can cut the cone with the hyperplane $x_{d+1} = t$ and obtain $t\mathcal{P}$, as suggested by the Figure 3.5.

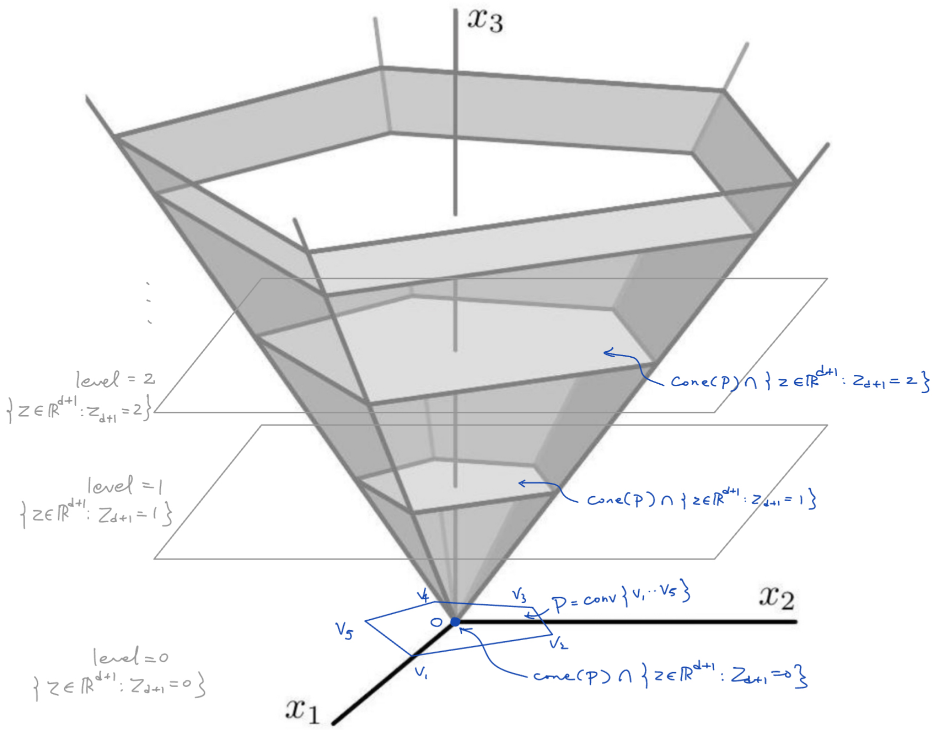


Figure 3.5: Integral dilates

Now let's form the integer-point transform $\sigma_{\operatorname{cone}(\mathcal{P})}$ of $\operatorname{cone}(\mathcal{P})$. By what we just said, we should look at different powers of z_{d+1} : there is one term (namely, 1), with z_{d+1}^0, corresponding to the origin; the terms with z_{d+1}^1 correspond to lattice points in $\mathcal{P}$ (listed as Laurent monomials in $z_1, z_2, \ldots, z_d$), the terms with z_{d+1}^2 correspond to points in $2\mathcal{P}$, etc. See Figure 3.5.

In short,

$$\begin{aligned} &\sigma_{\text{cone }(\mathcal{P})}\left(z_1, z_2, \ldots, z_{d+1}\right) \\ &\quad = \sigma_{\mathcal{P}}\left(z_1, \ldots, z_d\right) z_{d+1}^0 + \sigma_{\mathcal{P}}\left(z_1, \ldots, z_d\right) z_{d+1}^1 + \sigma_{2\mathcal{P}}\left(z_1, \ldots, z_d\right) z_{d+1}^2 + \cdots \\ &\quad = 1 + \sum_{t \geqslant 1} \sigma_{t\mathcal{P}}\left(z_1, \ldots, z_d\right) z_{d+1}^t \end{aligned}$$

Specializing further for enumeration purposes, we recall that $\sigma_{\mathcal{P}}(1, 1, \ldots, 1) = \#\left(\mathcal{P} \cap \mathbb{Z}^d\right)$, and so

$$\begin{aligned} \sigma_{\operatorname{cone}(\mathcal{P})}\left(1, 1, \ldots, 1, z_{d+1}\right) &= 1 + \sum_{t \geqslant 1} \sigma_{t\mathcal{P}}(1, 1, \ldots, 1) z_{d+1}^t \\ &= 1 + \sum_{t \geqslant 1} \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) z_{d+1}^t \end{aligned}$$

But by definition, the enumerators on the right-hand side are just evaluations of Ehrhart's counting function, that is, $\sigma_{\text{cone}(\mathcal{P})}(1,1,\ldots,1,z_{d+1})$ is nothing but the Ehrhart series of $\mathcal{P}$:

Proposition 3.4.1. For P integral, $\text{Ehr}_P(z)=\sigma_{C(P)}(1,\ldots,1,z)$.

Let us now focus on the case in which P is a simplex.

Proposition 3.4.2. For $P\subseteq\mathbb{R}^d$ an integral d-simplex we have that

$$\text{Ehr}_P(z)=\frac{h_0+h_1z+\cdots+h_dz^d}{(1-z)^{d+1}}$$

where each h_k is a non-negative integer. In particular, the numerator is a polynomial of degree at most d.

Proof. If $P=\text{conv}\{v_1,\ldots,v_{d+1}\}$, then $C(P)=\text{cone}\{w_1,\ldots,w_{d+1}\}$, where the last entry of each w_i is 1, is simplicial. By Theorem 3.3.4,

$$\sigma_{C(P)}(1,\ldots,1,z)=\frac{\sigma_\Pi(1,\ldots,1,z)}{(1-(1,\ldots,1,z)^{w_1})(1-(1,\ldots,1,z)^{w_d})}=\frac{\sigma_\Pi(1,\ldots,1,z)}{(1-z)^{d+1}}$$

It follows from Proposition 3.4.1 that

$$\text{Ehr}_P(z)=\frac{\sum_{k\geqslant 0}h_kz^k}{(1-z)^{d+1}}$$

where h_k counts the number of lattice points in Π with last entry k. Now,

$$\begin{bmatrix} x \\ x_{d+1}\end{bmatrix}\in\Pi\Leftrightarrow\begin{bmatrix} x \\ x_{d+1}\end{bmatrix}=\sum_{i=1}^{d+1}\lambda_i\begin{bmatrix} v_i \\ 1\end{bmatrix}$$

where each $\lambda_i\in[0,1)$. Note then that $x_{d+1}=\lambda_1+\cdots+\lambda_{d+1}<d+1$. It follows that $h_k=0$ whenever $k\geqslant d+1$. □

Lemma 3.4.3. A function $f:\mathbb{N}\to\mathbb{C}$ is a polynomial of degree d if and only if

$$\sum_{n\geqslant 0}f(n)z^n=\frac{g(z)}{(1-z)^{d+1}}$$

where g is a polynomial of degree $\leqslant d$ with $g(1)\neq 0$.

Proof. see [5] 4.1.4. □

We obtain the following consequences:

Corollary 3.4.4. If P is an integral d-simplex, then L_P is a polynomial of degree d.

Theorem 3.4.5 (Ehrhart's Theorem)**.** If P is an integral d-dimensional polytope, then $L_P(t)$ is a polynomial of degree d.

Proof. Triangulate P. □

Example 3.4.6. We compute $L_P(t)$ for $P_1=\text{conv}\{-3,2\}\subseteq\mathbb{R}$ and for $P_2=\text{conv}\{0,e_1,e_1+e_2,2e_2\}$ by interpolation.

$L_{P_1}(t)$ is a polynomial of degree 1 and is thus of the form $at+1$ (1 is because $\sigma_P(0)$ has to be 1). Now we pick say $t=1$ and count that $|P\cap\mathbb{Z}|=6$. We plug in $(1,6)$ for $at+1$ to get $a=5$. Then $L_{P_1}(t)=5t+1$.

Similarly, we plug in $(0,1),(1,5),(2,12)$ to at^2+bt+c to get $L_{P_2}(t)=\frac{3}{2}t^2+\frac{5}{2}t+1$.

3.4.1 More on Ehrhart theory of integral polytopes

The numerator of $\mathrm{Ehr}_P(z)$ is called the h^*-polynomial of P.

Theorem 3.4.7 (Stanley's non-negativity theorem)**.** Suppose P is an integral d-dimensional polytope with Ehrhart series

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \cdots + h_0}{(1-z)^{d+1}}$$

Then all h_i are non-negative integers.

Note that we have proven this in the case in which P is a simplex. The general proof triangulates, but this alone is not enough because $\mathrm{Ehr}_{P\cup Q} = \mathrm{Ehr}_P + \mathrm{Ehr}_Q - \mathrm{Ehr}_{P\cap Q}$. For the details you can see [4] Theorem 3.12.

Proposition 3.4.8. Let P be integral and $(h_0, \ldots, h_d)$ be its h^*-vector. We have that

$$L_P(t) = h_0\binom{t+d}{d} + h_1\binom{t+d-1}{d} + \cdots + h_d\binom{t}{d}$$

Proof. We have that

$$\sum_{t\geqslant 0} L_P(t) z^t = \mathrm{Ehr}_P(z) = \frac{h_d z^d + \cdots + h_0}{(1-z)^{d+1}} = \left(h_d z^d + \cdots + h_0\right)\left(\sum_{j\geqslant 0}\binom{j+d}{d}\right)$$

where the last equality follows from our computation of the h^*-polynomial of a simplex. Now, the coefficient of z^t on the RHS is $\sum_{i+j=t} h_i\binom{j+d}{d}$. □

There are many open problems about the h^*-polynomial, namely about their structure. Easy facts:

- $h_0 = 1$.
- $h_1 = \left|P \cap \mathbb{Z}^d\right| - d - 1$. (Exercise)

3.5 From Discrete to the Continuous Volume of a Polytope

Given a geometric object $S \subset \mathbb{R}^d$, its volume, defined by the integral vol $S := \int_S d\mathbf{x}$, is one of the fundamental data of S. By the definition of the integral, say in the Riemannian sense, we can think of computing vol S by approximating S with d-dimensional boxes that get smaller and smaller. To be precise, if we take the boxes with side length $\frac{1}{t}$, then they each have volume $\frac{1}{t^d}$. We might further think of the boxes as filling out the space between grid points in the lattice $\left(\frac{1}{t}\mathbb{Z}\right)^d$. This means that volume computation can be approximated by counting boxes, or equivalently, lattice points in $\left(\frac{1}{t}\mathbb{Z}\right)^d$:

$$\operatorname{vol} S = \lim_{t\to\infty} \frac{1}{t^d} \cdot \#\left(S \cap \left(\frac{1}{t}\mathbb{Z}\right)^d\right)$$

It is a short step to counting integer points in dilates of S, because

$$\#\left(S \cap \left(\frac{1}{t}\mathbb{Z}\right)^d\right) = \#\left(tS \cap \mathbb{Z}^d\right)$$

Let's summarize:

[1] From Wikipedia: "Eugène Ehrhart (29 April 1906 - 17 January 2000) was a French mathematician who introduced Ehrhart polynomials in the 1960s. Ehrhart received his high school diploma at the age of 22 . He was a mathematics teacher in several high schools, and did mathematics research on his own time. He started publishing in mathematics in his 40s, and finished his PhD thesis at the age of 60."

Lemma 3.5.1. Suppose $S \subset \mathbb{R}^d$ is d-dimensional. Then

$$\operatorname{vol} S=\lim_{t \rightarrow \infty} \frac{1}{t^d} \cdot \#\left(t S \cap \mathbb{Z}^d\right)$$

We emphasize here that S is d-dimensional, because otherwise (since S could be lower-dimensional although living in d-space), by our current definition, vol $S=0$. We will extend our volume definition later to give nonzero relative volume to objects that are not full-dimensional.

Part of the magic of Ehrhart's theorem lies in the fact that for an integral d-polytope $\mathcal{P}$, we do not have to take a limit to compute $\operatorname{vol} \mathcal{P}$; we need to compute "only" the $d+1$ coefficients of a polynomial.

Corollary 3.5.2. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex d-polytope with Ehrhart polynomial $c_d t^d+c_{d-1} t^{d-1}+\cdots+c_1 t+1$ (by Corollary 3.4.4). Then $c_d=\operatorname{vol} \mathcal{P}$.

Proof. By Lemma 3.5.1,

$$\operatorname{vol} \mathcal{P}=\lim_{t \rightarrow \infty} \frac{c_d t^d+c_{d-1} t^{d-1}+\cdots+c_1 t+1}{t^d}=c_d$$

Note that $+1$ is because $L_P(0)=1$. □

Remark 3.5.3. On the one hand, this should not come as a surprise, because the number of integer points in some object should grow roughly like the volume of the object as we make it bigger and bigger. On the other hand, the fact that we can compute the volume as one term of a polynomial should be very surprising: the polynomial is a counting function and as such is something discrete, yet by computing it (and its leading term), we derive some continuous data. Even more, we can - at least theoretically - compute this continuous datum (the volume) of the object by calculating a few values of the polynomial and then interpolating; this can be described as a completely discrete operation!

We finish this section by showing how to retrieve the continuous volume of an integral polytope from its Ehrhart series.

Corollary 3.5.4. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex d-polytope, and

$$\operatorname{Ehr}_{\mathcal{P}}(z)=\frac{h_d^* z^d+h_{d-1}^* z^{d-1}+\cdots+h_1^* z+1}{(1-z)^{d+1}}$$

Then $\operatorname{vol} \mathcal{P}=\frac{1}{d!}\left(h_d^*+h_{d-1}^*+\cdots+h_1^*+1\right)$.

Proof. Use the expansion of Lemma. The leading coefficient is

$$\frac{1}{d!}\left(h_d^*+h_{d-1}^*+\cdots+h_1^*+1\right)$$

□

Example 3.5.5 (Reeve's tetrahedron)**.** Let $\mathcal{T}_h$ be the tetrahedron with vertices $(0,0,0),(1,0,0),(0,1,0)$, and $(1,1, h)$, where h is a positive integer (see Figure 3.6).

To interpolate the Ehrhart polynomial $L_{\mathcal{T}_h}(t)$ from its values at various points, we use the figure to deduce the following:

$$\begin{gathered}
4=L_{\mathcal{T}_h}(1)=\operatorname{vol}\left(\mathcal{T}_h\right)+c_2+c_1+1 \\
h+9=L_{\mathcal{T}_h}(2)=\operatorname{vol}\left(\mathcal{T}_h\right) \cdot 2^3+c_2 \cdot 2^2+c_1 \cdot 2+1
\end{gathered}$$

Using the volume formula for a pyramid, we know that

$$\operatorname{vol}\left(\mathcal{T}_h\right)=\frac{1}{3}(\text { base area })(\text { height })=\frac{h}{6}$$

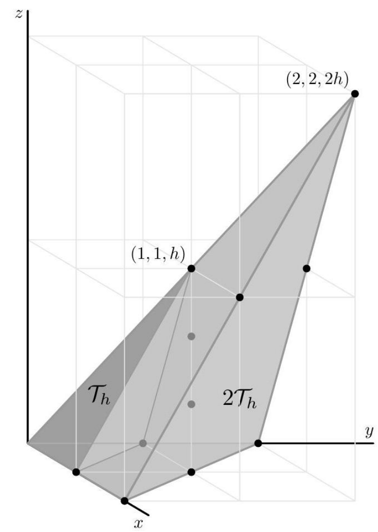


Figure 3.6: Reeve's tetrahedron $\mathcal{T}_h$ (and $2\mathcal{T}_h$)

Thus $h + 1 = h + 2c_2 - 1$, which gives us $c_2 = 1$ and $c_1 = 2 - \frac{h}{6}$. Therefore,

$$L_{\mathcal{T}_h}(t) = \frac{h}{6}t^3 + t^2 + \left(2 - \frac{h}{6}\right) t + 1$$

Chapter 4

Reciprocity of Ehrhart Polynomials

4.1 Introduction

Example 4.1.1. Let $P = [0,1]^d = \square_d$, then

$$\begin{aligned} \operatorname{int}(P) &= \{x \mid \forall i, 0 < x_i < 1\} \\ t\operatorname{int}(P) &= \{x \mid \forall i, 0 < x_i < t\}. \end{aligned}$$

It follows that $L_{\operatorname{int}(P)}(t) = (t-1)^d$ (we did this example at the beginning of last chapter). Note that $L_P(-t) = (-t+1)^d = (-1)^d L_{\operatorname{int}(P)}(t)$.

Example 4.1.2. Now, let $P = \Delta_d \subseteq \mathbb{R}^{d+1}$, then

$$\begin{aligned} \operatorname{int}(P) &= \left\{ x \Big| \sum_{i=1}^{d+1} x_i = 1, x_i > 0 \right\} \\ t\operatorname{int}(P) &= \left\{ x \Big| \sum_{i=1}^{d+1} x_i = t, x_i > 0 \right\} \end{aligned}$$

It follows tht $L_{\operatorname{int}(P)}(t)$ equals the number of positive integral solutions to $\sum_{i=1}^{d+1} x_i = t$. Note that x is such a solution if and only if $y = (x_1 - 1, \ldots, x_{d+1} - 1)$ is a non-negative integral solution to $\sum_{i=1}^{d+1} y_i = t - d - 1$. It follows that $L_{\text{int }(P)}(t) = \binom{t-1}{d}$. Note that

$$\begin{aligned} L_P(-t) = \binom{-t+d}{d} &= \frac{(-t+d)(-t+d-1)\cdots(-t+1)}{d!} \\ &= (-1)^d \frac{(t-1)(t-2)\cdots(t-d)}{d!} = (-1)^d L_{\operatorname{int}(P)}(t) \end{aligned}$$

Theorem 4.1.3 (Ehrhart-Macdonald Reciprocity (for Integral Polytopes))**.** Suppose $\mathcal{P}$ is a convex integral polytope. Then the evaluation of the polynomial $L_{\mathcal{P}}$ at negative integers yields

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t).$$

This theorem belongs to a class of famous reciprocity theorems. A common theme in combinatorics is to begin with an interesting object P, and

1. define a counting function $f(t)$ attached to P that makes physical sense for positive integer values of t;

2. recognize the function f as a polynomial in t;
3. substitute negative integral values of t into the counting function f, and recognize $f(-t)$ as a counting function of a new mathematical object Q.

For us, P is a polytope, and Q is its interior.

4.2 Ehrhart-Macdonald Reciprocity (for Integral Polytopes)

To prove Theorem 4.1.3, we will first look at reciprocity for cones.

Theorem 4.2.1 (Stanley's reciprocity for integral cones)**.** Let K be a d-dimensional cone generated by $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$. Then,

$$\sigma_K\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = (-1)^d \sigma_{\text{int}(K)}\left(z_1, \ldots, z_d\right)$$

(This holds as rational functions in $z_1, \ldots, z_d$.)

Example 4.2.2. Let $K = \operatorname{cone}\{e_1, e_2\}$. Then by Theorem 3.3.4, $\sigma_K(z) = \frac{1}{(1-z_1)(1-z_2)}$, and by Cor 3.3.5, $\sigma_{\text{int}(K)}(z) = \frac{\mathbf{z}^{(1,1)}}{(1-\mathbf{z}^{(1,0)})(1-\mathbf{z}^{(0,1)})} = \frac{z_1 z_2}{(1-z_1)(1-z_2)}$. Note that

$$\frac{1}{(1-(1/z_1))(1-(1/z_2))} = \frac{z_1 z_2}{(1-z_1)(1-z_2)}$$

Let us do the preparation to prove the Theorem 4.2.1.

Theorem 4.2.3. Let K be a d-dimensional cone generated by linearly independent $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$ (so K is simplicial), i.e., $K = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \ldots, \lambda_d \geqslant 0\}$. If $\mathbf{v}$ is such that $\partial(\mathbf{v}+K) \cap \mathbb{Z}^d = \varnothing$ (i.e., $\mathbf{v}$ for which the boundary of the shifted simplicial cone $\mathbf{v}+K$ contains no integer points), then

$$\sigma_{\mathbf{v}+K}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = (-1)^d \sigma_{-\mathbf{v}+K}\left(z_1, \ldots, z_d\right)$$

Proof. On the one hand,

$$\sigma_{\mathbf{v}+K}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = \frac{\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right)}{\left(1-\frac{1}{\mathbf{z}^{\mathbf{w}_1}}\right)\cdots\left(1-\frac{1}{\mathbf{z}^{\mathbf{w}_d}}\right)} = (-1)^d \frac{\mathbf{z}^{\mathbf{w}_1}\cdots\mathbf{z}^{\mathbf{w}_d}\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right)}{\left(1-\mathbf{z}^{\mathbf{w}_1}\right)\cdots\left(1-\mathbf{z}^{\mathbf{w}_d}\right)}.$$

On the other hand,

$$\sigma_{-\mathbf{v}+K}\left(z_1, \ldots, z_d\right) = \frac{\sigma_{-\mathbf{v}+\Pi}\left(z_1, \ldots, z_d\right)}{\left(1-\mathbf{z}^{\mathbf{w}_1}\right)\cdots\left(1-\mathbf{z}^{\mathbf{w}_d}\right)}$$

We leave as an exercise to show that

$$\mathbf{v}+\Pi = -(-\mathbf{v}+\Pi) + \mathbf{w}_1 + \cdots + \mathbf{w}_d$$

Figure 4.1 shows the situation in the case $d = 2$.

Translating this equation to generating functions implies that

$$\begin{aligned}\sigma_{\mathbf{v}+\Pi}\left(z_1, \ldots, z_d\right) &= \mathbf{z}^{\mathbf{w}_1}\cdots\mathbf{z}^{\mathbf{w}_d}\sigma_{-(-\mathbf{v}+\Pi)}\left(z_1 \ldots, z_d\right) \\ &= \mathbf{z}^{\mathbf{w}_1}\cdots\mathbf{z}^{\mathbf{w}_d}\sigma_{-\mathbf{v}+\Pi}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right),\end{aligned}$$

in other words,

$$\sigma_{\mathbf{v}+\Pi}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = \mathbf{z}^{-\mathbf{w}_1}\cdots\mathbf{z}^{-\mathbf{w}_d}\sigma_{-\mathbf{v}+\Pi}\left(z_1, \ldots, z_d\right).$$

The desired result now follows. ☐

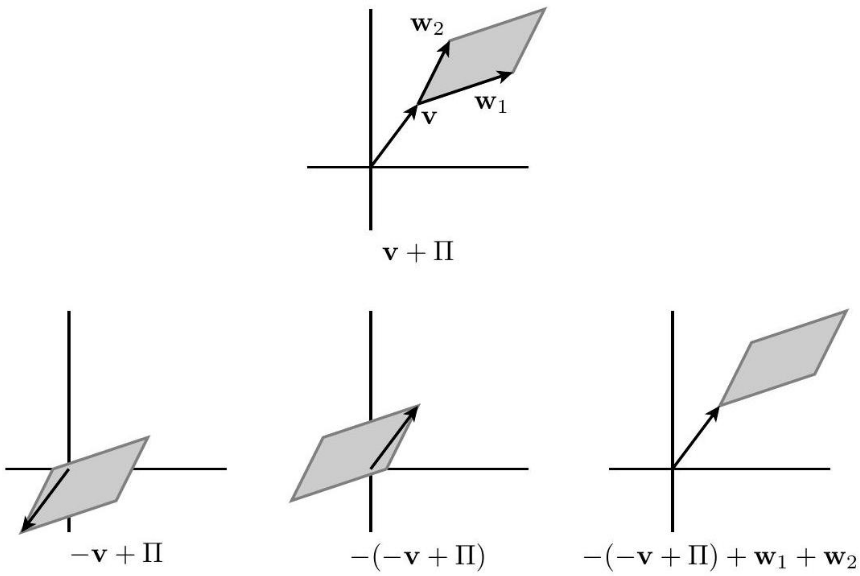


Figure 4.1: From $-\mathbf{v}+\Pi$ to $\mathbf{v}+\Pi$

We also need this lemma.

Lemma 4.2.4. Let K be as in the theorem 4.2.3. We triangulate it into simplicial cones $K_1, \cdots, K_m$. Then there exists $\mathbf{v}$ such that $\operatorname{int}(K) \cap \mathbb{Z}^d = (\mathbf{v}+K) \cap \mathbb{Z}^d$ and

$$\forall j, \partial\left(\mathbf{v}+K_j\right) \cap \mathbb{Z}^d = \varnothing \text{ and } \partial\left(-\mathbf{v}+K_j\right) \cap \mathbb{Z}^d = \varnothing.$$

Moreover, $K \cap \mathbb{Z}^d = (-\mathbf{v}+K) \cap \mathbb{Z}^d$.

Proof. Exercise. □

Proof of theorem 4.2.1. Triangulate K into simplicial cones $K_1, \ldots, K_m$. Shift K by a tiny vector v such that the previous lemma holds. Then, $K \cap \mathbb{Z}^d = (-\mathbf{v}+K) \cap \mathbb{Z}^d$. Now, putting all together,

$$\begin{aligned}
\sigma_K\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = \sigma_{-\mathbf{v}+K}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) &= \sum_{j=1}^m \sigma_{-\mathbf{v}+K_j}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) \\
&= (-1)^d \sum_{j=1}^m \sigma_{\mathbf{v}+K_j}\left(z_1, \ldots, z_d\right) = (-1)^d \sigma_{\mathbf{v}+K}\left(z_1, \ldots, z_d\right). \\
&= (-1)^d \sigma_{\operatorname{int}(K)}\left(z_1, \ldots, z_d\right).
\end{aligned}$$

□

We are almost ready to prove the Ehrhart-Macdonald reciprocity theorem.

Lemma 4.2.5. For any polynomial $f(t)$, we have that $\sum_{t \geqslant 0} f(t) z^t + \sum_{t<0} f(t) z^t = 0$.

We are now ready.

Proof of Ehrhart-Macdonald reciprocity. Let P be an integral d-dimensional polytope. Then, by Theorem 4.2.1, we have that

$$\sigma_{C(P)}\left(\frac{1}{z_1},\ldots,\frac{1}{z_{d+1}}\right)=(-1)^{d+1}\sigma_{\mathrm{int}(C(P))}\left(z_1,\ldots,z_{d+1}\right).$$

It follows that

$$\sigma_{C(P)}(1,\ldots,1,z)=(-1)^{d+1}\sigma_{\mathrm{int}(C(P))}(1,\ldots,1,z)$$

and so

$$\begin{aligned}\mathrm{Ehr}_P(1/z)&=(-1)^{d+1}\,\mathrm{Ehr}_{\mathrm{int}(P)}(z)\\&=(-1)^{d+1}\sum_{t\geqslant 0}L_{\mathrm{int}(P)}(t)z^t\\&=(-1)^d\sum_{t>0}L_{\mathrm{int}(P)}(-t)z^{-t}\end{aligned}$$

where the last equation holds by the lemma. The result follows from comparing the coefficients of z^{-t}. □

A polytope is **rational** if all its vertices are in $\mathbb{Q}^d$. A **quasipolynomial** $Q(t)$ is a function $\mathbb{N}\to\mathbb{N}$ such that there exists a positive integer k and polynomials $p_0,\ldots,p_{k-1}$ such that $Q(t)=p_i(t)$ if and only if $t\equiv i \bmod k$. The minimal such k is called the period of Q.

Theorem 4.2.6 (Ehrhart's theorem for rational polytopes)**.** If P is a rational d-dimensional polytope, then $L_P(t)$ is a quasipolynomial in t of degree d. Its period divides the least common multiple of the denominators of the coordinates of the vertices of P.

Proof. See [4] Theorem 3.23. □

Example 4.2.7. Let $P=\mathrm{conv}\{(0,0),(1/2,0),(0,1/2)\}$. Note then that

$$L_P(t)=\begin{cases}\frac{\frac{t}{2}\left(\frac{t}{2}+1\right)}{2}, & t \text{ even.}\\ \frac{\frac{t-1}{2}\left(\frac{t-1}{2}+1\right)}{2}, & t \text{ odd.}\end{cases}$$

Note that this polynomial has degree $d=2$ and period 2.

Theorem 4.2.8 (Stanley reciprocity)**.** Suppose $\mathcal{K}$ is a rational d-cone with the origin as apex. Then

$$\sigma_{\mathcal{K}}\left(\frac{1}{z_1},\frac{1}{z_2},\ldots,\frac{1}{z_d}\right)=(-1)^d\sigma_{\mathcal{K}^\circ}\left(z_1,z_2,\ldots,z_d\right).$$

Proof. See [4] Theorem 4.3. □

Theorem 4.2.9 (Ehrhart-Macdonald reciprocity)**.** Suppose $\mathcal{P}$ is a convex rational polytope. Then the evaluation of the quasipolynomial $L_{\mathcal{P}}$ at negative integers yields

$$L_{\mathcal{P}}(-t)=(-1)^{\dim\mathcal{P}}L_{\mathcal{P}^\circ}(t).$$

Proof. See [4] Theorem 4.1 and its proof in section 4.3. □

4.2.1 Application: From lattice points to faces

Theorem 4.2.10 (Euler-Poincaré equation). Let P be a d-dimensional polytope and f_k be the number of k-dimensional faces. Then $f_0 + \cdots + (-1)^d f_d = 1$.

Proof. We use Ehrhart-Macdonald reciprocity. Since $P = \bigsqcup_{F \text{ face}} \operatorname{int}(F)$,

$$L_P(t) = \sum_{F \text{ face}} L_{\text{int }(F)}(t) = \sum_{F \text{ face}} (-1)^{\dim(F)} L_F(-t).$$

Since $L_F(0) = 1$ it follows that

$$1 = \sum_{F \text{ face}} (-1)^{\dim(F)} = \sum_{k=0}^{d} (-1)^k f_k$$

□

4.3 Volumes

Let us start by computing the volumes of pyramids.

Proposition 4.3.1. Given a d-dimensional pyramid P with base B and height h we have that

$$\operatorname{Vol}_d(P) = \frac{h \operatorname{Vol}_{d-1}(B)}{d}.$$

Proof. It is equivalent to assume the base is $B \times \{h\}$ and apex 0 . We have that

$$\operatorname{Vol}_d(P) = \int_0^h \operatorname{Vol}_{d-1}\left(\frac{t}{h} B\right) dt = \int_0^h \frac{t^{d-1}}{h^{d-1}} \operatorname{Vol}_{d-1}(B) dt = \left.\frac{t^d}{dh^{d-1}} \operatorname{Vol}_{d-1}(B)\right|_0^h = \frac{h \operatorname{Vol}_{d-1}(B)}{d}.$$

□

Corollary 4.3.2. Let $P = \operatorname{conv}(0, v_1, \ldots, v_d) \subseteq \mathbb{R}^d$ be a simplex. Then

$$\operatorname{Vol}_d(P) = \frac{1}{d!} \left|\det \begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix}\right|.$$

Proof. Let $Q_d := \operatorname{conv}(0, e_1, \ldots, e_d)$ and note that Q is a d-dimensional pyramid with base Q_{d-1} and height 1. It follows that

$$\operatorname{Vol}_d(Q_d) = \frac{1}{d} \operatorname{Vol}_{d-1}(Q_{d-1}) = \cdots = \frac{1}{d!}.$$

For general P, use the change of basis $x = Au$, where $A = \begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix}$. We then have that

$$\operatorname{Vol}_d(P) = \int_P dx = \int_{Q_d} |\det(A)| du = \frac{1}{d!} |\det(A)|.$$

□

Definition 4.3.3. The **Minkowski sum** of P, Q is the sum $P + Q = \{p + q \mid p \in P, q \in Q\}$.

Exercise 4.3.4. Prove that if you add $P + \cdots + P$ in k times, you obtain a polytope equivalent to kP. Prove that if P, Q are polytopes, then $P + Q$ is again a polytope. What are the vertices of $P + Q$?

Example 4.3.5. Given $P = [0,1]^2, Q = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$, and $r, s \in \mathbb{R}_{\geqslant 0}$, consider the polytope $Z = rP + sQ = \{rp + sq \mid p \in P, q \in Q\}$. We then have that $\operatorname{Vol}_2(Z) = r^2 \operatorname{Vol}_2(P) + s^2 \operatorname{Vol}_2(Q) + 2rs$.

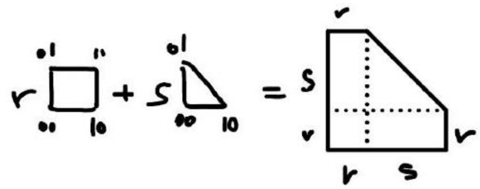


Figure 4.2: Minkowski sum.

Theorem 4.3.6. Let $P, Q \subseteq \mathbb{R}^d$ be polytopes. The function $\mathrm{Vol}_d(rP + sQ)$ is a homogeneous polynomial in r and s of degree d, i.e., $f(\lambda r, \lambda s) = \lambda^d f(r, s)$.

Definition 4.3.7. By the theorem, we write

$$\mathrm{Vol}_d(rP + sQ) = \sum_{i=0}^{d} \binom{d}{i} \mathrm{MV}\left(P^i, Q^{d-i}\right) r^i s^{d-i}$$

The scalars $\mathrm{MV}\left(P^i, Q^{d-i}\right)$ are called the **mixed volumes** of P and Q.

Example 4.3.8. In the example above, we have

$$\mathrm{Vol}_2(Z) = \mathrm{Vol}_2(P) r^2 s^0 + \binom{2}{1} \cdot 1 \cdot r^1 s^1 + \mathrm{Vol}(Q) r^0 s^2.$$

Thus, $\mathrm{MV}(P^2, Q^0) = \mathrm{Vol}_2(P)$, $\mathrm{MV}(P^0, Q^2) = \mathrm{Vol}_2(Q)$, and $\mathrm{MV}(P^1, Q^1) = 1$.

Let's do some preparatory work to prove the theorem.

Proposition 4.3.9. Let $P \subseteq \mathbb{R}^d$ be a d-dimensional polytope with facet description $P = \{x \mid \langle a_i, x\rangle \leqslant b_i, i \in [m]\}$ where each $\|a_i\| = 1$ and let F_i be the face of P defined by $\langle a_i, x\rangle = b_i$. Then, $\mathrm{Vol}_d(P) = \frac{1}{d}\sum_{i=1}^m b_i \mathrm{Vol}_{d-1}(F_i)$ and $\sum_{i=1}^m \mathrm{Vol}_{d-1}(F_i)\, a_i = 0$.

Proof. Let $q \in \mathrm{int}(P)$ and $P_i = \mathrm{conv}(F_i, q)$, i.e. P_i is the pyramid with base F_i and apex q. Let h_i be the height of P_i, then $h_i = b_i - \langle a_i, q\rangle$. Note that $P = \bigsqcup_{i=1}^m P_i$. Then,

$$\begin{aligned}\mathrm{Vol}_d(P) &= \sum_{i=1}^m \mathrm{Vol}_d(P_i) = \sum_{i=1}^m \frac{h_i}{d} \mathrm{Vol}_{d-1}(F_i) = \sum_{i=1}^m \frac{\mathrm{Vol}_{d-1}(F_i)}{d}\left(b_i - \langle a_i, q\rangle\right) \\ &= \frac{1}{d}\sum_{i=1}^m b_i \mathrm{Vol}_{d-1}(F_i) - \left\langle \frac{1}{d}\sum_{i=1}^m \mathrm{Vol}_{d-1} a_i (F_i), q\right\rangle.\end{aligned}$$

Note that this last equation holds for any $q \in \mathrm{int}(P)$. Since $\frac{1}{d}\sum_{i=1}^m b_i \mathrm{Vol}_{d-1}(F_i)$ is also a constant, then we must have that the last term doesn't depend on q. This is only possible if $\sum_{i=1}^m \mathrm{Vol}_{d-1}(F_i)\, a_i = 0$. □

Recall that $P_a = \{x \in P \mid \forall y \in P, \langle c, x\rangle \geqslant \langle c, y\rangle\}$ is the face of the polytope in direction a.

Lemma 4.3.10. $(rP + sQ)_a = rP_a + sQ_a$.

Proof. Suppose that $p = \max_{x\in P}\langle a, x\rangle$ and $q = \max_{x\in Q}\langle a, x\rangle$. Then, for any $rx + sy \in rP + sQ$ we have that

$$\langle a, rx + sy\rangle \leqslant rp + sq.$$

Also, if $x_* \in P$ and $y_* \in Q$ maximize these functions, then $rx_* + sy_*$ maximizes the left hand side. □

proof of Theorem 4.3.6. Proceed by induction on d.

Base case:
If $d = 1$, then $P = [x, y]$ and $Q = [z, w]$. Then, $rP + sQ = [rx + sz, ry + sw]$ and so

$$\binom{1}{0} \operatorname{Vol}_1(Q)s + \binom{1}{1} \operatorname{Vol}_1(P)r = \operatorname{Vol}_1(rP + sQ)$$

Inductive case:
Suppose the result holds for $d - 1$. Write the facet description $Z = rP + sQ = \{x \mid \langle a_i, x\rangle \leqslant b_i, i \in [m]\}$. Suppose

$$P_{a_i} = \{x \in P \mid \langle a_i, x\rangle = p_i\} \text{ and } Q_{a_i} = \{x \in Q \mid \langle a_i, x\rangle = q_i\},$$

then by the preceding lemma, we have that $rp_i + sq_i = b_i$. Now, by Proposition 4.3.9, we have that

$$\operatorname{Vol}_d(rP + sQ) = \frac{1}{d} \sum_{i=1}^{m} b_i \operatorname{Vol}_{d-1}(Z_{a_i})$$

Note that b_i is linear in r, s. By induction, $\operatorname{Vol}_{d-1}(Z_{a_i})$ is a homogeneous polynomial in r, s of degree $d - 1$. □

We recall the combinatorial (operational) and arithmetic defininition of **multinomial coefficient** before we give the general analog of Theorem 4.3.6.

Proposition 4.3.11. Let $b_1, \ldots, b_k$ be nonnegative integers, and let $n = b_1 + b_2 + \cdots + b_k$. The multinomial coefficient $\binom{n}{b_1, b_2, \ldots, b_k}$ is:

(1) the number of ways to put n interchangeable objects into k boxes, so that box i has b_i objects in it, for $1 \leqslant i \leqslant k$.

(2) the number of ways to choose b_1 interchangeable objects from n objects, then to choose b_2 from what remains, then to choose b_3 from what remains, ..., then to choose b_{k-1} from what remains.

(3) the number of unique permutations of a word with n letters and k distinct letters, such that the i th letter occurs b_i times.

(4) the product

$$\binom{n}{b_1}\binom{n-b_1}{b_2}\binom{n-b_1-b_2}{b_3}\cdots\binom{b_{k-1}+b_k}{b_{k-1}}\binom{b_k}{b_k}.$$

(5) the quotient

$$\frac{n!}{b_1! b_2! \cdots b_k!}.$$

Proof. (1) and (2) are clearly equivalent, and (2) and (4) are equivalent from the definition of the binomial coefficient. (4) and (5) are equivalent by simple algebra. There are a few ways to see that (3) is equivalent to the others. Arguing combinatorially, note that a permutation of a word as in (3) corresponds to choices of spots to put each of the repeated letters in; out of the spots $1, \ldots, n$, choose b_1 of those spots to put the first letter in, then b_2 spots out of the remaining $n - b_1$ to put the second letter in, and so on. So (3) is equivalent to (2). (One can also count permutations directly, by taking n ! permutations and dividing by factors that account for duplicates: divide by a factor of b_1 ! to account for the fact that permuting all of the first letters doesn't change the permutation, divide by b_2 ! to do the same for the second letters, and so on, which gives the formula from (5).) □

Theorem 4.3.12 (Multinomial Theorem)**.** Show the following equality: for $n, p \in \mathbb{N}$,

$$(x_1 + \cdots + x_p)^n = \sum_{\substack{n_1 + \cdots + n_p = n \\ n_1, \cdots, n_p \in \mathbb{N}_0}} \frac{n!}{n_1! \cdots n_p!} x_1^{n_1} \cdots x_p^{n_p},$$

where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ is the set of all nonnegative integers. Note that there are $\binom{n+p-1}{p}$ such tuples $(n_1, \cdots, n_p)$ solving the equation $n_1 + \cdots + n_p = n$, so there are $\binom{n+p-1}{p}$ terms in the RHS of above identity. If we force $n_i \in \mathbb{N} = \{1, 2, \cdots\}$ then this number of solutions changes to $\binom{n-1}{p-1}$.

Example 4.3.13. Let $p = 2$ and $n = 3$. Then the equation $n_1 + n_2 = 3$ has solution $(3, 0), (1, 2), (2, 1), (0, 3)$.

$$\begin{aligned}
(x_1 + x_2)^3 &= x_1^3 + 3x_1^2 x_2 + 3x_1 x_2^2 + x_2^3 \\
&= \frac{3!}{3!0!} x_1 x_1 x_1 + \frac{3!}{1!2!} x_1 x_2 x_2 + \frac{3!}{2!1!} x_1 x_1 x_2 + \frac{3!}{0!3!} x_2 x_2 x_2 \\
&= x_1 x_1 x_1 + (x_1 x_1 x_2 + x_1 x_2 x_1 + x_2 x_1 x_1) + (x_2 x_1 x_2 + x_2 x_2 x_1 + x_1 x_2 x_2) + x_2 x_2 x_2 \\
&= \sum_{\lambda_1, \lambda_2, \lambda_3 = 1}^{2} x_{\lambda_1} x_{\lambda_2} x_{\lambda_3}
\end{aligned}$$

Theorem 4.3.14 (H. Minkowski)**.** Let $P_1, \ldots, P_m$ be polytopes in $\mathbb{R}^d$, and $r_i \geqslant 0, i = 1, \ldots, m$. Then, $\mathrm{MV}(r_1 P_1 + \cdots + r_m P_m)$ is a homogeneous polynomial of degree n in $r_1, \ldots, r_m$,

$$\mathrm{Vol}_d (r_1 P_1 + \cdots + r_m P_m) = \sum_{\lambda_1, \ldots, \lambda_d = 1}^{m} \mathrm{MV}(P_{\lambda_1}, \ldots, P_{\lambda_d}) r_{\lambda_1} \cdots r_{\lambda_d},$$

the summation being carried out independently over the $\lambda_i, i = 1, \ldots, d$. There are m^d summands in the summation.

Proof. See [1] Theorem 3.2. ☐

Definition 4.3.15. Arranging the coefficients on the RHS of above equality such that $\mathrm{MV}\left(P_{\pi(\lambda_1)}, \ldots, P_{\pi(\lambda_d)}\right) = \mathrm{MV}(P_{\lambda_1}, \ldots, P_{\lambda_d})$ for any permutation π of $\lambda_1, \ldots, \lambda_d$, we call $\mathrm{MV}(P_{\lambda_1}, \ldots, P_{\lambda_d})$ the (d-dimensional) **mixed volume** of $P_{\lambda_1}, \ldots, P_{\lambda_d}$.

Note that there are in total $\frac{d!}{j_1! \cdots j_d!}$ such mixed volumes to be enforced to be equal, where j_k!'s are for those indices that are repeated. Since $r_{\pi(\lambda_1)} \cdots r_{\pi(\lambda_d)} = r_{\lambda_1} \cdots r_{\lambda_d}$, we see

$$\mathrm{Vol}_d (r_1 P_1 + \cdots + r_m P_m) = \sum_{i_1 + \cdots + i_m = d} \binom{d}{i_1, \cdots, i_m} \mathrm{MV}\left(P_1^{i_1}, \ldots, P_m^{i_m}\right) r_1^{i_1} \cdots r_m^{i_m}$$

Be awaring of the definition 4.3.15, one can deduce the basic properties of the mixed volume in the following theorem.

Theorem 4.3.16.

(1) The mixed volume $\mathrm{MV}(P_1, \cdots, P_n)$ is invariant if the P_i are replaced by their images under a volume-preserving transformation of $\mathbb{R}^n$ (for example, a translation).

(2) MV is symmetric and linear in each variable.

(3) $\mathrm{MV} \geqslant 0$. Furthermore, $\mathrm{MV}(P_1, \cdots, P_n) = 0$ if one of the P_i has dimension zero, and $\mathrm{MV}(P_1, \ldots, P_n) > 0$ if every P_i has dimension n.

(4) The mixed volume of any collection of polytopes can be computed as

$$\operatorname{MV}(P_1, \ldots, P_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{I \in \binom{[n]}{k}} \operatorname{Vol}_n \left(\sum_{i \in I} P_i \right)$$

Example 4.3.17. Let us verify that $\operatorname{MV}(P, Q) = 1$ for the P, Q from Example 4.3.5.

$$\operatorname{MV}(P, Q) = -\frac{1}{2} (\operatorname{Vol}_2(P) + \operatorname{Vol}_2(Q)) + \frac{1}{2} \operatorname{Vol}_2(P + Q) = -\frac{1}{2}(1 + 1/2) + \frac{1}{2}(7/2) = 1$$

Exercise 4.3.18. Prove that $\operatorname{MV}\left(P^d\right) = \operatorname{Vol}_d(P)$.

4.4 Volumes and polynomials.

Definition 4.4.1. The **Newton polytope** P_f of a Laurent polynomial $f = \sum_{\alpha \in \mathbb{Z}^d} c_\alpha x^\alpha \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_d^{\pm 1}\right]$ is the convex hull of the support of f, i.e. $P_f := \operatorname{conv}\left\{\alpha \in \mathbb{Z}^d \mid c_\alpha \neq 0\right\}$.

Example 4.4.2. For $f(x, y) = 3x^2 - y + 6$, P_f is depicted in Figure 4.3.

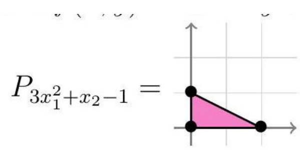


Figure 4.3: Newton Polynomial

Theorem 4.4.3 (Bernstein's Theorem)**.** If the system $f(x, y) = g(x, y) = 0$ has a finite number of solutions in $(\mathbb{C} \backslash \{0\}))^2$, then it has $\leqslant 2\operatorname{MV}(P_f, P_g)$ solutions. We have equality when the system is generic.

Example 4.4.4. Let $f = 3x^2 - y + 6$ and $g = x^2 + 2y - 3$ and note that $P_f = P_g$. Then, $2 * \operatorname{MV}(P_f, P_g) = 2 * \operatorname{Vol}(P_f) = 2$. This agrees with the intuition for the number of solutions of the system $f = g = 0$.

We can use Bernstein's theorem to prove a small case of Bezout's theorem.

Theorem 4.4.5 (Bezout's Theorem)**.** Let $f, g \in \mathbb{C}[x, y]$ be polynomials of degrees m, n respectively. If the system $f = g = 0$ has finitely many solutions then it has $\leqslant mn$ solutions. We have equality when the system is generic.

Proof. By Bernstein's theorem, the maximum happens when f, g are generic, so let's assume this. Let $\Delta = \operatorname{conv}\{(0,0), (1,0), (0,1)\}$. Then, $P_f = m\Delta$ and $P_g = n\Delta$. By

Bernstein's theorem, we have that the number of solutions is $2\operatorname{MV}(m\Delta, n\Delta)$. Now, by Theorem 4.3.16(4), we have that

$$\begin{aligned} 2\operatorname{MV}(m\Delta, n\Delta) &= (-1)(\operatorname{Vol}_2(n\Delta) + \operatorname{Vol}_2(m\Delta)) + \operatorname{Vol}_2(m\Delta + n\Delta) \\ &= \left[(n+m)^2 - n - m\right] \operatorname{Vol}_2(\Delta) = mn \end{aligned}$$

The general statement of Bernstein's Theorem is as follows. □

Theorem 4.4.6 (Bernstein's Theorem)**.** Let $f_1, \ldots, f_n \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_d^{\pm 1}\right]$ be Laurent polynomials. If the system $f_1 = \cdots = f_n = 0$ has finitely many solutions in $(\mathbb{C} \backslash \{0\})^n$, then it has $\leqslant n!\operatorname{MV}(P_{f_1}, \ldots, P_{f_n})$ solutions. We have equality when the system is generic.

4.5 The volume of the permutohedron

An equivalent way to describe the permutohedron is

$$\begin{aligned}\Pi_n &= \operatorname{conv}\left\{\left(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}\right) \mid \sigma \in S_n\right\} \\ &= \operatorname{conv}\{S_n\} - (1, \cdots, 1)\end{aligned}$$

where $\lambda = (0, 1, \ldots, n-1)$.

Example 4.5.1. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

Then the vertex is $(1, 4, 2, 3) - (1, 1, 1, 1) = (0, 3, 1, 2)$.

Remark 4.5.2. One can obtain other interesting polytopes by considering different λ, e.g. $\lambda = (0, 0, 1, 1, 2, 2)$.

Let us denote by $\Delta_{ij} = \operatorname{conv}\{e_i, e_j\}$.

Proposition 4.5.3.

$$\Pi_n = \sum_{1 \leqslant i < j \leqslant n} \Delta_{ij}.$$

Proof. Let $Q = \sum_{1 \leqslant i < j \leqslant n} \Delta_{ij}$.

First let us show that $\Pi_n \subseteq Q$.

To do so it suffices to show $\operatorname{Vtx}(\Pi_n) \subseteq Q$, i.e., every vertex of Π_n is in Q. Since Minkowski sum of polytopes is again a polytope and is thus convex, and Π_n is the smallest convex set containing its vertices.

Let $\sigma \in S_n$ and consider the vertex $v = (\sigma_1 - 1, \ldots, \sigma_n - 1) = \sigma - \mathbf{1}$. Note that

$$\begin{aligned}\left(\Delta_{ij}\right)_v &= \begin{cases} e_i, & \text{if } \sigma_i > \sigma_j \\ e_j, & \text{if } \sigma_i < \sigma_j \end{cases} \\ &\quad \text{(there's no } \sigma_i = \sigma_j \text{ as } \sigma \in S_n) \\ &= \text{face of } \Delta_{ij} \text{ in direction } v \\ &= \text{set of farthest points } P \text{ can reach in direction } v.\end{aligned}$$

See Figure 4.4 to get some intuition about how this is true.

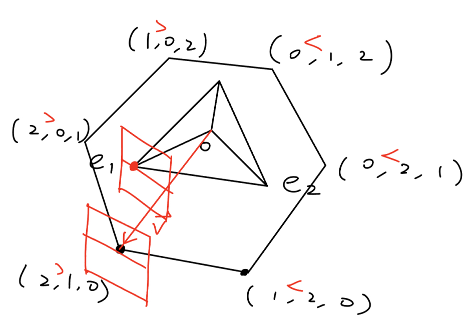


Figure 4.4: A clear dichotomy between σ_i and σ_j.

In particular, $(\Delta_{ij})_v = e_j$ if $\sigma_i < \sigma_j$ which implies $(\Delta_{i(\sigma^{-1})_k})_v = e_{(\sigma^{-1})_k}$ if $\sigma_i < \sigma_{(\sigma^{-1})_k} = k$. There are $(k-1)$ possible σ_i, i.e., there are $(k-1)$ possible ways for i to be mapped by σ.

Now, by Lemma 4.3.10, we have that

$$Q_v = \left(\sum_{1\leqslant i<j\leqslant n} \Delta_{ij}\right)_v \overset{\text{Lemma 4.3.10}}{=\!=\!=} \sum_{1\leqslant i<j\leqslant n} (\Delta_{ij})_v = \sum_{j=2}^{n}\sum_{i=1}^{j-1}(\Delta_{ij})_v$$
$$= \sum_{j=2}^{n} \sum_{\text{all } i<j=\sigma_{(\sigma^{-1})_j}} \left(\Delta_{i\sigma_{(\sigma^{-1})_j}}\right)_v = \sum_{j=2}^{n}(j-1)e_{(\sigma^{-1})_j}$$

which is a combination of unit vectors. Thus, $v \in Q$ and $\Pi_n \subseteq Q$.

For the opposite containment $Q \subseteq \Pi_n$, we need a lemma for the other representation of Π_n (Lemma 4.5.4):

$$\Pi_n = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} x_k = \binom{n}{2} \text{ and } \forall S \subseteq [n],\ \sum_{k\in S} x_k \geqslant \binom{|S|}{2} \right\}$$

To show $Q \subseteq \Pi_n$, we let $x \in Q$, i.e., $x = \sum_{1\leqslant i<j\leqslant n} \lambda_{ij}e_i + (1-\lambda_{ij}e_j) \in Q$, where $\lambda_{ij} \geqslant 0$. Then, we verify that it satisfies condition in the lemma.

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n}\sum_{1\leqslant i<j\leqslant n} \overbrace{(\lambda_{ij}e_i + (1-\lambda_{ij}e_j))_k}^{=\begin{cases}0, & k\neq i,j\\ \lambda_{ij}, & k=i\\ 1-\lambda_{ij}, & k=j\end{cases}}$$
$$= \sum_{k=1}^{n}\left[\sum_{j=k+1}^{n}\lambda_{kj} + \sum_{j=1}^{k-1}(1-\lambda_{ik})\right]$$
$$= \sum_{k=1}^{n}\sum_{j=k+1}^{n}\lambda_{kj} + \sum_{k=1}^{n}\sum_{i=1}^{k-1}(1-\lambda_{ik})$$
$$= \sum_{1\leqslant k<j\leqslant n}\lambda_{kj} + \sum_{1\leqslant i<k\leqslant n}(1-\lambda_{ik})$$
$$= \sum_{1\leqslant i<j\leqslant n}(\lambda_{ij} + (1-\lambda_{ij})) \overset{\text{binomial}}{=\!=\!=} \binom{n}{2}$$

For the second condition, let $S \subseteq [n]$. Then,

$$\sum_{k\in S} x_k = \sum_{k\in S}\left(\sum_{j=k+1}^{n}\lambda_{kj} + \sum_{i=1}^{k-1}(1-\lambda_{ik})\right)$$
$$= \sum_{\substack{1\leqslant i<j\leqslant n\\ i\in S,\, j\notin S}}\lambda_{ij} + \sum_{\substack{1\leqslant i<j\leqslant n\\ j\in S,\, i\notin S}}(1-\lambda_{ij}) + \underbrace{\sum_{\substack{1\leqslant i<j\leqslant n\\ i,j\in S}} 1}_{\binom{|S|}{2}} \geqslant \binom{|S|}{2}$$

Thus, $x \in \Pi_n$ and $Q \subseteq \Pi_n$. □

Lemma 4.5.4.

$$\Pi_n = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} x_k = \binom{n}{2} \text{ and } \forall S \subseteq [n],\ \sum_{k\in S} x_k \geqslant \binom{|S|}{2} \right\}$$

Proof. Exercise. □

Definition 4.5.5. A **zonotope** is the Minkowski sum of line segments.

Note that Π_d is a zonotope.

Proposition 4.5.6. Every face of a zonotope is a zonotope.

Proof. Suppose $Z = \sum_{i=1}^{n} [u_i, v_i]$. By Lemma 4.3.10 we have that $Z_c = \sum_{i=1}^{n} [u_i, v_i]_c$. Since

$$[u_i, v_i]_c = \begin{cases} [u_i, v_i], & \langle c, v_i \rangle = \langle c, u_i \rangle \\ \{u_i\}, & \langle c, v_i \rangle < \langle c, u_i \rangle, \\ \{v_i\}, & \langle c, v_i \rangle > \langle c, u_i \rangle \end{cases}$$

the result follows. □

Next, note that $\Pi_d \subseteq \mathbb{R}^d$, but Π_d is $(d-1)$-dimensional. Thus, $\mathrm{Vol}_d(\Pi_d) = 0$ which is not interesting. Instead, we consider

$$p : \mathbb{R}^d \to \mathbb{R}^{d-1}, p(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1})$$

and compute $\mathrm{Vol}_{d-1}(p(\Pi_d))$. This is called the relative volume of Π_d.

Theorem 4.5.7 (Stanley)**.** $(d-1)!\,\mathrm{Vol}_{d-1}(p(\Pi_d)) = d^{d-2}$ = number of spanning trees of the complete graph K_d.

We will prove this theorem using Bernstein's theorem. Since p is a linear function, it follows from Proposition 4.5.3 that

$$p(\Pi_d) = \sum_{1 \leqslant i < j \leqslant d} p(L_{ij}).$$

Then,

$$\mathrm{Vol}_{d-1}(p(\Pi_d)) = \sum_{i_1 j_1, \ldots, i_{d-1} j_{d-1}} \mathrm{MV}\left(p(L_{i_1 j_1}), \ldots, p\left(L_{i_{d-1} j_{d-1}}\right)\right),$$

where the sum is over all $(d-1)$-tuples of pairs ij such that $1 \leqslant i < j \leqslant d$.

Lemma 4.5.8. $\mathrm{MV}\left(p(L_{i_1 j_1}), \ldots, p\left(L_{i_{d-1} j_{d-1}}\right)\right) = \begin{cases} \frac{1}{(d-1)!}, & \text{if } i_1 j_1, \ldots, i_{d-1} j_{d-1} \text{ are the edges of a spanning tree of } K_d \\ 0, & \text{otherwise,} \end{cases}$

where K_d is the complete graph on d vertices.

Example 4.5.9.
(1) Let us describe the system of equations corresponding to

$$\mathrm{MV}(p(L_{12}), p(L_{23}), p(L_{34})).$$

We have tht $p(L_{i4}) = p([e_i, e_4]) = [e_i, 0]$ and this is the Newon polytope of $\lambda x_i + \mu$. The system is then

$$\begin{cases} \lambda_1 x_1 + \mu_1 x_2 = 0, \\ \lambda_2 x_1 + \mu_2 x_3 = 0, \\ \lambda_3 x_3 + \mu_3 = 0 \end{cases},$$

where all the λ_i, μ_i are non-zero. Note that we can solve for x_1, x_2, x_3 and so the system has a unique solution. By Bernstein's theorem we have that

$$4!\,\mathrm{MV}(p(L_{12}), p(L_{23}), p(L_{34})) = 1,$$

as anticipated by the lemma.

(2) The system of equations corresponding to $\mathrm{MV}\left(p\left(L_{12}\right), p\left(L_{14}\right), p\left(L_{24}\right)\right)$ is

$$\left\{\begin{array}{l}\lambda_1 x_1+\mu_1 x_2=0, \\ \lambda_2 x_1+\mu_2=0, \\ \lambda_3 x_2+\mu_3=0\end{array}\right.,$$

where all the λ_i, μ_i are non-zero. This sistem has no solution because $x_1=-\mu_2 / \lambda_2$ and $x_2=-\mu_3 / \lambda_3$ implies that $\lambda_1 x_1+\mu_1 x_2 \neq 0$. By Bernstein's theorem we have that $4!\mathrm{MV}\left(p\left(L_{12}\right), p\left(L_{14}\right), p\left(L_{24}\right)\right)=0$, as anticipated by the lemma.

(3) The system of equations corrresponding to $\mathrm{MV}\left(p\left(L_{12}\right), p\left(L_{13}\right), p(23)\right)$ is

$$\left\{\begin{array}{l}\lambda_1 x_1+\mu_1 x_2=0, \\ \lambda_2 x_1+\mu_2 x_3=0, \\ \lambda_3 x_2+\mu_3 x_3=0\end{array}\right.,$$

where all the λ_i, μ_i are non-zero. The only solution to this system is $x_1=x_2=x_3=0$, but this is not in $(\mathbb{C} \backslash\{0\})^3$. By Bernstein's theorem we have that $4!\mathrm{MV}(p(12), p(13), p(23))=0$, as anticipated by the lemma.

Sketch of proof. By Bernstein's theorem, we have that $d!\,\mathrm{MV}\left(p\left(i_1 j_1\right), \ldots, p\left(i_{d-1} j_{d-1}\right)\right)$ is the number of solutions in $(\mathbb{C} \backslash\{0\})^{d-1}$ of the system

$$\left\{\begin{array}{l}\lambda_{i_1} x_1+\mu_{i_1} x_{j_1}=0, \\ \lambda_{i_2} x_1+\mu_{i_2} x_{j_2}=0, \\ \vdots \\ \lambda_{i_{d-1}} x_{j_{d-1}}+\mu_{i_{d-1}} x_{j_{d-1}}=0 \\ x_d=1\end{array}\right.,$$

where all the λ_i, μ_i are non-zero. Suppose $i_1 j_1, \ldots, i_{d-1} j_{d-1}$ are the edges of a spanning tree of K_d Then, or each vertex v there is a unique path to d. Now, the edge $w d$ tells us that a solution must satisfy $x_w=-\frac{\mu_d}{\lambda_w}$. Substituting backwards along the path gives the unique value of x_v and this value is nonzero.

Instead, suppose that $i_1 j_1, \ldots, i_{d-1} j_{d-1}$ are not the edges of a spanning tree of K_d. Then, some of the vertices form a cycle. First, suppose that the graph is missing 2 or more vertices. Then the missing vertices give ∞-many solutions and there are no isolated solutions; thus the mixed volume is zero. (Alternatively, $\sum_{k=1}^{d-1} i_k j_k$ is $\leqslant(d-2)$ dimensional and by Theorem 4.3.16 (4) we have that the mixed volume is zero.) Next, ff the only missing vertex is d, then we obtain a system whose unique solution is $\{0\}^{d-1}$ and so the mixed volume is zero. Finally, if the the only missing vertex is $v \neq d$, then the system has no solution and so the mixed volume is zero. □

Proof. Proof of Theorem 4.5.7. The number of spanning trees of K_d is d^{d-2}. By Lemma 4.5.8, we have that

$$\operatorname{Vol}\left(p\left(\Pi_d\right)\right)=\sum_{T \text { spanning tree of } K_d} \frac{1}{(d-1)!}$$

and the result follows. □

Chapter 5

Zonotopes

We ended last chapter with some discussions on zonotopes, which are the Minkowski sums of line segments. In order to understand zonotopes (and polytopes in general) it is useful to look at the normal fan.

5.1 Normal fans

Definition 5.1.1. A **fan** $\mathcal{F} = C_1, \ldots, C_n$ in $\mathbb{R}^d$ is a collection of polyhedral cones

$$C_i = \operatorname{cone}(v_{i1}, \ldots, v_{ik_i}) \subseteq \mathbb{R}^d$$

such that every nonempty face of a cone is also a cone in Σ, and, the intersection of any two cones in Σ is a face of both.

Definition 5.1.2. The **normal fan** $\mathcal{N}(P) = \{N_F \mid F \text{ face of } P\}$ of a polytope P is the fan whose cones are

$$N_F = \left\{c \in \mathbb{R}^d \mid F \subseteq P_c\right\},$$

i.e., the directions that are maximized by F.

Example 5.1.3. Let $P = \operatorname{conv}\{(0,0), (2,0), (2,2), (1,2), (0,1)\}$. Then, the normal fan of P is depicted below. Normal fans of zonotopes are given by hyperplane arrangements.

Definition 5.1.4. A (central) **hyperplane arrangement** $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$ is a collection of hyperplanes

$$H_i = \left\{x \in \mathbb{R}^d \mid \langle a_i, x\rangle = 0\right\}.$$

Note that arrangements decompose $\mathbb{R}^d$ into a fan. Concretely, a cone C of the fan is determined by deciding for each H_i whether $\langle a_i, x\rangle > 0, \langle a_i, x\rangle < 0$, or $\langle a_i, x\rangle = 0$ for all $x \in \operatorname{int}(C)$. One can show that every zonotope is a translation of a zonotope of the form $Z = \sum_{i=1}^n [0, v_i]$, so let us restrict to these.

Proposition 5.1.5. The normal fan of a zonotope $Z = \sum_{i=1}^n [0, v_i]$ is the fan of the hyperplane arrangement $\{H_1, \ldots, H_n\}$, where

$$H_i = \left\{x \in \mathbb{R}^d \mid \langle v_i, x\rangle = 0\right\}.$$

Proof. Consider a face Z_c of Z. Recall that $Z_c = \sum_{i=1}^n [u_i, v_i]_c$, where

$$[0, v_i]_c = \begin{cases} [0, v_i], & \langle c, v_i\rangle = 0 \\ \{0\}, & \langle c, v_i\rangle < 0. \\ \{v_i\}, & \langle c, v_i\rangle > 0 \end{cases}$$

It follows that c, c' are in the same cone of $\mathcal{N}(Z)$ if and only if $\langle c, v_i\rangle$ and $\langle c', v_i\rangle$ have the same sign for all i. In turn, this holds if and only if c, c' are in the same cone of the hyperplane arrangement determined by $\mathcal{A}$. □

Theorem 5.1.6 (Shephard)**.** The zonotope $Z = \sum_{i=1}^{n}[0, v_i] \subseteq \mathbb{R}^d$ can be tiled into $\{0\}$ together with the translates of the half-open parallelepipeds $\sum_{j=1}^{k}(0, v_{i_k}]$, one for each linearly independent $v_{i_1}, \ldots, v_{i_k} \in \mathbb{R}^d$.

Example 5.1.7. Consider the permutohedron $\Pi_3 = [0,(1,-1,0)] + [0,(1,0,-1)] + [0,(0,1,-1)] + (1,2,3)$. The subdivision is depicted below.

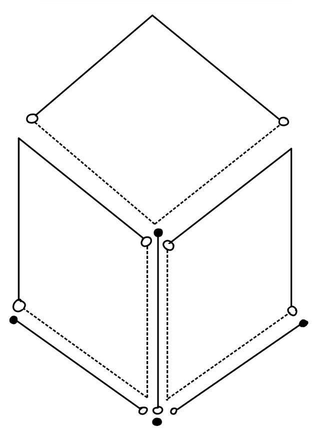


Figure 5.1: decomposition of cube

Proof. We proceed by induction on n. If $n = 1$, then $Z = [0, v_1] = \{0\} \cup (0, v_1]$, as desired. Now, suppose that $n > 1$. By induction, we have a decomposition as in the statement of $Z' = \sum_{i=1}^{n-1}[0, v_i]$. Define the hyperplane

$$H = \left\{x \in \mathbb{R}^d \mid \langle v_n, x\rangle = 0\right\},$$

and let $p : \mathbb{R}^d \to H$ be the orthogonal projection onto H. Then, $Z'' = \sum_{i=1}^{n-1}[0, p(v_i)]$ is a zonotope in H and so by induction we can decompose it as in the satement. Each open parallelepiped in the decomposition of Z'' is of the form $\sum_{i\in I}(0, p(v_i)]$, where the $p(v_i)$ are linearly independent. Now, given such a parallelepiped, lift it to a parallelepiped in Z by taking $(0, v_n] + \sum_{i\in I}(0, v_i]$. Consider a linear combination

$$\begin{aligned} \lambda_n v_n + \sum_{i\in I}\lambda_i v_i &= 0 \\ \Rightarrow 0 + \sum_{i\in I}\lambda_i p(v_i) &= 0 \end{aligned}$$

and since the $p(v_i)$ are linearly independent, we must have that $\lambda_i = 0$ for all $i \in I$. Since $v_n \neq 0$ then $\lambda_n = 0$ and so $\{v_n\} \cup \{v_i \mid i \in I\}$ are linearly independent. One can show that the union of the parallelepipeds obtained from Z' and Z'' is a tiling of Z. □

Lemma 5.1.8. Suppose $w_1, \ldots, w_d \in \mathbb{Z}^d$ are linearly independent, and let $\Pi = \sum_{i=1}^{d}(0, w_i]$. Then,

$$\mathrm{Vol}_d(\Pi) = \left|\Pi \cap \mathbb{Z}^d\right| = \left|\det(w_1, \ldots, w_d)\right|,$$

and for every positive $t \in \mathbb{Z}$,

$$\left|t\Pi \cap \mathbb{Z}^d\right| = (\mathrm{Vol}_d(\Pi))\, t^d.$$

Proof. Since Π is half-open, then we can tile $t\Pi$ using t^d translates of Π. It follows that

$$L_\Pi(t) = \left|t\Pi \cap \mathbb{Z}^d\right| = t^d \left|\Pi \cap \mathbb{Z}^d\right|.$$

Since $w_1, \ldots, w_d \in \mathbb{Z}^d$, we have that $L_\Pi(t)$ is a polynomial with leading coefficient

$$\mathrm{Vol}_d(\Pi) = \left|\det\left(w_1, \ldots, w_d\right)\right|.$$

Since the polynomials have to be equal, the result follows. □

Combining the Lemma and Shephard's theorem we obtain the following.

Corollary 5.1.9. Let $Z = \sum_{i=1}^n [0, v_i] \subseteq \mathbb{R}^d$. Then,

(1) $\mathrm{Vol}_d(Z) = \sum \left|\det\left(v_{i_1}, \ldots, v_{i_d}\right)\right|$, where the sum is over all bases $v_{i_1}, \ldots, v_{i_d}$.

(2) The Ehrhart polynomial $L_Z(t) = \sum_{W \text{ Wlin. ind.}} \mathrm{Vol}(Z(W)) t^{|W|}$, where $Z(W) = \sum_{v_i \in W} [0, v_i]$ and the $\mathrm{Vol}(Z(W))$ is taken in the affine span of $Z(W)$.

5.2 Generalized permutohedra

The permutohedron $\Pi_n \subseteq \mathbb{R}^n$ is the convex hull of $n\,!$ points obtained by permuting the coordinates of any vector $(a_1, \ldots, a_n)$ with strictly increasing coordinates $a_1 < \cdots < a_n$. Label the vertices of Π_n as $\pi_w = \left(a_{w^{-1}(1)}, \ldots, a_{w^{-1}(n)}\right)$, one for each $w \in S_n$. The edges of this permutohedron are $[\pi_w, \pi_{wsi}]$, where $s_i = (i, i+1)$ is an adjacent transposition.

Remark 5.2.1. For any $w \in S_n$ and s_i we have that $\pi_w - \pi_{wsi} = k_{w,i}\left(e_{w(i)} - e_{w(i+1)}\right)$, where $k_{w,i} \in \mathbb{Z}_{>0}$.

Definition 5.2.2. A **generalized permutohedron** P is the convex hull of $n\,!$ points $v_w \in \mathbb{R}^n$ such that for any $w \in S_n$ and adjacent transposition s_i we have that

$$v_w - v_{ws_i} = k_{w,i}\left(e_{w(i)} - e_{w(i+1)}\right),$$

where $k_{w,i} \in \mathbb{R}_{\geqslant 0}$ (i.e, $k_{w,i}$ can be zero).

Remark 5.2.3. This can be summarized as "P is a generalized permutohedron if and only if all of its edges are parallel to $e_i - e_j$ for some i, j ".

Example 5.2.4. The following are generalized permutohedra, Π_2, Q_1, Q_2.

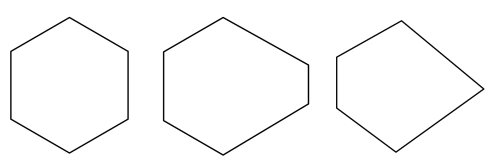


Figure 5.2: Examples of generalized permutohedra

Recall that:

- The **normal fan** $\mathcal{N}(P)$ of a polytope P is the fan whose cones are $N_F(P)$ where F is a face of P and $N_F(P) = \left\{c \in \mathbb{R}^d \mid F \subseteq P_c\right\}$.

- The **normal fan** of the zonotope $Z = \sum_{i=1}^n [0, v_i]$ is the fan of the hyperplane arrangement $\{H_1, \ldots, H_n\}$, where $H_i = \left\{x \in \mathbb{R}^d \mid \langle v_i, x\rangle = 0\right\}$.

Definition 5.2.5. We say that a fan $\mathcal{F}$ is **refined** by a fan $\mathcal{F}'$ if any cone in $\mathcal{F}$ is a union of cones in $\mathcal{F}'$.

Proposition 5.2.6. A polytope $P \subseteq \mathbb{R}^n$ is a generalized permutohedron if and only if $\mathcal{N}(P)$ is a coarsening of $\mathcal{N}(\Pi_n)$. (This fan is also known as the braid arrangement fan.)

Example 5.2.7. This can be verified for the polytopes in Example 5.2.4.

This proposition holds in a more general setting.

Proposition 5.2.8. Let $P, Q \subseteq \mathbb{R}^n$ be n-dimensional polytopes and assume Q is simple. The following are equivalent:

(1) $\mathcal{N}(Q)$ refines $\mathcal{N}(P)$.

(2) The vertices of P can be labelled $x_v, v \in \operatorname{verts}(Q)$ (possibly redundantly) so that for any edge $[u, v]$ of Q, there exist $k \in \mathbb{R}_{\geqslant 0}$ such that

$$x_u - x_v = k(u - v). \tag{5.1}$$

Exercise 5.2.9. Let $[u, v]$ be an edge of an n-dimensional polytope $Q \subseteq \mathbb{R}^n$. Show that $N_u(Q) \cap N_v(Q) \subseteq \{x \mid \langle x, u - v\rangle = 0\}$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathcal{N}(Q)$ refines $\mathcal{N}(P)$. Given a vertex $x \in P$, label it by x_v for every vertex $v \in Q$ such that $N_v(Q) \subseteq N_x(P)$. Now, consider an edge $[u, v]$ of Q. If $x_u = x_v$, then (15.1) holds trivially. On the other hand, if $x_u \neq x_v$, then $N_u(Q), N_v(Q)$ lie in different cones $N_{x_u}(P) \neq N_{x_v}(P)$. Since $N_u(Q)$ and $N_v(Q)$ are adjacent and $\mathcal{N}(Q)$ refines $\mathcal{N}(P)$, then $N_{x_u}(P)$ and $N_{x_v}(P)$ must share a codimension 1 face and the hyperplane defining it must be the same as the one separating $N_u(\Pi)$ from $N_v(\Pi)$. This hyperplane corresponds to an edge $[u, v]$ of Q and, by the exercise, the normal vector of this hyperplane is $u - v$. The exercise also implies that $x_u - x_v$ is a positive multiple of $u - v$, as desired.

(2) $\Rightarrow$ (1) To prove this result we need to recall the following. If λ is generic, then it induces an orientation of the graph of $P, G(P)$. Concretely, $u \to v$ if $\lambda(v) > \lambda(u)$. The orientation of $G(P)$ induced by this λ is acyclic and has a unique sink (a vertex with no outgoing edges). Moreover, λ is maximized over P at the sink.

Fix a vertex $u \in Q$ and let $c \in N_u(Q)$ be such that $\lambda(x) = \langle c, x\rangle$ is generic. Then, the orientation of $G(Q)$ induced by c is acyclic and has a unique sink u. For any other vertex $v \neq u$ there exists a unique directed path $(v_1 = v, v_2, \ldots, v_k = u)$ from v to u. Thus,

$$\lambda(v_1) < \cdots < \lambda(v_k) = \max_{x \in Q} \lambda(x).$$

Note that by (5.1) we have that

$$\lambda\left(x_{v_{i+1}}\right) - \lambda\left(x_{v_i}\right) = k\lambda\left(v_{i+1} - v_i\right) \geqslant 0$$

It follows that $\lambda(x_v) \leqslant \lambda(x_u)$. Since this inequality holds for any generic λ, we have that $N_u(Q) \subseteq N_{x_u}(P)$. Since the same statement is true for any vertex of Q, one deduces (1). ☐

Definition 5.2.10. A polytope P is a **deformation** of a simple polytope Q if it satisfies any of the conditions above.

Proposition 5.2.11. P is a deformation of Q if and only if P is a Minkowski summand of a dilation tQ, i.e., there exist a polytope P' and $t \in \mathbb{R}_{>0}$ such that $P + P' = tQ$.

Proof. See [8] Theorem 15.3. ☐

5.2.1 The normal fans of permutohedra

Recall that P is a generalized permutohedron if and only if $\mathcal{N}(P)$ is refined by the normal fan of the permutohedron.

Let us describe the normal fan of the permutohedron Π_d as a fan in $\mathbb{R}^d/\mathbb{R}\mathbb{1} \cong \mathbb{R}^{d-1}$. This is because $\Pi_d \subseteq \{x \in \mathbb{R}^d \mid \langle \mathbb{1}, x\rangle = n(n+1)/2\}$ and so it is natural to describe the normal fan in the orthogonal complement of this affine plane, translated to the origin.

Proposition 5.2.12. There is a cone $\sigma_{\mathcal{P}} \in \mathcal{N}(\Pi_d)$ for each ordered set partition $\mathcal{P} = (A_1, \ldots, A_k)$ of $[d]$. Concretely, the cone corresponding to $\mathcal{P}$ is

$$\begin{aligned}\sigma_{\mathcal{P}} &= \operatorname{cone}\left\{\sum_{i\in S} e_i \mid S = A_1 \cup \cdots \cup A_m \text{ for some } m < k\right\}\\ &= \{x \mid x_i \geqslant x_j \Leftrightarrow i \in A_m, j \in A_\ell, m \leqslant \ell\}.\end{aligned}$$

For example, the cone corresponding to the ordered set partition $(36, 124, 5)$ is

$$\operatorname{cone}\{e_3 + e_6, e_1 + e_2 + e_3 + e_4 + e_6\} = \{x \mid x_3 = x_6 \geqslant x_1 = x_2 = x_4 \geqslant x_5\}.$$

Proof. Since Π_d is a zonotope, the normal fan of this polytope is the fan for the hyperplane arrangement consisting of the hyperplanes

$$\{x \mid x_i = x_j\}$$

for each $i < j$. Concretely, the interiors of the cones of this fan are obtained by choosing for each $i < j$ one of the following

$$x_i = x_j, x_i < x_j, x_i > x_j.$$

In order to obtain a nonempty cone, we must be able to arrange these equations into a line, as in the inequality definition of $\sigma_{\mathcal{P}}$. □

5.3 Graphic zonotope

Definition 5.3.1. Let $G = (V = [n], E)$ be a graph without loops or multiple edges. The graphic zonotope $Z(G)$ is the Minkowski sum of the line segments $[e_i, e_j]$ for $(i, j) \in E$, $i < j$.

Example 5.3.2. The permutahedron Π_n is the graphic zonotope of the complete graph K_n.

Proposition 5.3.3. The zonotopal generalized permutohedra are exactly the graphic zonotopes.

It is useful to note that

$$Z(G) = \sum_{\substack{(i,j)\in E\\ i<j}} [e_i, e_j] = \sum_{\substack{(i,j)\in E\\ i<j}} (e_i + [0, e_j - e_i]).$$

Proof. We have seen that the normal fan of the zonotope $Z = \sum_{i=1}^n [0, v_i]$ is the fan of the hyperplane arrangement $\{H_1, \ldots, H_n\}$, where $H_i = \{x \in \mathbb{R}^d \mid \langle v_i, x\rangle = 0\}$. It follows that the $\mathcal{N}(Z(G))$ is the fan of the hyperplane arrangement $\{H_{ij} \mid (i, j) \in E\}$, where $H_{ij} = \{x \in \mathbb{R}^d \mid \langle e_i - e_j, x\rangle = 0\}$. This is a coarsening of the normal fan of the permutohedron Π_n. □

Proposition 5.3.4. Let G be a connected graph. The volume of $Z(G)$ equals the number of spanning trees of G. The number of lattice points of $Z(G)$ equals the number of forests in G.

The proof of this proposition is a direct application of Corollary 5.1.9. To do so we prove this result for the translated polytope $\sum_{\substack{(i,j)\in E\\ i<j}} [0, e_j - e_i]$.

Proof. Let $U = \operatorname{span}\{e_j - e_i \mid (i,j) \in E\}$. To prove the volume claim by applying Corollary 5.1.9 we need to show that $e_{j_1} - e_{i_1}, \ldots, e_{v_d} - e_{i_d}$ form a basis for U if and only the edges $(i_1, j_1), \ldots, (i_d, j_d)$ form a spanning tree of G. First, note the following (1) If some collection of edges $(i_1, j_1), \ldots, (i_k, j_k)$ forms a cycle, then $\sum (e_{j_1} - e_{i_1}) + \cdots (e_{j_d} - e_{i_d}) = 0$, i.e. the corresponding vectors are linearly dependent. (2) If the sub-graph formed by some collection of edges $(i_1, j_1), \ldots, (i_k, j_k)$ is missing vertex v of G, then since G is connected there is an edge $(u, v) \in E$ and the corresponding $e_v - e_u$ is linearly independent from the vectors $e_{j_1} - e_{i_1}, \ldots, e_{v_k} - e_{i_k}$.

Suppose the edges $(i_1, j_1), \ldots, (i_d, j_d)$ do not form a spanning tree of G. Then, either there is a cycle or a vertex is missing. In the first case we have, by (1), that the vectors are linearly dependent. In the second case we have, by (2), that the vectors are not spanning. In either case we see that the vectors do not form a basis for U.

Conversely, suppose that the edges $(i_1, j_1), \ldots, (i_d, j_d)$ form a spanning tree of G. First, note that the corresponding vectors must be spanning. Concretely, given $e_u - e_v$ not in the list if we add the edge (u, v) to the spanning tree this creates a cycle containing (u, v). By (1) we must have that $e_u - e_v$ is in the span of the vectors. Now, if the vectors are spanning but not linearly independent, then we can delete some of the vectors until we obtain a basis. However, since we have a spanning tree we must have disconnected a vertex from the graph, a contradiction to spanning independence, by (2). The volume claim follows from showing that

$$|\det(e_{j_1} - e_{i_1}, \ldots, e_{v_d} - e_{i_d})| = 1,$$

which is left as an exercise. The argument above can be adapted to show that $e_{j_1} - e_{i_1}, \ldots, e_{v_d} - e_{i_d}$ are linearly independent if and only the edges $(i_1, j_1), \ldots, (i_d, j_d)$ form a forest in G. Applying Corollary 5.1.9(2) with $t = 1$ we obtain the number of lattice points claim. □

5.3.1 Minkowski sums of simplices

Given $I \subseteq [d]$, let $\Delta_I := \operatorname{conv}\{e_i \mid i \in I\}$ which is a standard simplex. Given $\mathcal{I}$ a collection of subsets of $[d]$ and list of positive real numbers $\bar{y} = (y_I \mid I \in \mathcal{I})$ consider the polytope

$$\Delta_{\mathcal{I},\bar{y}} := \sum_{I \in \mathcal{I}} y_I \Delta_I.$$

If $|I| = 2$ and $y_I = 1$ for all $I \in \mathcal{I}$, then $\Delta_{\mathcal{I}}$ is a graphic zonotope.

Proposition 5.3.5. $\Delta_{\mathcal{I},\bar{y}}$ is a generalized permutohedron.

Proof. Recall that for any pair of polytopes P, Q and scalars r, s we have $(rP + sQ)_a = rP_a + sQ_a$. Thus, every edge of $\Delta_{\mathcal{I},\bar{y}}$ is parallel to a sum of an edge of exactly one Δ_I together with a collection of vertices of the remaining Δ_J. Note that the edges of Δ_I are parallel to $e_i - e_j$ for some $i, j \in I$. It follows that $\Delta_{\mathcal{I},\bar{y}}$ is a deformation of the permutahedron, i.e., a generalized permutohedron. □

Recall that Given $I \subseteq [d]$, let $\Delta_I := \operatorname{conv}\{e_i \mid i \in I\}$ which is a standard simplex. Given $\mathcal{I}$ a collection of subsets of $[d]$ and list of positive real numbers $\bar{y} = (y_I \mid I \in \mathcal{I})$ consider the polytope

$$\Delta_{\mathcal{I},\bar{y}} := \sum_{I \in \mathcal{I}} y_I \Delta_I.$$

We showed last time that these polytopes are generalized permutohedra.

Definition 5.3.6. We say that a collection $\mathcal{B}$ of nonempty subsets of a finite set S is a **building set** if it satisfies the following conditions.

(B1) If $I, J \in \mathcal{B}$ and $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.

(B2) For all $i \in S, \{i\} \in \mathcal{B}$.

Example 5.3.7. Let G be a graph with no loops or multiple edges with vertex set S. The set

$$\mathcal{B}(G) = \{J \subseteq S | J \neq \varnothing, G|_J \text{ is connected } \}$$

is a building set.

Sums of simplices where $\mathcal{I}$ is a building set are called **nestohedra**. Note that condition (B2) does not impose any additional restrictions on the structure of $\Delta_{\mathcal{I},\bar{y}}$ since it only translates the polytope. This condition is there only for convenience.

The next result computes the dimension and face lattice of a nestohedron. It also shows that these polytopes are simple. To understand the statement we need some definitions.

Definition 5.3.8. Let $\mathcal{B}$ be a building set on $[d]$. The collection $\mathcal{B}_{\max}$ consists of the inclusion-maximal elements of $\mathcal{B}$. A subset $\mathrm{N} \subset \mathcal{B} \backslash \mathcal{B}_{\max}$ is a **nested** if it satisfies the following conditions:

(N1) For any $I, J \in \mathrm{N}$, either $I \subseteq J, J \subseteq I$, or $I \cap J = \varnothing$.

(N2) For any $k \geqslant 2$ and $I_1, \ldots, I_k \in \mathrm{N}$ pairwise disjoint, the union $I_1 \cup \cdots \cup I_k \notin \mathcal{B}$.

Define the **nested complex** $\mathcal{C}_{\mathcal{B}}$ as the simplicial complex whose faces are the nested sets of $\mathcal{B}$.

A building set on S in **connected** if $\mathcal{B}_{max} = \{S\}$.

Example 5.3.9. The smallest connected building set on $[d]$ is $\mathcal{B} = \{\{1\}, \{2\}, \ldots, \{d\}, [d]\}$. The corresponding nestohedron is $\Delta_{\mathcal{B},1}$ is the standard simplex.

Example 5.3.10. Consider the building set $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. The resulting nestohedron is depicted below.

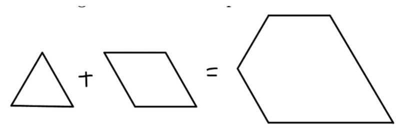


Figure 5.3: nestohedra

The maximal elements are $\mathcal{B}_{\max} = \{\{1, 2, 3\}\}$. The nested complex is depicted below.

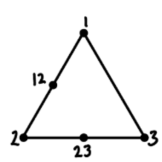


Figure 5.4: nested complex

Remark 5.3.11. One way to obtain the nested complex is as follows.

(1) Start with the simplicial complex whose facets are $\left\{ \binom{[d]}{d-1} \right\}$.

(2) Choose an ordering of the non-singleton elements of $\mathcal{B} \backslash \mathcal{B}_{\max}$ from larger sets to smaller sets.

(3) For each $I \in \mathcal{B} \backslash \mathcal{B}_{\max}$ and following this order perform a stellar subdivision of the simplex $\Delta = \{\{i\} \mid i \in I\}$. This means, add the vertex I to the complex and replace Δ by the $|I|$ simplices $\Delta_j = \{I\} \cup (\{\{i\} \mid i \in I \backslash \{j\}\})$ for each $j \in I$ (and their subsets).

The resulting complex is the nested complex.

Theorem 5.3.12. Let $\mathcal{B}$ is a building set on [d]. The nestohedron $\Delta_{\mathcal{B},\bar{y}}$ is a simple polytope of dimension $d - |\mathcal{B}_{\max}|$. The dual of $\Delta_{\mathcal{B},\bar{y}}$ is isomorphic, as a simplicial complex, to the nested complex $\mathcal{C}_{\mathcal{B}}$.

Proof. Let $P = \Delta_{\mathcal{B},\bar{y}}$. We want to show that each $\mathrm{N} \in \mathcal{C}_{\mathcal{B}}$ corresponds to a cone of the normal fan of P. First, start with a cone C in $\mathcal{N}(P)$. Since P is a generalized permutohedron, then C is a union of the cones in Proposition ref16.1. Pick a $\sigma_{\mathcal{P}} \subset C$ of maximal dimension, where $\mathcal{P} = (A_1, \ldots, A_k)$ is a partition of $[d]$, and let $c \in \sigma_{\mathcal{P}}$. Now,

$$(\Delta_I)_c = \Delta_{I \cap A_{j(I)}},$$

where $j(I)$ is the minimal index such that $I \cap A_j \neq \varnothing$. Note that if $I \subseteq J$, then $j(I) \geqslant j(J)$. Define

$$\mathrm{N} = \{I \in \mathcal{B} \backslash \mathcal{B}_{\max} \mid j(I) > j(J) \text{ for any } J \supsetneq I, J \in \mathcal{B}\}.$$

We claim that N is a nested set. (N1) Suppose $I, J \in \mathrm{N}$ are such that $I \cap J \neq \varnothing, I \nsubseteq J$, and $J \nsubseteq I$. By (B1) it follows that $I \cup J \in \mathcal{B}$. Since $A_{j(I \cup J)} \cap (I \cup J) \neq \varnothing$, then either $A_{j(I \cup J)} \cap I \neq \varnothing$ or $A_{j(I \cup J)} \cap J \neq \varnothing$. However, this contradicts that $j(I) > j(I \cup J)$ and $j(J) > j(I \cup J)$. (N2) Can be proven in a very similar way to (N1).

It follows that N is a nested set, as desired. For the converse, let N be a nested set. In order to obtain a cone of $\mathcal{N}(P)$ we need to provide an ordered set partition. For each $I \in \mathrm{N} \cup \mathcal{B}_{\max}$, let

$$A_I = I \backslash \bigcup_{\substack{J \subseteq I \\ J \in \mathrm{N}}} J.$$

We claim these sets partition $[d]$.

(1) If $x \in A_I \cap A_J$ then $x \in I \cap J$ and so by (N1) (WLOG) $J \subseteq I$. If $J \subsetneq I$, then $x \notin A_I$, which is not possible.

(2) Let $x \in [d]$. Note then that $x \in I$ for some $I \in \mathcal{B}_{\max}$. If $x \notin A_I$, then $x \in J$ for some $J \in \mathrm{N}$ such that $J \subsetneq I$. Let $\hat{J}$ be the smallest such set. Then, $x \in A_{\hat{J}}$.

Pick a linear order of $A_{I_1} < \cdots < A_{I_k}$ of these sets satisfying that $A_{I_i} < A_{I_j}$ if $I_i \subsetneq I_j$. It follows that there is a cone $\sigma_{\mathcal{P}} \in \mathcal{N}(\Pi_d)$ corresponding to this ordered set partition. The face corresponding to N is then given by the smallest cone of $\mathcal{N}(P)$ containing $\sigma_{\mathcal{P}}$.

There are some missing details that are left to the reader:

- The two processes are inverses of each other.

- $\mathrm{N} \subseteq \mathrm{N}'$ if and only if the cone corresponding to N is contained in the cone corresponding to N'. □

5.4 Catalan Numbers and Triangulations

Tom Davis's pdf gives a a set of combinatorial problems equivalently defining Catalan numbers. We recall that the Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$. For more of such equivalence, one may consult [11] section 6.2.

Proposition 5.4.1. The number of lattice paths from $(0,0)$ to (n,n), which only use the steps $(1,0),(0,1)$, and which do not pass above the diagonal equals the n-th Catalan number C_n.

Proof. First, all paths can be encoded in a sequence of nN 's and nE 's. Thus, the number of latttice paths from $(0,0)$ to (n,n) (which may go above the diagonal) is $\binom{2n}{n}$. Next, let us count the number of bad paths. To do so, let us show that the bad paths are in bijection with the lattice paths from $(0,0)$ to $(n-1,n+1)$. Given a bad path, it must cross the diagonal and touch the next diagonal $y = x+1$. The first time it touches $y = x+1$, reflect the remaining path over $y = x+1$. Note that in the section of the path that is not reflected, there is one more N step than E steps. It follows that in the section of the path that is reflected, there is one more E step than N steps. Since reflecting swaps $E \Leftrightarrow N$, it follows that the reflected path has $n+1$ N steps and $n-1$E steps. So, instead of reaching (n,n), all bad paths after reflection end at $(n-1,n+1)$, as desired. Now, the count for number of paths ending at $(n-1,n+1)$ is

$$\binom{n-1+n+1}{n-1} = \binom{2n}{n-1} = \binom{2n}{n+1}.$$

We conclude that the number of good paths is

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

□

Proposition 5.4.2. The Catalan numbers satisfy the recurrence $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for $n \geqslant 0$.

Proof. To obtain each good path from $(0,0)$ to $(n+1,n+1)$, we can follow the procedure below.

(1) Start with a good path from $(0,0)$ to (i,i) for some $i \in [n]$. There are C_i such paths.

(2) Take an E step to reach $(i+1,i)$.

(3) Take a path from $(i+1,i)$ to $(n,n-1)$ that stays above $y = x-1$. There are C_{n-i} such paths.

(4) Take an N step to reach $(n+1,n+1)$.

It follows that the number of good paths from $(0,0)$ to $(n+1,n+1)$ is $\sum_{i=0}^{n} C_i C_{n-i}$. □

Proposition 5.4.3. The number of triangulations of an $(n+2)$-gon that only use diagonals is C_n.

Proof. Let T_n be the number of triangulations of an $(n+2)$-gon. We will show that T_n satisfies the same recurrence as the Catalan numbers. Clearly, $T_0 = 1$ and $T_1 = 1$. Now, let $n \geqslant 1$ and consider an $(n+1+2)$-gon P. Pick one side of the P and call it the base. Label the vertices of P as $0, \ldots, n+2$ counterclockwise so that the base is the side between $n+1$ and $n+2$. To obtain a triangulation of P, we can follow the procedure below.

(1) Pick a vertex v among the ones labeled $0, \ldots, n$ and add the triangle with vertices $v, n+1, n+2$.

(2) The remaining part of P that needs to be triangulated consists of two polygons, one with $n+1$ as a vertex and the other with $n+2$ as a vertex. Note that the polygon P_1 with $n+1$ as a vertex is an $(v+2)$-gon and the polygon P_2 with $n+2$ as a vertex is an $(n+2-v)$-gon.

(3) Triangulate each of these polygons separatedly.

This shows that $T_{n+1} = \sum_{i=0}^{n} T_i T_{n-i}$. □

The collection of polygonal subdivisions of an $(n+2)$-gon tile a sphere.

Question:
Is there a polytope that agrees with this tiled sphere?

Theorem 5.4.4 (Loday)**.** Let $\mathcal{B}$ be the building set corresponding to the graph which is a path. The nestohedron $\Delta_{\mathcal{B},\mathbb{1}}$ agrees with this tiled sphere. This polytope is called **Loday's associahedron**.

Example 5.4.5. Consider the path with 3 vertices. The resulting nestohedron is given in example 5.3.10.

Remark 5.4.6. This is not the only polytope that agrees with this tiled sphere. There are many polytopes that do so and not all of them are isomorphic.

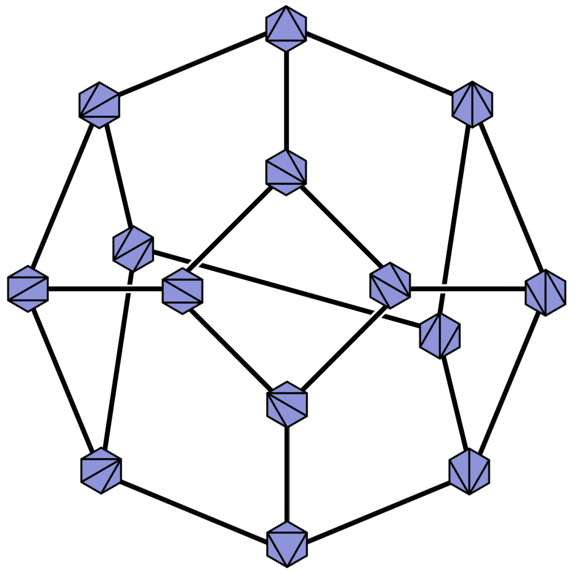
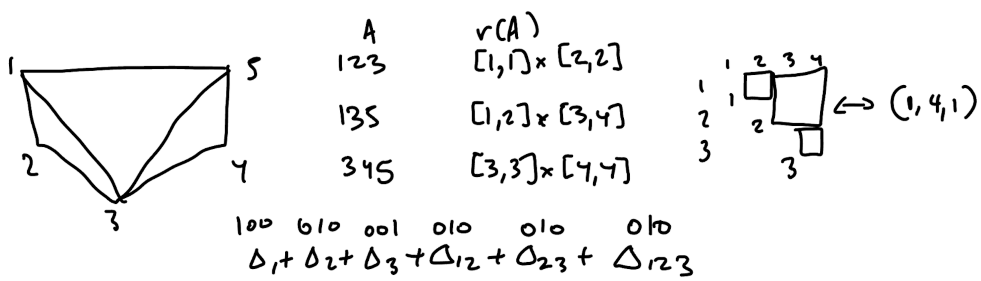


Figure 5.5: Polygonal subdivisions.

Lemma 5.4.7. The tilings of the staircase shape $(n, n-1, \dots, 1)$ with n rectangles are in bijection with the triangulations of an $(n+2)$-gon.

Example 5.4.8.



Sketch of proof. Label the vertices of an ($n+2$)-gon P as $1, \dots, n$ counterclockwise. Given a triangulation T of P we can obtain a tiling of the staircase shape as follows. For each triangle in T with vertices $i < j < k$, add the rectangle $[i, j-1] \times [j, k-1]$. This gives a bijection. □

Proof of Loday's theorem. Let $\mathcal{B}$ be the building set corresponding to the graph which is a path with n vertices. Let us prove that the vertices of $P = \Delta_{\mathcal{B},\mathbb{1}}$ are in bijection with the tilings of the staircase shape $(n, n-1, \dots, 1)$ with n rectangles. Label the corners of the steps $1, \dots, n$ and fix a tiling of the staircase shape. The i-th rectangle in such a subdivision is the rectangle that contains the i-th corner of the triangular shape. Associate to the tiling the vector $t = (t_1, \dots, t_n)$ where t_i equals the number of boxes in the i-th rectangle. The face P_t is the vertex t.

One can show that every vertex of P can be obtained in this way. Roughly, given c such that P_c is a vertex do the following.

(1) Let c_i be the largest entry. Then place the rectangle with corners at i and $(1, n)$.

(2) Remove this rectangle from the shape as well as entry c_i from c and repeat the process with each connected component.

One can then show that the higher dimensional faces of this polytope agree with those of the tiled sphere. □

Chapter 6

Polytopes in Algebraic Geometry

6.1 Polytopes arising from a torus action

The **projective space** is $\mathbb{P}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$, where $\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n)$. Denote the coordinates of $\mathbb{P}^n$ by $[a_0 : \cdots : a_n]$.

A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is **homogeneous** if $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$ for all $\lambda \in \mathbb{C}^*$. This is equivalent to saying that f is a sum of monomials of the same degree. An ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is **homogeneous** if it is generated by homogeneous polynomials.

A **projective variety** is $V(I) = \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in I\}$, where I is a homogeneous ideal. A **morphism** between two varieties $\phi : X \to Y$ is a map such that each entry is a polynomial. A projetive variety is irreducible if its defining ideal is prime. Today we will only work with irreducible varieties.

An (algebraic) **torus** is $T = (\mathbb{C}^*)^d$. Note that this is a group with respect to point-wise multiplication:

$$(t_1, \ldots, t_d) \cdot \left(t_1', \ldots, t_d'\right) = \left(t_1 t_1', \ldots, t_d t_d'\right).$$

The torus T acts on $\mathbb{P}^d$ by

$$(t_1, \ldots, t_d) \cdot [a_0 : \cdots : a_d] = [a_0 : t_1 a_1 \cdots : t_d a_d]. \tag{6.1}$$

However, there are many other ways in which a torus can act on $\mathbb{P}^n$. For example, given $T = (\mathbb{C}^*)^d$ and $w_0, \ldots, w_n \in \mathbb{Z}^d$ we have the action

$$(t_0, \ldots, t_n) \cdot [a_0 : \cdots : a_n] = [t^{w_0} : a_0 : t^{w_1} a_1 \cdots : t^{w_n} a_n].$$

An action of T on X is **algebraic** if $T \times X \to X$ is a morphism (i.e. a polynomial map). In fact, every algebraic action of T on $\mathbb{P}^n$ is of this form, see [7] section 1.1.

We are interested in projective varieties $X \subseteq \mathbb{P}^n$ that are invariant under an action of a torus T, i.e.

$$x \in X, t \in T \Longrightarrow t \cdot x \in X.$$

Let us call them T**-varieties**.

Example 6.1.1. Suppose T acts on $\mathbb{P}^n$ and $a \in \mathbb{P}^n$. The orbit of a is the set $T \cdot a = \{t \cdot a \mid t \in T\}$. Denote by $\overline{T \cdot a}$ the smallest projective variety containing $T \cdot a$, then $\overline{T \cdot a}$ is a T-variety. In fact, it is a projective toric variety. Let us look at some instances of for the action on $\mathbb{P}^2$ in (6.1).

(1) If $a = [1:1:1]$, then $T \cdot a = \{b \in \mathbb{P}^2 \mid \forall i, b_i \neq 0\}$. Moreover, $\overline{T \cdot a} = \mathbb{P}^2$. This is clearly T-invariant.

(2) If $a = [1:0:0]$, then $T \cdot a = \{a\}$. Moreover, $\overline{T \cdot a} = V(\langle x_1, x_2 \rangle) = \{a\}$. It is straightforward that this is T-invariant.

(3) If $a = [1:1:0]$, then $T \cdot a = \{b \in \mathbb{P}^2 \mid b_2 = 0, b_1, b_2 \neq 0\}$. We have that $\overline{T \cdot a} = V(\langle x_2 \rangle)$. One can verify that $[1:0:0] \in \overline{T \cdot a}$ by noting that

$$[1:0:0] = \lim_{t_2 \to 0} (t_1, t_2) \cdot [1:1:0].$$

Since $(t_1, t_2) \cdot [a_0 : a_1 : 0]$ satisfies that $x_2 = 0$, then this orbit closure is T-invariant.

Proposition 6.1.2. If T acts on $\mathbb{P}^n$ and $a \in \mathbb{P}^n$, then $\overline{T \cdot a}$ is T-invariant.

Proof. Since $T \times X \to X$ is a morphism, then it is continuous. Since every point in $\overline{T \cdot a}$ is obtained as a limit it follows that for any $s \in T, b \in \overline{T \cdot a}$,

$$s \cdot b = s \cdot \left(\lim_{t \to t*} t \cdot a \right) = \lim_{t \to t*} (st) \cdot a.$$

□

Definition 6.1.3.

- An action of T on $X \subseteq \mathbb{P}^n$ is **effective** (a.k.a. faithful) if

$$t = \mathbb{1} \iff \forall x \in X, t \cdot x = x.$$

- Suppose we have an effective action ot T on $\mathbb{P}^n$. The **weights** of this action are the $\omega_0, \ldots, \omega_n \in \mathbb{Z}^d$ such that

$$t \cdot [a_0, \ldots, a_n] = \left[a_0 t^{\omega_0}, \ldots, a_n t^{\omega_n}\right].$$

- If $X \subseteq \mathbb{P}^n$ is T-invariant, the weights of this action are the ω_i from above such that there exists $a \in X$ with $a_i \neq 0$.
- The **moment polytope** of $X \subseteq \mathbb{P}^n$ with respect to a given T-action is $\mu(X) = \mathrm{conv}\{$ weights of the T-action on $X\}$.

We saw how to obtain a polytope from a projective toric variety, via moment polytopes. We can see many properties of the variety encoded into the polytope.

Example 6.1.4. Consider the orbit closure $\overline{T \cdot [1:1:0]}$. Note that

$$\overline{T \cdot [1:1:0]} = V(\langle x_2 \rangle) = T \cdot [1:1:0] \sqcup T \cdot [1:0:0] \sqcup T \cdot [0:1:0].$$

This is reflected in the decomposition

$$\mu(\overline{T \cdot [1:1:0]}) = \mathrm{conv}\{0, e_1\} = \{0\} \sqcup \{e_1\} \sqcup (0, e_1).$$

Proposition 6.1.5. Suppose T is an action on a projective toric variety X.

(1) $\dim(X) = \dim(\mu(X))$.

(2) If X is a smooth, then the h-polynomial of $\mu(X)$ equals the Poincaré polynomial of X.

A consequence of (1) above is a criterion for when a T-variety is toric with respect to the T-action.

Corollary 6.1.6. Let X be a T-variety. The variety X is a projective toric variety if and only if $\dim(X) = \dim(\mu(X))$.

Proof. If X is a toric variety then, by (1) above, $\dim(X) = \dim(\mu(X))$. Conversely, if $\dim(X) = \dim(\mu(X))$, let $a \in X$ be generic. Such a point has the property that $\dim(\overline{T \cdot a})$ is as large as possible. Since $\overline{T \cdot a}$ and X are irreducible varieties and have the same dimension they must agree. It follows that X is a toric variety. □

Definition 6.1.7. The degree of a projective variety $X \subseteq \mathbb{P}^n$ is the number of points in $X \cap H$, where H is a generic linear space such that $\dim(H) = n - \dim(X)$.

Example 6.1.8. This notion generalizes the notion of degree of a polynomial. Consider the polynomial $f(x_0, x_1, x_2) = x_0^2 x_2 - x_1^2$ and the variety $V(f) \subseteq \mathbb{P}^2$. Then to obtain de degree we can look at the intersection of $V(f)$ with a line. In the chart $x_0 = 1$, the equation is $x_2 = x_1^2$ and we can see that there are two points in such an intersection. Thus, the degree of $V(f)$ is 2 .

Theorem 6.1.9. The degree of a projective toric variety X is equal to $\dim(X)! \operatorname{Vol}(\mu(X))$.

One can show that this is equivalent to Bernstein's theorem.

Definition 6.1.10. Let $X = V(I) \subseteq \mathbb{P}^n$ be a projective variety. The **Hilbert series** of I is

$$H_I(t) = \sum_{m \geqslant 0} \dim_{\mathbb{C}} \left(\left(\mathbb{C}[x_0, \ldots, x_n] / I \right)_m \right) t^m .$$

Here $\left(\mathbb{C}[x_0, \ldots, x_n] / I \right)_m$ denotes the m-th graded piece of $\mathbb{C}[x_0, \ldots, x_n] / I$, which is a $\mathbb{C}$ vector space.

Example 6.1.11. Let I be the trivial ideal so that $\mathbb{C}[x_0, \ldots, x_n] / I = \mathbb{C}[x_0, \ldots, x_n]$. Then, $\left(\mathbb{C}[x_0, \ldots, x_n] \right)_m$ consists of the homogeneous polynomials of degree m and this is generated as a $\mathbb{C}$-vector space by the monomials of degree m. It follows that $\dim \left(\mathbb{C}[x_0, \ldots, x_n] \right)_m = \left(\begin{array}{c} n+m \\ m \end{array} \right)$. We conclude that

$$H_I(t) = \sum_{m \geqslant 0} \binom{n+m}{m} t^m = \frac{1}{(1-t)^n} .$$

This agrees with the Ehrhart series of the simplex (see Example 8.7) and this is no accident.

Proposition 6.1.12. Let $X = V(I) \subseteq \mathbb{P}^n$ be a projective toric variety. The Hilbert series of I agrees with Ehrhart series of $\mu(X)$.

6.2 Moment polytopes in the Grassmannian

Definition 19.4. The Grassmannian $\operatorname{Gr}(k, d)$ is the set of k-dimensional linear subspaces of $\mathbb{C}^d$. For example, $\operatorname{Gr}(1, d) = \mathbb{P}^{d-1}$.

We give the Grassmannian the structure of a variety as follows. Given $v_1, \ldots, v_k \in \mathbb{C}^d$ linearly independent form the matrix $V = (v_1, \ldots, v_k)$, i.e. the v_i s are the columns of V. Given $I = \{i_1 < i_2 < \cdots < i_k\} \in \left(\begin{array}{c} [d] \\ k \end{array} \right)$, let

$$p_I(v_1, \ldots, v_k) = \det(\text{ rows of } V \text{ indexed by } I).$$

The Plücker embedding is the map

$$\begin{aligned} p : \operatorname{Gr}(k, d) &\hookrightarrow \mathbb{P}^{\binom{d}{k}-1} \\ \operatorname{span}(v_1, \ldots, v_k) &\mapsto \left[p_I(v_1, \ldots, v_k) \right]_{I \in \left(\begin{array}{c} [d] \\ k \end{array} \right)} . \end{aligned}$$

Exercise 6.2.1. The Plücker embedding is well-defined and one-to-one.

Theorem 6.2.2. The image of $\mathrm{Gr}(k,d)$ under this map is a variety.

Proof. The image is cut out by an ideal called the **Plücker ideal**. See [2] section 9.1 for a proof. □

Example 6.2.3. Consider the Grassmannian $\mathrm{Gr}(2,4)$ and the matrix $V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}^T$. The Plücker coordinates are

$$p_{12}(V) = 1, p_{13}(V) = 2, p_{14}(V) = 3, p_{23}(V) = 1, p_{24}(V) = 2, p_{34}(V) = 1.$$

The point in $\mathbb{P}^5$ corresponding to V is $[1:2:3:1:2:1]$.

The Grassmannian is a T-variety. The torus $T = (\mathbb{C}^*)^d$ acts on $\mathrm{Gr}(k,d)$ by any of the following equivalent ways:

- $t \cdot \left[p_I \middle| I \in \binom{[d]}{k} \right] = \left[t_{i_1} \cdots t_{i_k} p_I \middle| I \in \binom{[d]}{k} \right]$.
- $t \cdot \mathrm{colspan}(V) = \mathrm{colspan}\left(\mathrm{diag}\left(t_1, \ldots, t_d\right) V\right)$.
- $\mathrm{Gr}(k,d) = GL_d/P_k$, where P_k is a maximal parabolic subgroup. Identifying T with the subgroup of diagonal matrices in GL_d, the action of T of GL_d of left multiplication induces an action of T on $\mathrm{Gr}(k,d)$.

Note that $t_{i_1} \cdots t_{i_k} = t^{e_{i_1} + \cdots + e_{i_k}}$, where e_i is the i-th standard basis vector of $\mathbb{R}^d$. It follows that the weights of the action are the $e_{i_1} + \cdots + e_{i_k}$ for $I = \{i_1, \ldots, i_k\}$. Thus,

$$\mu(\mathrm{Gr}(k,d)) = \mathrm{conv}\left\{ e_{i_1} + \cdots + e_{i_k} \middle| I = \{i_1, \ldots, i_k\} \in \binom{[d]}{k} \right\}.$$

This polytope is called a **hypersimplex** and is denoted by $\Delta_{k,d}$.

Example 6.2.4. The moment polytope $\mu(\mathrm{Gr}(2,4))$ is depicted below.

Definition 6.2.5. A **matroid polytope** is a generalized permutohedron such that the vertices are $0/1$-vectors.

Exercise 6.2.6. The hypersimplex $\Delta_{k,d}$ is a matroid polytope.

Remark 6.2.7. Given $V \in \mathrm{Gr}(k,d)$, there is a combinatorial object called a matroid associated to V. One of the ways to describe this object is as a polytope, called a matroid polytope. It is a theorem of Gelfand-Goresky-MacPherson-Serganova that the moment polytope of $\overline{T \cdot V}$ is the matroid polytope for the matroid associated to V.

6.3 The moment polytope of the flag variety

Definition 6.3.1. The (complete) **flag variety** is

$$\mathrm{Fl}(d) = \left\{ (V_1, \ldots, V_{d-1}) \in \prod_{k=1}^{d-1} \mathrm{Gr}(k,d) \mid \forall i, V_i \subseteq V_{i+1} \right\}.$$

An embedding of $\prod_{k=1}^{d-1} \mathrm{Gr}(k,d)$ into $\mathbb{P}^N$ yields an embedding of $\mathrm{Fl}(d)$ into $\mathbb{P}^N$. To obtain the former note the following.

(1) The **Segre embedding** is the morphism

$$\begin{gathered}\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{rs-1} \\ ([x_0 : \cdots : x_{r-1}], [y_0 : \cdots : y_{s-1}]) \mapsto [x_i y_j \mid 0 \leqslant i < r, 0 \leqslant j < s].\end{gathered}$$

(2) Applying this map multiple times gives the embedding

$$\mathbb{P}^{r_1-1} \times \cdots \times \mathbb{P}^{r_d-1} \to \mathbb{P}^{r_1 \cdots r_d - 1}.$$

We obtain an embedding as follows

$$\prod_{k=1}^{d-1} \mathrm{Gr}(k,d) \hookrightarrow \mathbb{P}^{\binom{d}{1}-1} \times \cdots \times \mathbb{P}^{\binom{d}{d-1}-1} \hookrightarrow \mathbb{P}^{\binom{d}{1}\cdots\binom{d}{d-1}-1}.$$

Theorem 6.3.2. The image of $\mathrm{Fl}(d)$ under these maps is a variety.

Proof. See [2] section 9.1 for a proof. □

We consider the action of T on $\mathrm{Fl}(d)$ given diagonally by the action on each $\mathrm{Gr}(k,d)$, i.e.,

$$t \cdot (V_1, \ldots, V_{d-1}) = (t \cdot V_1, \ldots, t \cdot V_{d-1}).$$

Following the embedding carefully one can verify that the weights of the T-action on $\mathrm{Fl}(d)$ are

$$\left\{ \sum_{j=1}^{d} \sum_{i=1}^{j} e_{w_i} \mid w \in S_d \right\}.$$

Exercise 6.3.3. Do this computation for $d = 3$ and verify that $\mu(\mathrm{Fl}(3)) = \Pi_3$.

Proposition 6.3.4. The moment polytope of the flag variety $\mathrm{Fl}(d)$ is the permutohedron Π_d.

Chapter 7

Computing the Discrete Continuously

7.1 Brion's Theorem

We denote the **Laurent polynomials** in d variables with coefficients in C as

$$\mathbb{C}[\mathbf{z}^{\pm}] := \mathbb{C}[z_1^{\pm}, \cdots, z_d^{\pm}].$$

and the field of fractions of polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$ (i.e., the set of all rational functions $\{f(\mathbf{z})/g(\mathbf{z}) : f, g \in \mathbb{C}[\mathbf{z}], g \neq 0\}$) is denoted as $\mathbb{C}(z)$. We also let CL_d be the set of all (formal) Laurent series $\sigma_S(\mathbf{z})$ of rational cones S in $\mathbb{R}^d$. Recall that

$$\sigma_S(\mathbf{z}) = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} \cdots z_d^{m_d}.$$

Since the multiplication of a rational function and a Laurent series makes sense, we realize CL_d as a module over $\mathbb{C}[\mathbf{z}^{\pm}]$. $\mathbb{C}(\mathbf{z})$ is also a module over $\mathbb{C}[\mathbf{z}^{\pm}]$ in the obvious sense.

We notice that Theorem 3.3.4 evaluates the Laurent series $\sigma_{\mathcal{K}}(\mathbf{z}) \in \mathrm{CL}_d$ of a simplicial points cone $\mathcal{K}$ as a rational function in $\mathbb{C}(\mathbf{z})$. We want to extend that result of evaluation as a rational function in $\mathbb{C}(\mathbf{z})$.

Lemma 7.1.1. There is a unique linear map $\phi : \mathrm{CL}_d \to \mathbb{C}(\mathbf{z})$ that maps an integer-point transform $\sigma_S(\mathbf{z})$ (viewed as a Laurent series) of a rational simplicial cone

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_k \mathbf{w}_k : \lambda_1, \lambda_2, \ldots, \lambda_k \geqslant 0\} \subseteq \mathbb{R}^d$$

to the rational function

$$\frac{\sigma_\Pi(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_k})},$$

where $\sigma_\Pi(\mathbf{z})$ is the integer-point transform of the half-open parallelepiped

$$\Pi := \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_k \mathbf{w}_k : 0 \leqslant \lambda_1, \lambda_2, \ldots, \lambda_k < 1\}.$$

Proof. As we said, we proved in Theorem 3.3.4 that

$$(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_k})\, \sigma_{\mathcal{K}}(\mathbf{z}) = \sigma_\Pi(\mathbf{z}) \tag{7.1}$$

We remark that (7.1) is an identity in the module CL_d over $\mathbb{C}[\mathbf{z}^{\pm}]$: the factor $(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_k})$ on the left-hand side and the right-hand side $\sigma_\Pi(\mathbf{z})$ are Laurent polynomials in $\mathbb{C}[\mathbf{z}^{\pm}]$, whereas $\sigma_{\mathcal{K}}(\mathbf{z})$ is in CL_d.

If $\mathcal{K}$ is now a general rational cone, we can triangulate it into simplicial cone, each of which comes with a version of (7.1). The integer-point transform $\sigma_{\mathcal{K}}(\mathbf{z}) \in \mathrm{CL}_d$ of our general cone $\mathcal{K}$ can naturally be written in an inclusion-exclusion form as a sum (with positive and negative terms) of integer-point transforms of these simplicial cones and their faces, which are also simplicial cones. Applying the same sum to the identities of the form (7.1) for these simplicial cones gives an identity

$$g(\mathbf{z})\sigma_{\mathcal{K}}(\mathbf{z}) = f(\mathbf{z})$$

for some Laurent monomials $f(\mathbf{z})$ and $g(\mathbf{z})$. This yields our sought-after linear map: we define

$$\phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right) := \frac{f(\mathbf{z})}{g(\mathbf{z})} \in \mathbb{C}(\mathbf{z})$$

That this map ϕ is linear follows by construction, and that it is unique follows from the uniqueness of the rational-function form of $\sigma_{\mathcal{K}}(\mathbf{z})$ when $\mathcal{K}$ is simplicial. □

We will prove the following result in this section.

Theorem 7.1.2 (Brion's theorem for simplices)**.** Suppose Δ is a rational simplex. Then we have the following identity of rational functions:

$$\sigma_{\Delta}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \Delta} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$$

The notation $\mathcal{K}_{\mathcal{F}}$ stands for the **tangent cone** of a face $\mathcal{F}$ of $\mathcal{P}$, defined as

$$\mathcal{K}_{\mathcal{F}} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x} \in \mathcal{F}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}_{\geqslant 0}\}.$$

We note that $\mathcal{K}_{\mathcal{P}} = \operatorname{span} \mathcal{P}$. For a vertex $\mathbf{v}$ of $\mathcal{P}$, the tangent cone $\mathcal{K}_{\mathbf{v}}$ is often called a **vertex cone**; it is pointed, and we show an example in Figure 7.1. For a k-face $\mathcal{F}$ of $\mathcal{P}$ with $k > 0$, the tangent cone $\mathcal{K}_{\mathcal{F}}$ is not pointed. For example, the tangent cone of an edge of a 3-polytope is a wedge.

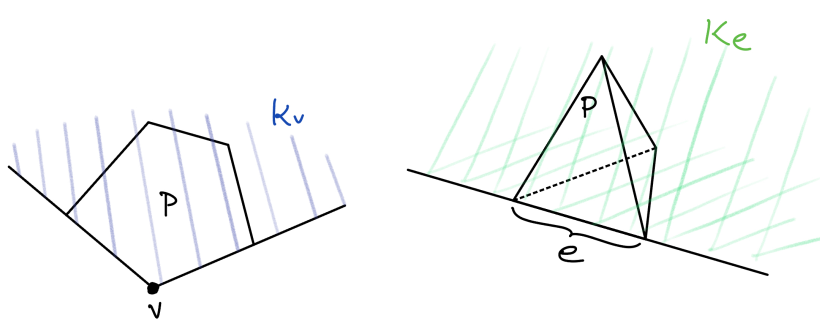


Figure 7.1: Tangent cones.

The summation over vertices comes from the summation over all faces of Δ where the summand for the faces that are not vertices are zero due to the following proposition.

Proposition 7.1.3. Let $\phi : \mathrm{CL}_d \to \mathbb{C}(\mathbf{z})$ be the linear map in Lemma 7.1.1, and let $\mathcal{K} \subseteq \mathbb{R}^d$ be a rational cone that contains a line. Then

$$\phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right) = 0$$

In particular, if $\mathcal{F}$ is a face of a polytope $\mathcal{P}$ that is not a vertex. Then

$$\phi\left(\sigma_{\mathcal{K}_{\mathcal{F}}}(\mathbf{z})\right) = 0.$$

Proof. Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a rational cone that contains a line. This implies that there exists a vector $\mathbf{w} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ such that $\mathbf{w} + \mathcal{K} = \mathcal{K}$. Translated into the language of Laurent series, this means that $\mathbf{z}^{\mathbf{w}} \sigma_{\mathcal{K}}(\mathbf{z}) = \sigma_{\mathcal{K}}(\mathbf{z})$, and thus, since ϕ is linear,

$$\mathbf{z}^{\mathbf{w}} \phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right) = \phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right).$$

But this gives the identity $(1 - \mathbf{z}^{\mathbf{w}})\, \phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right) = 0$ in the world $\mathbb{C}(\mathbf{z})$ of rational functions. Since $1 - \mathbf{z}^{\mathbf{w}}$ is not a zero divisor in this world, we conclude that $\phi\left(\sigma_{\mathcal{K}}(\mathbf{z})\right) = 0$.

The particular case where $\mathcal{F}$ is a face of a polytope $\mathcal{P}$ that is not a vertex is due to the observations that for every face $\mathcal{F}$ of $\mathcal{P}$, the tangent cones $\mathcal{K}_{\mathcal{F}}$ contains the affine space $\operatorname{span} \mathcal{F} = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$ (called the **apex** of the tangent cones $\mathcal{K}_{\mathcal{F}}$; see Figure 7.2 and Figure 7.1) and that affine space contains no line if and only if it has dimension 0.

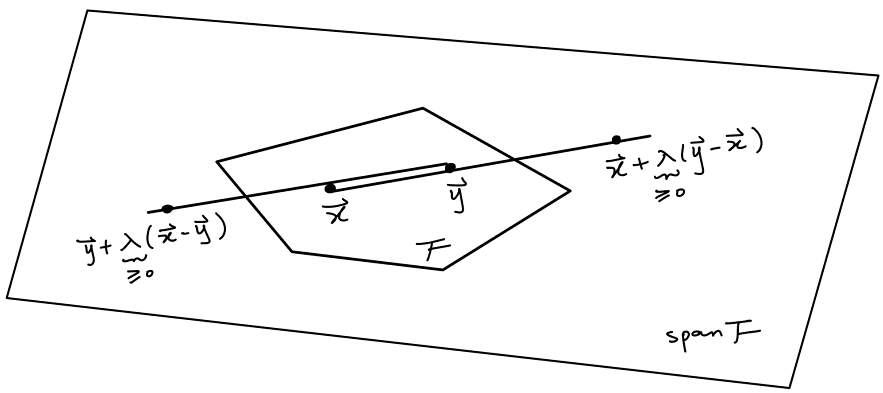


Figure 7.2: span of face

□

proof of theorem 7.1.2. We shall assume the **Brianchon-Gram identity** (which holds for rational polytopes in general and was proved in the case of simplices in [4] section 11.4; it's generally a result of reciprocity): for a d-simplex Δ,

$$1_{\Delta}(\mathbf{x}) = \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}} 1_{\mathcal{K}_{\mathcal{F}}}(\mathbf{x})$$

where the sum is taken over all nonempty faces $\mathcal{F}$ of Δ. Then,

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^d} 1_{\Delta}(\mathbf{m}) \mathbf{z}^{\mathbf{m}} &= \sum_{\mathbf{m} \in \mathbb{Z}^d} \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}} 1_{\mathcal{K}_{\mathcal{F}}}(\mathbf{m}) \mathbf{z}^{\mathbf{m}} \\ \sigma_{\Delta}(\mathbf{z}) &= \sum_{\mathcal{F} \subseteq \Delta} (-1)^{\dim \mathcal{F}} \sigma_{\mathcal{K}_{\mathcal{F}}}(\mathbf{z}) \quad \text{by defn. of } \sigma_S(\mathbf{z}). \end{aligned}$$

Applying the linear map $\phi : \mathrm{CL}_d \to \mathbb{C}(\mathbf{z})$ to above equality and noticing that ϕ evaluates $\sigma_S(\mathbf{z})$ as $\sigma_S(\mathbf{z})$ for simplicial S:

$$\begin{aligned}\sigma_\Delta(\mathbf{z}) = \phi\left(\sigma_\Delta(\mathbf{z})\right) &= \sum_{\mathcal{F}\subseteq\Delta}(-1)^{\dim\mathcal{F}}\phi\left(\sigma_{\mathcal{K}_\mathcal{F}}(\mathbf{z})\right) \\ &\overset{\text{prop. 7.1.3}}{=\!=\!=} \sum_{\mathbf{v}\in\mathrm{Vtx}\Delta}(-1)^{\overbrace{\dim\mathbf{v}}^{=0}}\phi\left(\sigma_{\mathcal{K}_\mathbf{v}}(\mathbf{z})\right) \\ &\overset{\mathcal{K}_\mathbf{v}\text{ simplicial}}{=\!=\!=} \sum_{\mathbf{v}\in\mathrm{Vtx}\Delta}\sigma_{\mathcal{K}_\mathbf{v}}(\mathbf{z})\end{aligned}$$

□

In fact, Brion's theorem holds for rational convex polytopes in general.

Theorem 7.1.4 (Brion's theorem)**.** Suppose $\mathcal{P}$ is a rational convex polytope. Then we have the following identity of rational functions:

$$\sigma_\mathcal{P}(\mathbf{z}) = \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}}\sigma_{\mathcal{K}_\mathbf{v}}(\mathbf{z})$$

Proof. See [4] Theorem 11.7. □

7.2 Fourier-Poisson and Euler-Maclaurin

We return to our recurring theme of computation of volume of polytopes. Recall that the **discrete volume** of a polytope $\mathcal{P}$ has the following form

$$|P\cap\mathbb{Z}^d| = L_\mathcal{P}(1) = \sum_{\mathbf{m}\in\mathbb{Z}^d}1_\mathcal{P}(\mathbf{m}) = \sum_{\mathbf{m}\in\mathcal{P}\cap\mathbb{Z}^d}1, \tag{7.2}$$

where $L_\mathcal{P}(t) = |t\mathcal{P}\cap\mathbb{Z}^d|$. The **continuous volume** of $\mathcal{P}$ is

$$\mathrm{Vol}_d(\mathcal{P}) = \int_{\mathbb{R}^d}d\mathbf{y} = \hat{1}_\mathcal{P}(0), \tag{7.3}$$

where $\hat{1}_\mathcal{P}(\mathbf{y}) = \int_{\mathbb{R}^d}1_\mathcal{P}(\mathbf{y})e^{-2\pi i\mathbf{y}\cdot\mathbf{x}}d\mathbf{x}$. [10] and [4] then introduce two approaches of exponentiation for summation to compute the discrete volumes. We can name them as Fourier-Poisson approach and Euler-Maclaurin approach.

The **Poisson summation formula** tells us that for any "sufficiently nice" function $f:\mathbb{R}^d\to\mathbb{C}$ we have:

$$\sum_{n\in\mathbb{Z}^d}f(n) = \sum_{\xi\in\mathbb{Z}^d}\hat{f}(\xi).$$

In particular, if we were to naively set $f(n) := 1_\mathcal{P}(n)$, the indicator function of a polytope $\mathcal{P}$, then we would get:

$$\sum_{n\in\mathbb{Z}^d}1_\mathcal{P}(n) = \sum_{\xi\in\mathbb{Z}^d}\hat{1}_\mathcal{P}(\xi), \tag{7.4}$$

which is technically false for functions, due to the fact that the indicator function $1_\mathcal{P}$ is discontinuous on $\mathbb{R}^d$. But when we do counting, Donald Knuth says we sometimes don't need to take care of those requirements to use some formulae because they serve as guessing tools. The end justifies the means.

[10] in chapter 10 defines the integer point transform of a rational polytope $\mathcal{P}$ by

$$\sigma_\mathcal{P}(z) := \sum_{n\in\mathcal{P}\cap\mathbb{Z}^d}e^{2\pi i\langle n,z\rangle},$$

a discretization of the Fourier transform of $\mathcal{P}$. Under the change of variable $q_1 := e^{2\pi i z_1}, \ldots, q_d := e^{2\pi i z_d}$, it is able to use the notations $q_1^{n_1} q_2^{n_2} \cdots q_d^{n_d} = e^{2\pi i n_1 z_1 + \cdots + 2\pi i n_d z_d} := e^{2\pi i \langle n, z \rangle}$ to define the multinomial notation for a monomial in several variables

$$q^n := q_1^{n_1} q_2^{n_2} \cdots q_d^{n_d}.$$

and recover the original defininition used in [4]

$$\sigma_{\mathcal{P}}(q) := \sum_{n \in \mathcal{P} \cap \mathbb{Z}^d} q^n.$$

One may check [10] chapter 8, 10 for more on Fourier-Poisson approach, where Brion's theorem is also written in Fourier series.

We return to Euler-Maclaurin approach in [4]. We consider the following exponentiation of the difference between (7.2) and (7.3):

$$\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} e^{\mathbf{m} \cdot \mathbf{x}} - \int_{\mathcal{P}} e^{\mathbf{y} \cdot \mathbf{x}} d\mathbf{y}. \tag{7.5}$$

where we have replaced the variable $\mathbf{z}$ that we have commonly used in generating functions by the exponential variable $(z_1, z_2, \ldots, z_d) = (e^{x_1}, e^{x_2}, \ldots, e^{x_d})$. Note that on setting $\mathbf{x} = 0$ in (7.5), we get quantity

$$\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} 1 - \int_{\mathcal{P}} d\mathbf{y} \tag{7.6}$$

7.3 A continuous version of Brion's Theorem

We give an integral analogue of Theorem 7.1.4 for simple rational polytopes. We begin by translating Brion's integer-point transforms

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$$

into an exponential form:

$$\sigma_{\mathcal{P}}(\exp \mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\exp \mathbf{z})$$

where we used the notation $\exp \mathbf{z} = (e^{z_1}, e^{z_2}, \ldots, e^{z_d})$. For the continuous analogue of Brion's theorem, we replace the sum on the left-hand side,

$$\sigma_{\mathcal{P}}(\exp \mathbf{z}) = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} (\exp \mathbf{z})^{\mathbf{m}} = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} \exp(\mathbf{m} \cdot \mathbf{z})$$

by an integral.

Theorem 7.3.1 (Brion's theorem: continuous form)**.** Suppose $\mathcal{P}$ is a simple rational convex d-polytope. For each vertex cone $\mathcal{K}_{\mathbf{v}}$ of $\mathcal{P}$, fix a set of generators $\mathbf{w}_1(\mathbf{v}), \mathbf{w}_2(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v}) \in \mathbb{Z}^d$. Then

$$\int_{\mathcal{P}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x} = (-1)^d \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \left|\det\left(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})\right)\right|}{\prod_{k=1}^d \left(\mathbf{w}_k(\mathbf{v}) \cdot \mathbf{z}\right)}$$

for all $\mathbf{z}$ such that the denominators on the right-hand side do not vanish.

Proof. We begin with the assumption that $\mathcal{P}$ is an integral polytope; we will see in the process of the proof that this assumption can be relaxed. Let's write out the exponential form of Brion's theorem (Theorem 7.1.4), using the assumption that the vertex cones are simplicial (because $\mathcal{P}$ is simple). By Theorem 3.3.4,

$$\sigma_{\mathcal{P}}(\exp \mathbf{z}) = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} \exp(\mathbf{m} \cdot \mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \sigma_{\Pi_{\mathbf{v}}}(\exp \mathbf{z})}{\prod_{k=1}^d \left(1 - \exp\left(\mathbf{w}_k(\mathbf{v}) \cdot \mathbf{z}\right)\right)} \tag{7.7}$$

where

$$\Pi_{\mathbf{v}} = \{\lambda_1 \mathbf{w}_1(\mathbf{v}) + \lambda_2 \mathbf{w}_2(\mathbf{v}) + \cdots + \lambda_d \mathbf{w}_d(\mathbf{v}) : 0 \leqslant \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}$$

is the fundamental parallelepiped of the vertex cone $\mathcal{K}_{\mathbf{v}}$. We would like to rewrite (7.7) with the lattice $\mathbb{Z}^d$ replaced by the refined lattice $\left(\frac{1}{n}\mathbb{Z}\right)^d$, because then, the left-hand side of (7.7) will give rise to the sought-after integral by letting n approach infinity. The right-hand side of (7.7) changes accordingly; now every integral point has to be scaled down by $\frac{1}{n}$:

$$\sum_{\mathbf{m} \in \mathcal{P} \cap \left(\frac{1}{n}\mathbb{Z}\right)^d} \exp(\mathbf{m} \cdot \mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \sum_{\mathbf{m} \in \Pi_{\mathbf{v}} \cap \mathbb{Z}^d} \exp\left(\frac{\mathbf{m}}{n} \cdot \mathbf{z}\right)}{\prod_{k=1}^d \left(1 - \exp\left(\frac{\mathbf{w}_k(\mathbf{v})}{n} \cdot \mathbf{z}\right)\right)} \tag{7.8}$$

The proof of this identity is in essence the same as that of Theorem 3.3.4; we leave it as an exercise. Now our sought-after integral is

$$\begin{aligned} \int_{\mathcal{P}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x} &\overset{\text{Riemann integral}}{=\!=\!=\!=} \lim_{n \to \infty} \frac{1}{n^d} \sum_{\mathbf{m} \in \mathcal{P} \cap \left(\frac{1}{n}\mathbb{Z}\right)^d} \exp(\mathbf{m} \cdot \mathbf{z}) \\ &= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \sum_{\mathbf{m} \in \Pi_{\mathbf{v}} \cap \mathbb{Z}^d} \exp\left(\frac{\mathbf{m}}{n} \cdot \mathbf{z}\right)}{\prod_{k=1}^d \left(1 - \exp\left(\frac{\mathbf{w}_k(\mathbf{v})}{n} \cdot \mathbf{z}\right)\right)} \end{aligned} \tag{7.9}$$

At this point, we can see that our assumption that $\mathcal{P}$ has integral vertices can be relaxed to the rational case, since we may compute the limit only for n 's that are multiples of the denominator of $\mathcal{P}$. The numerators of the terms on the right-hand side have a simple limit:

$$\begin{aligned} \lim_{n \to \infty} \exp(\mathbf{v} \cdot \mathbf{z}) \sum_{\mathbf{m} \in \Pi_{\mathbf{v}} \cap \mathbb{Z}^d} \exp\left(\frac{\mathbf{m}}{n} \cdot \mathbf{z}\right) &= \exp(\mathbf{v} \cdot \mathbf{z}) \sum_{\mathbf{m} \in \Pi_{\mathbf{v}} \cap \mathbb{Z}^d} 1 \\ &= \exp(\mathbf{v} \cdot \mathbf{z}) \left|\det\left(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})\right)\right| \end{aligned}$$

where the last identity follows from Lemma 5.1.8. Hence (7.9) simplifies to

$$\int_{\mathcal{P}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x} = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \left|\det\left(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})\right)\right|}{\prod_{k=1}^d \lim_{n \to \infty} n \left(1 - \exp\left(\frac{\mathbf{w}_k(\mathbf{v})}{n} \cdot \mathbf{z}\right)\right)}$$

Finally, using L'Hôpital's rule,

$$\lim_{n \to \infty} n \left(1 - \exp\left(\frac{\mathbf{w}_k(\mathbf{v})}{n} \cdot \mathbf{z}\right)\right) = -\mathbf{w}_k(\mathbf{v}) \cdot \mathbf{z}$$

and the theorem follows. □

It is an exercise to show that for each vertex cone $\mathcal{K}_{\mathbf{v}}$,

$$\int_{\mathcal{K}_{\mathbf{v}}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x} = (-1)^d \frac{\exp(\mathbf{v} \cdot \mathbf{z}) \left|\det\left(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})\right)\right|}{\prod_{k=1}^d \left(\mathbf{w}_k(\mathbf{v}) \cdot \mathbf{z}\right)}$$

and above theorem shows that the Fourier-Laplace transform of $\mathcal{P}$ equals the sum of the Fourier-Laplace transforms of the vertex cones. In other words,

$$\int_{\mathcal{P}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x} = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \int_{\mathcal{K}_{\mathbf{v}}} \exp(\mathbf{x} \cdot \mathbf{z}) d\mathbf{x}$$

We also remark that $\left|\det\left(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})\right)\right|$ has a geometric meaning: it is the volume of the fundamental parallelepiped of the vertex cone $\mathcal{K}_{\mathbf{v}}$.

The curious reader might wonder what happens to the statement of Theorem 7.3.1 if we scale each of the generators $\mathbf{w}_k(\mathbf{v})$ by a different factor. It is immediate ([4] Exercise 12.7) that the right-hand side of Theorem 7.3.1 remains invariant.

There is an important difference between the vertex cone generating functions (integrals) that appear in the continuous version of Brion's theorem (Theorem 7.3.1) and the vertex cone generating functions (sums) that appear in the discrete Brion theorem (Theorem 7.1.4). To see the difference, consider the following example:

Let K_0 be the first quadrant in $\mathbb{R}^2$, having generators $(1,0)$ and $(0,1)$. Let K_1 be the cone defined as the nonnegative real span of $(1,0)$ and $(1,k)$. For $k = 2^{100}$, say, we see that for all practical purposes, K_1 is very close to K_0 in its geometry, in the sense that their angles are almost the same for computational purposes, and thus their continuous Brion generating functions are almost the same, computationally.

However, $\sigma_{K_0}(z)$ is quite far from $\sigma_{K_1}(z)$, since the latter has 2^{100} terms in its numerator, while the former has only 1 as its trivial numerator. Thus, tangent cones that are "arbitrarily close" geometrically may simultaneously be "arbitrarily far" from each other in the discrete sense dictated by the integer points in their fundamental domains.

Exercise 7.3.2. Given a unimodular cone

$$\mathcal{K} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_d\mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geqslant 0\},$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$ such that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ are a basis for $\mathbb{Z}^d$ show that

$$\sigma_{\mathcal{K}}(\mathbf{z}) = \frac{\mathbf{z}^{\mathbf{v}}}{\prod_{k=1}^{d}\left(1 - \mathbf{z}^{\mathbf{w}_k}\right)}$$

and $|\det(\mathbf{w}_1, \ldots, \mathbf{w}_d)| = 1$.

7.4 Computing the Discrete Continuously

Our reference [4] bears the name *Computing the Continuous Discretely*, but we shall in this section do the reverse. We will prove Khovanskii-Pukhlikov Theorem for a certain class of polytopes, namely the unimodular polytopes in subsection 7.4.1.

7.4.1 Unimodular polytopes

We refer to [6] for unimodular polytopes.

Definition 7.4.1 (Unimodular Polytope)**.** A convex polytope $\Delta \subset \mathbb{R}^n$ is called unimodular if

- (Simplicity) there are n edges meeting at each vertex,
- (Rationality) the edges meeting at the vertex τ are rational in the sense that every edge E_k is of the form $\tau + tu_k$ where $t \in [0, T]$ and $u_k \in \mathbb{Z}^n$,
- (Smoothness) for each vertex with edges $E_1, \ldots, E_n$ the corresponding vectors $u_1, \ldots, u_n$ spanning the edges can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The following lemma will prove very useful for proving that a given set of vectors $u_1, \ldots, u_n$ is indeed a $\mathbb{Z}$-basis:

Lemma 7.4.2. The vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if

$$\det\begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} = \pm 1$$

Proof. Since $u_1, \ldots, u_n \subset \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ iff the matrix is invertible, and the matrix is invertible iff its determinant is a unit, which in $\mathbb{Z}$ are exactly ± 1, the result follows. $\square$

Remark 7.4.3. The name unimodular comes from the fact that a square integer matrix having determinant $+1$ or -1 is called a **unimodular matrix**. Unimodular matrices form a subgroup of the general linear group under matrix multiplication. Pascal matrices and permutation matrices are unimodular.

Permutation matrices are unimodular, although there are only two elements in S_2:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Pascal matrices are unimodular too. Recall that Pascal triangle can be put into a lower-triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

That is, $L_{ij} = \binom{i}{j} = \frac{i!}{j!(i-j)!}, j \leqslant i$. We use L_n to denote its $n \times n$ truncated version. Then observe that the determinant of a triangular matrix is the product of its diagonal. In this case, the determinant is then just 1. The matrix $A_n = L_n L_n^T$ has $\binom{i+j}{i} = \binom{i+j}{j} = \frac{(i+j)!}{i!j!}$ and $|A_n| = 1$. Consider $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and see Figure 7.3 for the lattice generated by $(1, 1)$ and $(1, 2)$.

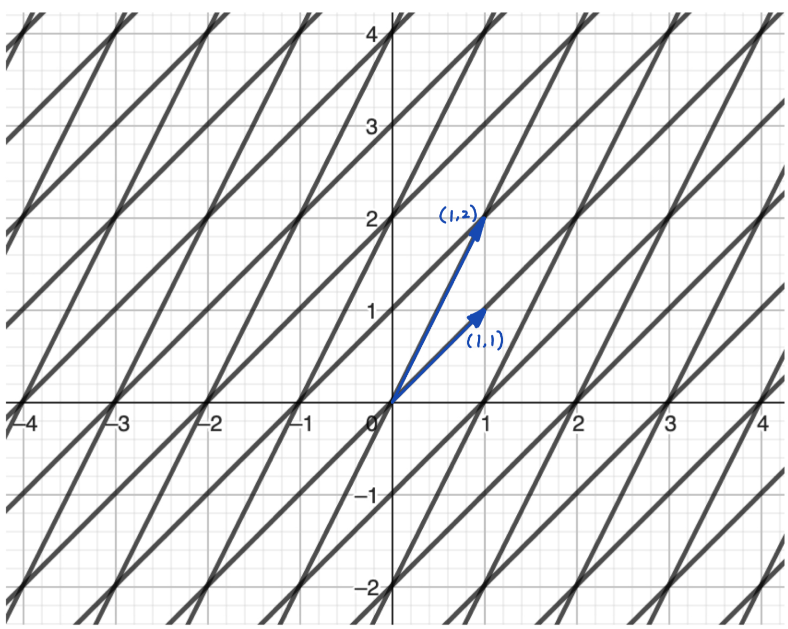
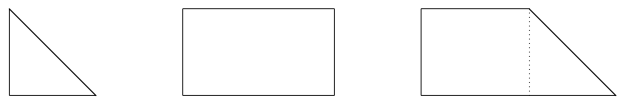


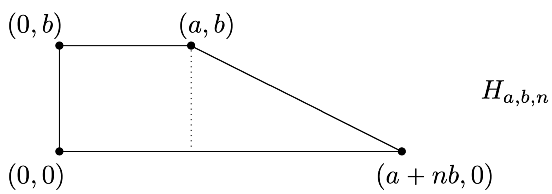
Figure 7.3: Lattice generated by $(1, 1)$ and $(1, 2)$

Unimodular polytopes are, in the context of symplectic toric manifolds, sometimes also referred to as **Delzant polytopes**.

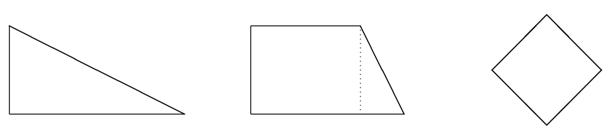
More examples of unimodular polytopes in $\mathbb{R}^2$:



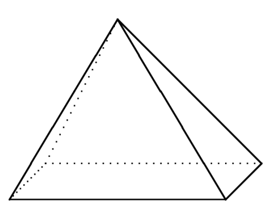
The pictures above represent polytopes in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$, i.e., standard horizontal and vertical cartesian axes with same scale. The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an isosceles triangle. For ”taller” triangles, smoothness would be violated. ”Wider” triangles may still be unimodular as in the examples below, denoted $H_{a,b,n}$, as long as the slope of the hypothenuse satisfies an integrality condition given by $n = 0, 1, 2, \ldots$ The positive real parameters a and b are the width and height of the left rectangle. We call these examples **Hirzebruch trapezoids**. In particular, $H_{a,b,0}$ is just a rectangle.



Examples of polytopes that are not unimodular: Once again, the pictures above represent polytopes in $\mathbb{R}^2$ with



standard lattice $\mathbb{Z}^2$. The picture on the left fails the smoothness condition on the upper vertex (see Figure 7.4), whereas the one in the middle fails the smoothness condition on the two right vertices, and the one on the right fails the smoothness condition on all vertices. Moreover, the following pyramid in $\mathbb{R}^3$ fails the simplicity condition.



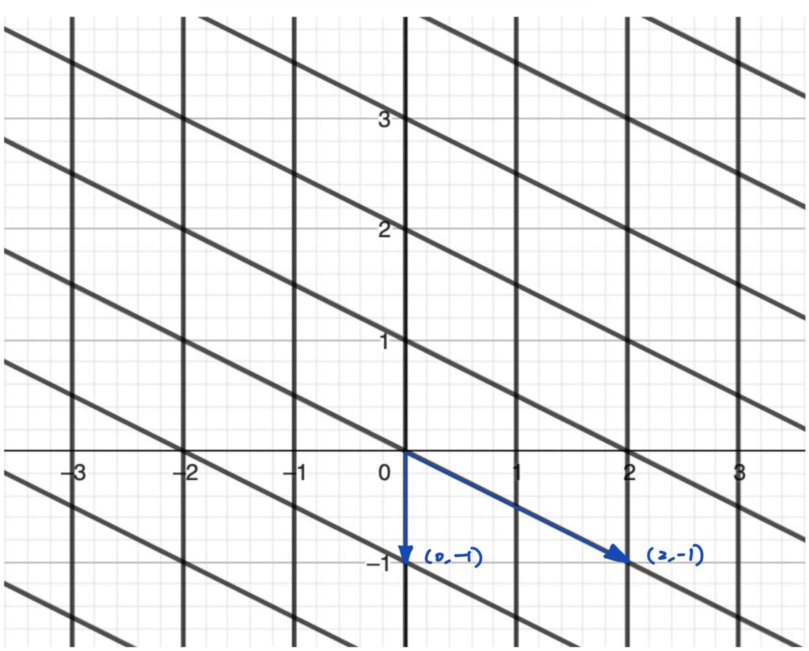


Figure 7.4: Lattice generated by $(0,-1)$ and $(2,-1)$.

7.4.2 Todd operator

Recall the **Bernoulli numbers** B_k defined by the generating function

$$\frac{z}{e^z-1}=\sum_{k\geqslant 0}\frac{B_k}{k!}z^k$$

We now introduce a differential operator via essentially the same generating function, namely

$$\operatorname{Todd}_h:=\sum_{k\geqslant 0}(-1)^k\frac{B_k}{k!}\left(\frac{d}{dh}\right)^k. \tag{12.3}$$

This **Todd operator** is often abbreviated as

$$\operatorname{Todd}_h=\frac{\frac{d}{dh}}{1-e^{-\frac{d}{dh}}} \tag{7.10}$$

but we should keep in mind that this is only a shorthand notation for the infinite series (7.10). We first show that the exponential function is an eigenfunction of the Todd operator.

Lemma 7.4.4. For $z\in\mathbb{C}\backslash\{0\}$ with $|z|<2\pi$,

$$\operatorname{Todd}_h e^{zh}=\frac{ze^{zh}}{1-e^{-z}}$$

Proof.

$$\begin{aligned}\operatorname{Todd}_h e^{zh} &= \sum_{k\geqslant 0}(-1)^k \frac{B_k}{k!}\left(\frac{d}{dh}\right)^k e^{zh}\\ &= \sum_{k\geqslant 0}(-1)^k \frac{B_k}{k!} z^k e^{zh}\\ &= e^{zh}\sum_{k\geqslant 0}(-z)^k \frac{B_k}{k!}\\ &= e^{zh}\frac{-z}{e^{-z}-1}.\end{aligned}$$

The condition $|z| < 2\pi$ is needed in the last step, by [4] Exercise 2.14. □

7.4.3 Khovanskii-Pukhlikov Theorem

we apply the Todd operator to a perturbation of the continuous volume. Namely, consider a simple full-dimensional polytope $\mathcal{P}$, which we may write as

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\right\}$$

Then we define the perturbed polytope

$$\mathcal{P}(\mathbf{h}) := \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}+\mathbf{h}\right\}$$

for a small vector $\mathbf{h} \in \mathbb{R}^m$ (we will quantify the word small in a moment). A famous theorem due to Askold Khovanskiì and Aleksandr Pukhlikov says that the integer-point count in $\mathcal{P}$ can be obtained by applying the Todd operator to $\operatorname{vol}(\mathcal{P}(\mathbf{h}))$. Here we prove the theorem for a certain class of polytopes, which we need to define first.

Theorem 7.4.5 (Khovanskii-Pukhlikov theorem)**.** For a unimodular d polytope $\mathcal{P}$,

$$\#\left(\mathcal{P} \cap \mathbb{Z}^d\right) = \operatorname{Todd}_{\mathbf{h}} \operatorname{vol}(\mathcal{P}(\mathbf{h}))\big|_{\mathbf{h}=0}$$

More generally,

$$\sigma_{\mathcal{P}}(\exp \mathbf{z}) = \operatorname{Todd}_{\mathbf{h}} \int_{\mathcal{P}(\mathbf{h})} \exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}\bigg|_{\mathbf{h}=0}$$

Proof. We use Theorem 7.3.1, the continuous version of Brion's theorem; note that if $\mathcal{P}$ is unimodular, then $\mathcal{P}$ is automatically simple. For each vertex cone $\mathcal{K}_{\mathbf{v}}$ of $\mathcal{P}$, denote its generators by $\mathbf{w}_1(\mathbf{v}), \mathbf{w}_2(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v}) \in \mathbb{Z}^d$. Then Theorem 7.3.1 states that

$$\begin{aligned}\int_{\mathcal{P}} \exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x} &= (-1)^d \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})\left|\det\left(\mathbf{w}_1(\mathbf{v}),\ldots,\mathbf{w}_d(\mathbf{v})\right)\right|}{\prod_{k=1}^d\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\\ &= (-1)^d \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})}{\prod_{k=1}^d\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\end{aligned} \tag{7.11}$$

where the last identity follows from Exercise 7.3.2. A similar formula holds for $\mathcal{P}(\mathbf{h})$, except that we have to account for the shift of the vertices. The vector $\mathbf{h}$ shifts the facet-defining hyperplanes. This shift of the facets induces a shift of the vertices; let's say that the vertex $\mathbf{v}$ gets moved along each edge direction $\mathbf{w}_k$ (the vectors that generate the vertex cone $\mathcal{K}_{\mathbf{v}}$) by $h_k(\mathbf{v})$, so that $\mathcal{P}(\mathbf{h})$ has now the vertex $\mathbf{v} - \sum_{k=1}^d h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})$. If $\mathbf{h}$ is small enough, $\mathcal{P}(\mathbf{h})$ will still be simple, and we can apply Theorem 7.3.1 to $\mathcal{P}(\mathbf{h})$:

[1]The cautious reader may consult [12] p. 66 to confirm this fact.

$$\begin{aligned}\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x} &= (-1)^d \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \frac{\exp\left(\left(\mathbf{v}-\sum_{k=1}^{d} h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})\right)\cdot\mathbf{z}\right)}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\\
&= (-1)^d \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \frac{\exp\left(\mathbf{v}\cdot\mathbf{z}-\sum_{k=1}^{d} h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\\
&= (-1)^d \sum_{\mathbf{v}\text{ a vertex of }\mathcal{P}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})\prod_{k=1}^{d}\exp\left(-h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\end{aligned}$$

Strictly speaking, this formula holds only for $\mathbf{h}\in\mathbb{Q}^m$, so that the vertices of $\mathcal{P}(\mathbf{h})$ are rational. Since we will eventually set $\mathbf{h}=0$, this is a harmless restriction. Now we apply the Todd operator:

$$\begin{aligned}&\operatorname{Todd}_{\mathbf{h}}\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}\Big|_{\mathbf{h}=0}\\
&= (-1)^d \sum_{\mathbf{v}\text{ vertex of }\mathcal{P}} \operatorname{Todd}_{\mathbf{h}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})\prod_{k=1}^{d}\exp\left(-h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\Bigg|_{\mathbf{h}=0}\\
&= (-1)^d \sum_{\mathbf{v}\text{ vertex of }\mathcal{P}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\\
&\times\prod_{k=1}^{d}\operatorname{Todd}_{h_k(\mathbf{v})}\exp\left(-h_k(\mathbf{v})\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)\Big|_{h_k(\mathbf{v})=0}\end{aligned}$$

By a multivariate version of Lemma 7.4.4,

$$\begin{aligned}&\operatorname{Todd}_{\mathbf{h}}\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}\Big|_{\mathbf{h}=0}\\
&= (-1)^d \sum_{\mathbf{v}\text{ vertex of }\mathcal{P}} \frac{\exp(\mathbf{v}\cdot\mathbf{z})}{\prod_{k=1}^{d}\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\prod_{k=1}^{d}\frac{-\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}}{1-\exp\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\\
&= \sum_{\mathbf{v}\text{ vertex of }\mathcal{P}} \exp(\mathbf{v}\cdot\mathbf{z})\prod_{k=1}^{d}\frac{1}{1-\exp\left(\mathbf{w}_k(\mathbf{v})\cdot\mathbf{z}\right)}\end{aligned}$$

However, Brion's theorem (Theorem 7.1.4), together with the fact that $\mathcal{P}$ is unimodular, says that the right-hand side of this last formula is precisely the integer-point transform of $\mathcal{P}$ (see also (7.11)), and thus

$$\operatorname{Todd}_{\mathbf{h}}\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}\Big|_{\mathbf{h}=0} = \sigma_{\mathcal{P}}(\exp\mathbf{z})$$

Finally, setting $\mathbf{z}=0$ gives

$$\operatorname{Todd}_{\mathbf{h}}\int_{\mathcal{P}(\mathbf{h})}d\mathbf{x}\Big|_{\mathbf{h}=0} = \sum_{\mathbf{m}\in\mathcal{P}\cap\mathbb{Z}^d}1$$

as claimed. □

We note that $\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}$ is, by definition, the continuous FourierLaplace transform of $\mathcal{P}(\mathbf{h})$. Upon being acted on by the discretizing operator Todd $_{\mathbf{h}}$, the integral $\int_{\mathcal{P}(\mathbf{h})}\exp(\mathbf{x}\cdot\mathbf{z})d\mathbf{x}$ gives us the discrete integer-point transform $\sigma_{\mathcal{P}}(\mathbf{z})$.

7.5 Note

1. The classical Euler-Maclaurin formula states that

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{f(0)+f(n)}{2} + \sum_{m=1}^{p} \frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(x)\right]_0^n + \frac{1}{(2p+1)!}\int_0^n B_{2p+1}(\{x\})f^{(2p+1)}(x)dx$$

   where $B_k(x)$ denotes the k^{th} Bernoulli polynomial. It was discovered independently by Leonhard Euler and Colin Maclaurin. This formula provides an explicit error term, whereas [4] Theorem 12.2 provides a summation formula with no error term.

2. The Todd operator was introduced by Friedrich Hirzebruch in the 1950s [9], following a more complicated definition by John A. Todd some twenty years earlier. The Khovanskii-Pukhlikov theorem can be interpreted as a combinatorial analogue of the algebrogeometric Hirzebruch-Riemann-Roch theorem, in which the Todd operator plays a prominent role.

3. Theorem 7.3.1, the continuous form of Brion's theorem, was generalized by Alexander Barvinok to every polytope [3]. In fact, [3] contains a certain extension of Brion's theorem to irrational polytopes as well.

Bibliography

[1] Ewald, Günter. *Combinatorial Convexity and Algebraic Geometry*, Springer Science & Business Media, vol. 168, 1996.

[2] Fulton, William. *Young Tableaux,* vol. 35, London Mathematical Society Student Texts, with applications to representation theory and geometry, Cambridge University Press, 1997, pp. x+260. ISBN: 0-521-56144-2; 0-521-56724-6.

[3] Barvinok, Alexander. *Exponential Integrals and Sums over Convex Polyhedra*, Funktsional. Anal. i Prilozhen. 26 (1992), no. 2, pp. 64-66.

[4] Beck, Matthias and Robins, Sinai. *Computing the Continuous Discretely*, Springer, vol. 61, 2007.

[5] Beck, Matthias and Sanyal, Raman. *Combinatorial Reciprocity Theorems*, American Mathematical Soc., vol. 195, 2018.

[6] Cannas, Ana. *Seminar on Symplectic Toric Manifolds*. Notes available at https://people.math.ethz.ch/~acannas/Papers/stm_seminar.pdf

[7] Cox, David A. and Little, John B. and Schenck, Henry K. *Toric Varieties*, vol. 124, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7. DOI: 10.1090/gsm/124. URL: https://doi.org/10.1090/gsm/124.

[8] Postnikov, Alex and Reiner, Victor and Williams, Lauren. *Faces of Generalized Permutohedra*. In: Doc. Math. 13 (2008), pp. 207–273. ISSN: 1431-0635,1431-0643.

[9] Pukhlikov, A. V. and Khovanskii, A. G. *A Riemann-Roch Theorem for Integrals and Sums of Quasipolynomials over Virtual Polytopes*, Algebra i Analiz, vol. 4 (1992), no. 4; St. Petersburg Math. J., vol. 4 (1993), no. 4.

[10] Robins, Sinai. *A Friendly Introduction to Fourier Analysis on Polytopes*, arXiv preprint arXiv:2104.06407, 2021.

[11] Stanley, Richard P. *Enumerative Combinatorics, Volume 2*, vol. 62, Cambridge Studies in Advanced Mathematics, with a foreword by Gian–Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, 1999, pp. xii+581. ISBN: 0-521-56069-1; 0-521-78987-7.

[12] Ziegler, Günter M. *Lectures on Polytopes*, Vol. 152, Springer Science & Business Media, 2012.