Symplectic Geometry: Reduction, Convexity, and Unimodularity

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Abstract

In this thesis, we review the classical results of symplectic geometry, focusing on the Marsden-Weinstein-Meyer Theorem, the Atiyah-Guillemin-Sternberg Theorem, and Delzant's classification of the symplectic toric manifolds. We begin with preliminaries covering symplectic manifolds, compatible triples, Morse theory, Lie groups, and Hamiltonian actions, including examples like circle actions and complex projective space with Fubini-Study form. The thesis concludes with some applications and generalizations of these classical theorems: Horn's conjecture on Hermitian spectra, Kirwan [Kir84] and Weinstein [Wei01]'s generalization of the convexity theorem, and the principle of quantization commuting with reduction.

Contents

1 Introduction

Symplectic geometry, a field that emerged from classical mechanics, provides the geometric framework for Hamiltonian mechanics. Its origins can be traced to the study of phase space and canonical transformations, which found formal structure through the works of Poincaré, Lie, and Hamilton. The development of symplectic geometry accelerated in the 20th century, notably with the introduction of the Marsden-Weinstein-Meyer Reduction Theorem [MW74; Mey73] and the Atiyah-Guillemin-Sternberg Convexity Theorem [Ati82; GS82a]. These theorems illustrate how symmetry and integrable structures shape the topology and geometry of symplectic manifolds.

The reduction theorem, discovered independently by Meyer [Mey73] and Marsden and Weinstein [MW74], concerns the symplectic reduction M_{red} of a Hamiltonian G-space M. It demonstrates how conserved quantities arising from symmetry allow us to reduce the dimension of the phase space, yielding simpler manifolds that retain crucial geometric and dynamical information. This theorem has applications in classical mechanics [Mar+07; MMR90] and in geometric representation theory, where symplectic reduction parallels the construction of representation spaces.

The convexity theorem, developed independently by Atiyah [Ati82] and Guillemin and Sternberg [GS82a], describes the image of moment maps associated with torus actions on compact symplectic manifolds. The theorem asserts that these images are convex polytopes, establishing a striking link between geometry and combinatorics. Delzant [Del88] extended these ideas by classifying symplectic toric manifolds in terms of rational, simple, and smooth polytopes—now called Delzant polytopes. In particular, given such a polytope Δ, one constructs a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ via symplectic reduction from $(\mathbb{C}^d, \omega_0)$ by a subtorus $N \subset \mathbb{T}^d$, yielding a moment map μ_Δ such that $\mu_\Delta(M_\Delta) = \Delta$. This classification also laid the foundation for the study of toric varieties using symplectic geometry.

Karshon and Lerman later classified the non-compact analogues of symplectic toric manifolds using manifolds with corners equipped with degree-two cohomology classes and unimodular local embeddings [KL15]. My interest in this topic was sparked in Professor Laura Escobar's course on polytopes, where I encountered Pukhlikov and Khovanskii's paper [PK93]. Their work gives a combinatorial formula for counting lattice points in Delzant polytopes using Todd operators, which relates to a representation-theoretic conjecture posed by Guillemin and Sternberg in the 1980s [GS82b].

This *multiplicity conjecture* asserts that for a Hamiltonian G-space (M, ω, G, μ) where G is a compact connected Lie group, and for a prequantizable coadjoint orbit O corresponding to an irreducible unitary representation ρ_O in Bott-Borel-Weil sense, the multiplicity of ρ_O in the quantized Hilbert space $Q(M)$ equals the Riemann-Roch number of the reduced space $M_O = (M \times O^-) /\!\!/ G$, where O^- is the symplectic manifold O equipped with its negative Kirillov-Kostant-Souriau form. This principle is now referred to as *quantization commutes with reduction*, and was proven independently by Meinrenken and by Tian and Zhang in the 1990s. For a survey and some reformulations of this principle, see for example [Woo10; Rod20; Ma21].

In the toric case, where $M = \mathbb{C}^d$ and $G = \mathbb{T}^n$ is abelian, coadjoint orbits reduce to points $\lambda^0 \in (\mathbb{R}^n)^*$, and due to Delzant's construction the reduced space $M_O = M_\Delta$ corresponds to a given Delzant polytope Δ. The multiplicity of ρ_{λ^0} in $Q(\mathbb{C}^d)$ equals the number of lattice points in Δ, i.e., $\#(\Delta \cap \mathbb{Z}^n)$. On the other hand, the Riemann-Roch number is computed by the application of a Todd operator to a perturbed volume polynomial:

$$\mathrm{Todd}_h(\mathrm{vol}(\Delta_h))|_{h=0}\,, \quad \text{where} \quad \Delta_h = \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq \lambda_i + h_i\}.$$

Pukhlikov and Khovanskii [PK93] verified the identity

$$\#(\Delta \cap \mathbb{Z}^n) = \mathrm{Todd}_h(\mathrm{vol}(\Delta_h))|_{h=0}\,,$$

confirming the conjecture in the symplectic toric case.

We will not delve into the details of prequantization or the full proof from [PK93], as these are covered in other sources. Instead, this paper presents the construction of the symplectic toric manifold M_Δ from a given Delzant polytope Δ, following [ACL12]. We begin with preliminaries on symplectic structures, outline the convexity and reduction theorems, and then present Delzant's construction. We conclude by discussing examples, applications, and potential generalizations.

I would like to express my sincere gratitude to Professor Xiang Tang for his invaluable guidance throughout this project, Professor Laura Escobar for introducing me to the fascinating world of polytopes, and Professor Renato Feres for his inspiring teaching of Riemannian geometry.

2 Preliminaries

2.1 Geometric Structures

Given a vector bundle $E \to M$ over smooth manifold M, a geometric structure attached to M amounts to the assignment of some smooth section $T \in \Gamma\left(E^{\otimes_{\mathbb{F}}^k} \otimes_{\mathbb{F}} (E^*)^{\otimes_{\mathbb{F}}^j}\right)$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Three geometric structures will be needed in this thesis: symplectic structure, Riemannian structure, and almost complex structure.

Symplectic Manifolds

Let V be a finite-dimensional real vector space and $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by $\widetilde{\Omega}(v)(u) = \Omega(v, u)$. If its kernel $\ker(\widetilde{\Omega}) = \{v \in V \mid \forall u \in V, \widetilde{\Omega}(v)(u) = 0\}$ is the zero subspace of V, or equivalently $\widetilde{\Omega}$ is bijective, then we say Ω is **nondegenerate**. The pair (V, Ω) is called a **symplectic vector space**.

Let $\omega \in \Omega^2(M) = \Gamma\left(\bigwedge^2 T^*M\right)$ be an alternating 2-form on a manifold M. ω is **closed** if it satisfies $d\omega = 0$, where d is the exterior derivative. ω is **symplectic** if ω is closed and ω_p is nondegenerate for all $p \in M$. A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form. A **symplectomorphism** $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a diffeomorphism between two symplectic manifolds such that $\varphi^*\omega_2 = \omega_1$.

Example 2.1. Let $M = \mathbb{R}^{2n}$ with coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$. It is a symplectic manifold with form $\omega_0 = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$, called the **standard symplectic form**. Under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$, we see this symplectic manifold is the same as $(\mathbb{C}^n, \omega_0)$ where $\omega_0 = \frac{i}{2}\sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$. ♣

Riemannian Manifolds

A **Riemannian metric** on a smooth manifold M is a smooth section $g \in \Gamma(\Sigma^2 T^*M)$, meaning that at each point $x \in M$, $g_x : T_xM \times T_xM \to \mathbb{R}$ is a symmetric, positive-definite bilinear form. The pair (M, g) is then called a **Riemannian manifold**. A Riemannian metric always exists for a smooth manifold.

A fundamental object associated with a Riemannian manifold is the **Levi-Civita connection** ∇, which is the unique affine connection on M satisfying **metric property** (for all $X, Y, Z \in \mathfrak{X}(M)$, $\nabla_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$) and **symmetric property** (for all $X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$). A key consequence of the Levi-Civita connection is the existence of **geodesics**, which are curves $\gamma : I \to M$ satisfying: $\nabla_{\dot\gamma}\dot\gamma = 0$. These curves represent locally distance-minimizing paths and generalize the notion of "straight lines" to curved spaces. The **exponential map** $\exp_p : T_pM \to M$ is then defined by mapping a tangent vector $v \in T_pM$ to the point reached by traveling along the unique geodesic with initial velocity v for unit time: $\exp_p(v) = \gamma_v(1)$, where γ_v is the unique (maximal) geodesic satisfying $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. The exponential map provides local normal coordinates and is a fundamental tool in Riemannian geometry.

Almost Complex Structures

An **almost complex structure** J on a manifold M is a smooth field of complex structures on the tangent spaces:

$$x \longmapsto J_x : T_xM \to T_xM \quad \text{linear,} \quad \text{and} \quad J_x^2 = -\mathrm{Id}.$$

The pair (M, J) is then called an **almost complex manifold.** Let (M, ω) be a symplectic manifold. An almost complex J on M is called **compatible** (with ω or ω-compatible) if $g_x(u, v) := \omega_x(u, J_x v)$ defines a Riemannian metric on M. The triple (ω, g, J) is called a **compatible triple** when $g(\,\cdot\,,\,\cdot\,) = \omega(\,\cdot\,, J\cdot)$. Given ω and g on M, there exists a canonical almost complex structure J on M which is compatible to them. Since Riemannian metrics always exist, we see that any symplectic manifold has a compatible almost complex structure.

2.2 Morse Theory

We recall a concept from general Riemannian manifold theory and then examine that for our symplectic manifold.

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $f \in C^\infty(M)$ be a smooth function. Then $\nabla f \in \Gamma(T^{(0,1)}TM) = \Omega^1(M)$ is just the 1-form $\mathrm{d}f$:

$$\nabla f(X) = \nabla_X f = Xf = \mathrm{d}f(X). \tag{1}$$

The 2-tensor $\nabla^2 f = \nabla(\mathrm{d}f)$ is called the **(covariant) Hessian of** f. By adopting a notational stipulation that $\nabla^2_{X,Y}F(\cdots) := \nabla^2 F(\cdots, X, Y)$ (this is different from [Lee18, pp.99]), we see

$$\nabla^2 f(X,Y) = \nabla^2_{X,Y} f = \nabla_X(\nabla_Y f) - \nabla_{(\nabla_X Y)} f = X(Yf) - (\nabla_X Y)f = X(Yf) - \mathrm{d}f(\nabla_X Y).$$

Proposition 2.2. *Show that* $\nabla^2 f(X,Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle = \langle \nabla_X \operatorname{grad} f, Y\rangle$.

Proof.

$$\begin{aligned}
\nabla^2 f(X,Y) &= \nabla_X \langle \operatorname{grad} f, Y\rangle - \langle \operatorname{grad} f, \nabla_X Y\rangle \\
&= \langle \nabla_X \operatorname{grad} f, Y\rangle + \langle \operatorname{grad} f, \nabla_X Y\rangle - \langle \operatorname{grad} f, \nabla_X Y\rangle \quad \text{by metric property} \\
&= \langle \nabla_X \operatorname{grad} f, Y\rangle.
\end{aligned}$$

■

Now let M be a compact manifold and $f \in C^\infty(M)$. The collection of all critical points is called the **critical set** $\mathrm{Crit}(f) = \{p \in M \mid \mathrm{d}f(p) = 0\}$. Let $x \in \mathrm{Crit}(f)$ and consider the Hessian map $\mathrm{Hess}_x(f) : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathrm{Hess}_x(f)(X_x, Y_x) = X(Yf)(p) = X_x(Yf). \tag{2}$$

where X and Y are any vector fields whose value at x are X_x and Y_x.

Remark 2.3.

(a) Recall that if $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, then Xf is a smooth function $M \to \mathbb{R}$ defined as $p \mapsto X_x f$. X_x is a tangent vector, or a derivation. $\mathrm{d}f(X)$ is also a smooth function defined by $p \mapsto \mathrm{d}f_x(X_x)$. These two functions are the same $Xf = \mathrm{d}f(X)$.

(b) The gradient $\operatorname{grad} f$ vanishes exactly at critical points: $(\operatorname{grad} f)_x = \hat{g}^{-1}(\mathrm{d}f_x) = 0 \overset{\hat{g}\text{ iso}}{\iff} \mathrm{d}f_x = 0 \iff p \in \mathrm{Crit}(f)$.

(c) For $p \in \mathrm{Crit}(f)$, $\mathrm{Hess}_x(f)$ is well-defined, symmetric, and bilinear: $\mathrm{Hess}_x(f)$ is symmetric as $X(Yf)(p) - Y(Xf)(p) = ([X,Y]f)(p) = \mathrm{d}f_x[X,Y]_x = 0 \implies X(Yf)(p) = Y(Xf)(p)$. Since $X(Yf)(p) = X_x(Yf)$, the RHS of eq.(2) is independent of the choice of X. Since $Y(Xf)(p) = Y_x(Xf)$, the RHS of eq.(2) is also independent of the choice of Y. Bilinearity is trivial.

(d) Note that if ∇ is a connection on the manifold M, then for $x \in \mathrm{Crit}(f)$,

$$(\nabla^2 f)_x(X_x, Y_x) = X_x(Yf) - \underbrace{\mathrm{d}f_x((\nabla_X Y)_x)}_{=0 \text{ as } x \in \mathrm{Cirt}(f)} = X_x(Yf).$$

♠

Definition 2.4. *A critical point $x \in \mathrm{Crit}(f)$ is called* ***nondegenerate*** *if the bilinear form $\mathrm{Hess}_x(f)$ is nondegenerate. A function $f \in C^\infty(M)$ is called a* ***Morse function*** *if $\mathrm{Crit}(f)$ is discrete and each $x \in \mathrm{Crit}(f)$ is nondegenerate. Since any infinite set of points in a compact space must have an accumulation point, a Morse function on a compact manifold M must only have finitely many critical points.*

A function $f \in C^\infty(M)$ is called a ***Morse-Bott function*** *if $\mathrm{Crit}(f)$ decomposes into finitely many connected submanifolds of M, which we shall call the* ***critical manifolds****, and for each $x \in \mathrm{Crit}(f)$,*

$$T_x\mathrm{Crit}(f) = \ker \widehat{\mathrm{Hess}_x(f)} = \{v \in T_xM \mid \mathrm{Hess}_x(f)(v,w) = 0, \forall w \in T_xM\}.$$

In this definition, we identified the bilinear form $\mathrm{Hess}_x(f) : T_xM \times T_xM \to \mathbb{R}$ with an operator $\widehat{\mathrm{Hess}_x(f)} : T_xM \to T_xM$ using a Riemannian metric $\langle \cdot\,, \cdot \rangle$ on M. That is,

$$\mathrm{Hess}_x(f)(v,w) = \left\langle \widehat{\mathrm{Hess}_x(f)}(v), w \right\rangle_x$$

Then

$$\begin{aligned}\{v \in T_xM \mid \mathrm{Hess}_x(f)(v,w) = 0, \forall w \in T_xM\} &= \left\{ v \,\middle|\, \left\langle \widehat{\mathrm{Hess}_x(f)}(v), w \right\rangle_x = 0, \forall w \right\} \\ &= \{v | \widehat{\mathrm{Hess}_x(f)}(v) = 0\} \\ &= \ker \widehat{\mathrm{Hess}_x(f)}(v).\end{aligned}$$

It is also easy to see this operator is self-adjoint:

$$\left\langle \widehat{\mathrm{Hess}_x(f)}(v), w \right\rangle_x = \left\langle v, \widehat{\mathrm{Hess}_x(f)}(w) \right\rangle_x \quad \forall v, w \in T_xM,$$

so the matrix of $\mathrm{Hess}_x(f)$ is symmetric and its eigenvectors comtribute to a basis of T_xM.

Notice that the definition of a Morse function is a special case of a Morse-Bott function where the critical manifolds are all zero dimensional, and hence for any $x \in \mathrm{Crit}(f)$ we have $\ker \widehat{\mathrm{Hess}_x(f)} = 0$, and therefore the Hessian is nondegenerate.

It is useful to consider the following definition from dynamics to understand the Morse-Bott function.

Definition 2.5. *Let M be a compact Riemannian manifold, let $f : M \to M$ be a diffeomorphism, and let L be an f-invariant subset of M. We say that L is a* ***normally hyperbolic invariant manifold*** *if for any point $x \in L$ the tangent space T_xM splits as a direct sum of three subbundles:*

$$T_xM = T_xL \oplus E_x^+ \oplus E_x^-$$

where, with respect to some Riemannian metric on M:

(1) the restriction of $\mathrm{d}f$ to E^+, called the ***stable bundle****, is a contraction;*

(2) the restriction of $\mathrm{d}f$ to E^-, called the ***unstable bundle****, is an expansion;*

(3) the restriction of $\mathrm{d}f$ to TL is ***relatively neutral****.*

In other words, there must exist constants $0 < \kappa < \delta^{-1} < 1$ and $0 < c$ such that:

(1) $\mathrm{d}f_x E_x^+ = E_{f(x)}^+$ *and* $\mathrm{d}f_x E_x^- = E_{f(x)}^-$ *for all* $x \in L$

(2) $\|\mathrm{d}f^n v\| \leq c\kappa^n \|v\|$ *for all* $v \in E^+$ *and* $n > 0$

(3) $\|\mathrm{d}f^{-n} v\| \leq c\kappa^n \|v\|$ *for all* $v \in E^-$ *and* $n > 0$

(4) $\|\mathrm{d}f^{-n} v\| \leq c\delta^n \|v\|$ *for all* $v \in TL$ *and* $n > 0$.

If f is a Morse-Bott function then its critical manifolds are all normally hyperbolic invariant manifolds with respect to the negative gradient flow. The negative gradient flow is the family of diffeomorphisms $\phi_t : M \to M$ defined by $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} \phi^{(x)}(t) = (-\operatorname{grad} f)(\phi_{t_0}(x))$ and $\phi_0 = \mathrm{id}$ for $t_0 \in \mathbb{R}$. Remark 2.3 (b) implies that for $x \in C$, $(-\operatorname{grad} f)(x) = 0$. Thus, at time zero, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \phi^{(x)}(t) = (-\operatorname{grad} f)(\phi_0(x)) = (-\operatorname{grad} f)(\mathrm{id}(x)) = (-\operatorname{grad} f)(x) = 0$. The trajectory $\phi_t(x)$ never moves, i.e., $\phi_t(x) = x$, and $\mathrm{Crit}(f)$ is first of all a ϕ_t-invariant subset of M.

Then for any critical manifold C, and for any point $x \in C$, the tangent space $T_x M$ decomposes as a direct sum:

$$T_x M = T_x C \oplus E_x^+ \oplus E_x^-$$

where $T_x C = \ker \widehat{\mathrm{Hess}_x(f)}$ =zero eigenspace (eigenspace with zero eigenvalue); E_x^+ is spanned by the positive eigenspaces of $\widehat{\mathrm{Hess}_x(f)}$; and E_x^- is spanned by the negative eigenspaces of $\widehat{\mathrm{Hess}_x(f)}$. Note that we can use eigenvectors of $\widehat{\mathrm{Hess}_x(f)}$ as a basis for $T_x M$ because $\widehat{\mathrm{Hess}_x(f)}$ is self-adjoint.

Lemma 2.6 (Morse-Bott Lemma, [BH04])**.** *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact n-dim manifold M, C a connected component of Crit (f) of dimension d, and $x \in C$. Then there exists an open neighborhood U of p and a smooth chart $\varphi : U \to \mathbb{R}^d \times \mathbb{R}^{n-d}$, such that:*

(a) $\varphi(x) = 0$;

(b) $\varphi(U \cap C) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{n-d} \mid y = 0\}$; *and*

(c) $(f \circ \varphi^{-1})(x, y) = f(C) - y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_{n-d}^2$ *where $k \leq n - d$ is the index of* $\mathrm{Hess}_x(f)$ *(the dimension of the subspace of $T_x M$ on which the form is negative definite) and $f(C)$ is the common value of f on C.*

Thus, the Hessian is locally a quadratic form with eigenvalues $0, \pm 2$. †

Let ϕ_t be the negative gradient flow on compact manifold M. Let $x \in C \subseteq \mathrm{Crit}(f)$. We introduce the following definition.

Definition 2.7. *The set of points $x \in M$ whose trajectories $\phi_t(x)$ converge to some point in C as $t \to \infty$ form a manifold called the **stable manifold**, denoted $W^s(C)$. Additionally, for any point $x \in C, T_x W^s(C) = T_x C \oplus E_x^+$. Similarly, the set of points $x \in M$ whose trajectories $\phi_t(x)$ converge to some point in C as $t \to -\infty$ form a manifold called the **unstable manifold**, denoted $W^u(C)$. Additionally, for any point $x \in C, T_x W^u(C) = T_x C \oplus E_x^-$.*

*The **index** of a connected critical submanifold C is defined by*

$$n^-(C) = \dim W^u(C) - \dim C = \operatorname{codim} W^s(C)$$

*and agrees with the dimension of the negative eigenspace of the Hessian of f on the normal bundle of C. Similarly, the **coindex** of C is defined by*

$$n^+(C) = \dim W^s(C) - \dim C = \operatorname{codim} W^u(C)$$

†In their local form, the ovariant Hessian $\nabla^2 f$ is given by $\mathrm{Hess}_x(f) = \nabla_i \partial_j f \mathrm{d}x^i \otimes \mathrm{d}x^j = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right) \mathrm{d}x^i \otimes \mathrm{d}x^j$; the symmetric bilinear form $\mathrm{Hess}_x(f)$ is given by $\frac{\partial^2 f}{\partial x^i \partial x^j} \mathrm{d}x^i \otimes \mathrm{d}x^j$. The matrix $(\partial^i \partial^j f)_{ij}$ is of the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2I_k & 0 \\ 0 & 0 & 2I_{n-k-d} \end{bmatrix}$.

and agrees with the dimension of the positive eigenspace of the Hessian on the normal bundle of C.

A simple calculation shows that [†]

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} f \circ \phi^{(x)}(t) = -\| \operatorname{grad} f(\phi_{t_0}(x))\|^2 \leq 0.$$

By differentiating the gradient equation $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = -\operatorname{grad} f(\phi_t(x))$, one can see $\| \operatorname{grad} f(\phi_t(x))\|^2 \to 0$ as $t \to \infty$, which gives a limit x_∞ for a given point critical point x. It is nontrivial to show that this limit point lies in the critical set. [†] Therefore,

$$M = \bigcup_{C \subseteq \mathrm{Cirt}(f)} W^s(C).$$

Similarly, M is the union of all the unstable manifolds:

$$M = \bigcup_{C \subseteq \mathrm{Cirt}(f)} W^u(C).$$

The reasons for developing these Morse-Bott function theory is to show the following lemma, which will be directly responsible for showing connectedness of preimage of moment map of Hamiltonian $\mathbb{T}^m$-actions over compact symplectic manifold, i.e., the connectedness part of Atiyah-Guillemin-Sternberg theorem.

Lemma 2.8. *Suppose M is a compact connected manifold and $f : M \to \mathbb{R}$ is a Morse-Bott function such that for any of the critical manifolds C of f we have $n^{\pm}(C) \neq 1$. Then for every $c \in \mathbb{R}$ the level set $f^{-1}(c)$ is connected.*

Proof. See [MS17, Lemma 5.51]. ∎

2.3 Lie Groups

We recall some notions from Lie groups that will be used later. The **Lie algebra** $\mathfrak{g}$ of a Lie group G is the tangent space at the identity element $e \in G$, equipped with a Lie bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying bilinearity, antisymmetry, and the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

The **exponential map** $\exp : \mathfrak{g} \to G$ provides a local diffeomorphism near zero, mapping a Lie algebra element X to the corresponding one-parameter subgroup $\exp tX$ in G.

Denote $\mathfrak{g}^*$ as the dual space of the Lie algebra. Let $a \in G$ be an arbitrary element. The differential of the conjugation map $g \mapsto g \cdot a \cdot g^{-1}$ at e is denoted as Ad_g, which is an element of the general linear group $\mathrm{GL}(\mathfrak{g})$. The map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ is called the **adjoint representation** of G on its Lie algebra $\mathfrak{g}$. The **coadjoint representation** is then defined on the dual space $\mathfrak{g}^*$ by:

$$\langle \mathrm{Ad}_g^* \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle, \quad \forall \xi \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

where $\langle \cdot , \cdot \rangle$ is the natural pairing.

Compact Lie groups will be of primary interest in this paper. In particular, any compact connected abelian Lie group is a torus $\mathbb{T}^n$, whose action on compact symplectic manifolds will be analyzed in the Atiyah-Guillemin-Sternberg theorem in the next section.

[†]Letting $\gamma(t) = f \circ \phi^{(x)}(t)$, see that LHS= $\mathrm{d}\gamma_{t_0}(d/dt|_{t=t_0}) = \mathrm{d}f_{\phi_{t_0}(x)} \circ \mathrm{d}\phi^{(x)}_{t_0}(d/dt|_{t=t_0}) = \mathrm{d}f_{\phi_{t_0}(x)}(d/dt|_{t=t_0}\phi^{(x)}(t))$ =RHS, where we plugged in $X = \operatorname{grad} f$ into $\mathrm{d}f(X) = \langle \operatorname{grad} f, X \rangle$ for the last step.

[†]See Otis Chodosh's reply on Stackexchange for a proof.

Orbits

We recall some more properties of orbit map $\theta^{(p)} : G \to M;\ g \mapsto g \cdot p$ for an action G on M:

Proposition 2.9.

(1) *[Lee12, Proposition 7.26, Problem 21-17]: for a smooth left action θ of a Lie group G on smooth manifold M and a point $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank, so the isotropy group $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ is a properly embedded Lie subgroup of G. Each orbit $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is an immersed submanifold of M, which is embedded if the action is proper. Note that if the action is transitive, then $\theta^{(p)}$ is a surjective and thus a smooth submersion by the global rank theorem ([Lee12, Theorem 4.14]), which by submersion Level set theorem ([Lee12, Corollary 5.13]) implies that $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ also has dimension$=\dim M - \dim G$. If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion.*

(2) *[Lee12, Corollary 21.6] says that continuous action by compact Lie group on manifold is proper, and [Lee12, Proposition 21.7] says that the orbit map is then a proper map and $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is closed in M. Since (1) claims that if we have $G_p = \{e\}$ then $\theta^{(p)}$ is an injective smooth immersion, [Lee12, Proposition 4.22] shows that $\theta^{(p)}$ is an embedding and thus its image $G \cdot p = \operatorname{Im}\left(\theta^{(p)}\right)$ is a properly embedded submanifold (where we also used [Lee12, Theorem 5.5].)*

Remark 2.10. Suppose θ is a smooth left action of Lie group G on smooth manifold M and $p \in M$. Then by (1) above (specifically [Lee12, Problem 21-17]), $\mathcal{O}_p = G \cdot p$ is a smooth manifold. Consider a new orbit map $\varphi : G \times \mathcal{O}_p \to \mathcal{O}_p$ defined by restricting the action θ. Then by the smoothness and contant rank of $\varphi^{(p)}$ given by (1) above (specifically [Lee12, Proposition 7.26]), we see this surjective map φ is a smooth submersion due to [Lee12, Theorem 4.14]. Thus, $T_p\mathcal{O}_p = T_{\varphi^{(p)}(e)}\mathcal{O}_p = \operatorname{Im}\left(\mathrm{d}\varphi_e^{(p)}\right)$.

Another way to show this claim is by using [Lee12, Theorem 21.18] applied to G-homogeneous space $\mathcal{O}_p$ and considering the commutative diagram below:

$$\begin{tikzcd} & G \ni g \arrow[dl, "\pi"'] \arrow[d, "\varphi^{(p)}"] \\ G/G_p \arrow[r, "F"] & G \cdot p \end{tikzcd} \qquad g \cdot G_p \longmapsto g \cdot p$$

Then $\mathrm{d}\varphi_e^{(p)} = \mathrm{d}(F \circ \pi)_e = \underbrace{dF_{\pi(e)}}_{\text{diffeo}} \circ \underbrace{\mathrm{d}\pi_e}_{C^\infty \text{ subm-}}$ is a smooth submersion. Thus, $T_p\mathcal{O}_p = \operatorname{Im}\left(\mathrm{d}\varphi_e^{(p)}\right)$. ♠

The following results will also be useful.

Lemma 2.11. *If an action $\theta : G \times M \to M$ is free then $g_1 \cdot p = g_2 \cdot p \implies g_1 = g_2$. In this case, the orbit map $\theta^{(p)} : G \to G \cdot p;\ g \mapsto g \cdot p$ is bijective.*

Theorem 2.12. *[Quotient Manifold Theorem, [Lee12, Theorem 21.10]] Suppose G is a Lie group acting smoothly, freely, and properly on smooth manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Proposition 2.13. *[[Lee12, Proposition 5.38]] Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, i.e., $S \cap U$ is a regular level set of Φ, then $T_pS = \ker \mathrm{d}\Phi_p : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap N$.*

We note a convenient alternative characterization of the vector field $X^{\#}(p)$, which is called the **infinitesimal generator of group action** $\Psi : G \times M \to M$ as at [Lee12, pp.529]. Consider the orbit map $\Psi^{(p)} : G \to M;\ g \mapsto g \cdot p$. Then the orbit $\mathcal{O}_p$ through p is the image of $\Psi^{(p)}$. Since $\gamma(t) = \exp tX$ is a smooth curve in G

whose velocity is $\gamma'(0) = X_e$, it follows from [Lee12] corollary 3.25 that for each $p \in M$ we have

$$\mathrm{d}\Psi_e^{(p)}(X_e) = (\Psi^{(p)} \cdot \gamma)'(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Psi_{\exp tX}(p) = X^{\#}(p) \tag{3}$$

Let $\Psi : G \times M \to M$ be a smooth action of a Lie group over smooth manifold M.

- The orbit
$$\mathcal{O}_p = G \cdot p = \{g \cdot p \mid g \in G\}$$
is an embedded submanifold with tangent space at p equals
$$T_p\mathcal{O}_p = \left\{X^{\#}(p) \mid X \in \mathfrak{g}\right\} \tag{4}$$
due to Remark 2.10 and eq.(3).

- The stabilizer subgroup of each $p \in M$,
$$G_p = \{g \in G \mid g \cdot p = p\}$$
is a Lie subgroup $H = \left(\Psi^{(p)}\right)^{-1}(p)$ of G which by [Lee12] Theorem 8..46 has Lie algebra $\mathfrak{h}$ equal to
$$\begin{aligned} \mathfrak{g}_p :=& \mathfrak{h} = \{X \in \mathfrak{g} : X_e \in T_e H\} \\ &= \left\{X \in \mathfrak{g} : X_e \in \ker \mathrm{d}\Psi_e^{(p)}\right\} \\ &= \left\{X \in \mathfrak{g} \mid X^{\#}(p) = 0\right\} \subseteq \mathfrak{g} \end{aligned} \tag{5}$$
where the next-to-last step is due to property 2.9 (1) and proposition 2.13; and the last step is due to eq.(3).

Principal G-bundle

Definition 2.14. *A smooth fiber bundle $\pi : P \to M$ with fiber Lie group G is a smooth **principal G-bundle** if G acts smoothly and freely on P and the fiber-preserving local trivializations*

$$\Phi_U : \pi^{-1}(U) \to U \times G$$

*are G-**equivariant**:*

$$\Phi_U(g \cdot x) = g \cdot \Phi_U(x), \quad \forall x \in \pi^{-1}(U)$$

where on the RHS, G acts on $U \times G$ by

$$g \cdot (x, h) = (x, gh)$$

Example 2.15.

1. A **trivial fiber bundle** is one that admits a local trivialization over the entire base space (a global trivialization). It is said to be **smoothly trivial** if it is a smooth bundle and the global trivialization is a diffeomorphism.

2. Every product space $M \times F$ is a fiber bundle with projection $\pi_1 : M \times F \to M$, called a **product fiber bundle**. It has a global trivialization given by the identity map $M \times F \to M \times F$, so every product bundle is trivial. If $F = G$ a Lie group, then it is a **product G-bundle**, a principal G-bundle.

3. Every rank-k vector bundle is a fiber bundle with model fiber $\mathbb{R}^k$.

4. If G is a Lie group and H is a closed subgroup, then the quotient G/H can be given the structure of a manifold such that the projection map $\pi : G \to G/H$ is a **principal H-bundle**.

5. The group $\mathbb{S}^1$ of unit complex numbers acts on the complex vector space $\mathbb{C}^{n+1}$ by multiplication. This action induces an action of $\mathbb{S}^1$ on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$. The complex projective space $\mathbb{C}P^n$ can be defined as the orbit space of $\mathbb{S}^{2n+1}$ by $\mathbb{S}^1$. The natural projection $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$ with fiber $\mathbb{S}^1$ turn out to be a principal $\mathbb{S}^1$-bundle. When $n = 1, \mathbb{S}^3 \to \mathbb{C}P^1$ with fiber $\mathbb{S}^1$ is called the **Hopf bundle**.

♣

Theorem 2.16. *If a Lie group G acts smoothly, properly, and freely on a smooth manifold M, then M/G is a manifold and the map $\pi : M \to M/G$ is a principal G-bundle.*

Proof. Use quotient manifold theorem. ∎

Haar Measure

Recall that the **Haar measure** is the unique positively oriented left-invariant n-form $\mu(g)$ on compact connected Lie group G such that $\int_G \mu(g) = 1$ if we choose a left-invariant orientation (see [Lee12] Proposition 15.19 and 16.10).

Consider the symplectic action of compact connected Lie group G on (M, ω). Let m' be an arbitrary Riemannian metric on M. For each $p \in M$ and $X_p, Y_p \in T_pM$, define the new metric m by averaging over G:

$$m_p\left(X_p, Y_p\right) := \int_{g\in G} \left(\psi_g^* m'\right)_p \left(X_p, Y_p\right) \mathrm{d}\mu(g) \tag{6}$$

The property of this metric that we shall use is its ψ_h-invariance, or simply G-invariance:

$$\forall h \in G, \quad \left(\psi_h^* m\right)_p \left(X_p, Y_p\right) = m_p\left(X_p, Y_p\right).$$

Indeed,

$$\begin{aligned}
\left(\psi_h^* m\right)_p \left(X_p, Y_p\right) &= m_{\psi_h(p)}\left(\mathrm{d}(\psi_h)_p\left(X_p\right), \mathrm{d}(\psi_h)_p\left(Y_p\right)\right) \\
&= \int_{g\in G} m'_{\psi_g(\psi_h(p))}\left(\mathrm{d}(\psi_g)_{\psi_h(p)}\mathrm{d}(\psi_h)_p\left(X_p\right), \mathrm{d}(\psi_g)_{\psi_h(p)}\mathrm{d}(\psi_h)_p\left(Y_p\right)\right) \mathrm{d}\mu(g) \\
&= \int_{g\in G} m'_{\psi_{gh}(p)}\left(\mathrm{d}(\psi_{gh})_p(X_p), \mathrm{d}(\psi_{gh})_p(Y_p)\right) \mathrm{d}\mu(g) \\
&= \int_{k\in G} \left(\psi_k^* m'\right)_p \left(X_p, Y_p\right) \mathrm{d}\mu(k) \text{ by letting } k = gh \\
&= m_p(X_p, Y_p).
\end{aligned}$$

Proposition 2.17. *Show that there exists a compatible almost complex structure J on (M, ω) which is invariant under the G-action, that is, $\psi_g^* J = J\psi_g^*$, for all $g \in G$.*

Proof. Let m' be any Riemannian metric on M and define m as in eq.(6). We have shown the ψ_h-invariance of this metric. There is a canonical compatible structure (ω, m, J). Now, using ψ_g as symplectomorphism and compatibility, we have

$$\begin{array}{l}
m_p(X, Y) = \omega_p\left(X, J_pY\right) = \psi_g^*\omega_p\left(X, J_pY\right) = \omega_{\psi_g(p)}\left(\mathrm{d}(\psi_g)_pX, \mathrm{d}(\psi_g)_pJ_pY\right) \\
\quad \| \\
(\psi_g^* m)_p(X, Y) = m_{\psi_g(p)}\left(\mathrm{d}(\psi_g)_pX, \mathrm{d}(\psi_g)_pY\right) = \omega_{\psi_g(p)}\left(\mathrm{d}(\psi_g)_pX, J_{\psi_g(p)}\mathrm{d}(\psi_g)_pY\right)
\end{array}$$

for any vectors X, Y in any tangent space T_pM. By the nondegeneracy of ω, we must have $\mathrm{d}(\psi_g)_pJ_pY = J_{\psi_g(p)}\mathrm{d}(\psi_g)_pY$. That is, $\psi_g^* J = J\psi_g^*$ for all $g \in G$. ∎

2.4 Hamiltonian G-spaces

Let (M, ω) be a symplectic manifold and let G be a Lie group acting on it by symplectomorphisms $\Psi : G \to \mathrm{Sympl}(M, \omega)$, which is a group homomorphism such that the evaluation map $\mathrm{ev}_\Psi(g, p) := \Psi_g(p)$ is smooth.

Definition 2.18. *The action Ψ is a **Hamiltonian action** if there exists a map, called **moment map**, $\mu : M \longrightarrow \mathfrak{g}^*$ satisfying (i) Hamiltonian condition, i.e., $\forall X \in \mathfrak{g}$, $\mathrm{d}\mu^X = \iota_{X^\#}\omega$, where μ^X is a smooth function $\mu^X(p) := \langle \mu(p), X\rangle$; (ii) Equivariance condition: $\forall g \in G$, $\mu \circ \Psi_g = \mathrm{Ad}_g^* \circ \mu$. The tuple (M, ω, G, μ) is then called a **Hamiltonian G-space**.*

Remark 2.19. We note that now $X^\#$ does not arise from any $X \in \mathfrak{g}$ like what the general case does. $\mathbb{R}$ as an additive Lie group has exponential map $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$. $\mathbb{S}^1$ has exponential map $\mathbb{R} \ni \theta \mapsto e^{i\theta} \in \mathbb{S}^1$. The only difference between $\mathbb{S}^1$ and $\mathbb{R}$ actions is that the former is 2π-periodic.

When $G = \mathbb{R}$, we have $\mathfrak{g} = \mathfrak{g}^* = \mathbb{R}$ For the generator $X = 1$ of $\mathfrak{g}$, we have $\mu^X(p) = \mu(p) \cdot 1$, i.e., $\mu^X = \mu$. The vector field $X^\#(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Psi_{\exp t\cdot 1}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Psi_t(p)$ is the standard vector field on M generated by $\mathbb{S}^1$ (or $\mathbb{R}$). Then $\mathrm{d}\mu = \iota_{X^\#}\omega$. ♠

Remark 2.20. For abelian group actions, the conjugations become identity maps. So do their differential Ad_g at identity and thus the coadjoint Ad_g^*. So the equivariance condition becomes invariance $\mu \circ \Psi_g = \mu$. We shall see this in the next remark. ♠

Proposition 2.21. *For connected Lie groups, there is also an equivalent characterization of Hamiltonian actions via **comoment map** $\mu^*(X) := \mu^X$ with the two conditions rephrased as: (i) $\mu^*(X) = \mu^X$ is a Hamiltonian function for the vector field $X^\#$; (ii) μ^* is a Lie algebra homomorphism: $\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)\}$ where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$.*

Proof. See Appendix. ■

Let's look at a classical exmaple.

Example 2.22 (Circle Action on $\mathbb{S}^2$)**.** Consider the sphere $\mathbb{S}^2$. A point p on it can be written in spherical coordinates as

$$(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

so it has "height" $\cos\phi$. We thus define the height function as $H(\theta, h) = h$ on the sphere with symplectic form $\omega = \mathrm{d}\theta \wedge dh$, the standard form for chart $(U, (\theta, h))$.

Consider the circle action on the sphere by horizontal rotation by the circle:

$$\begin{aligned} \Psi : \quad \mathbb{S}^1 &\longrightarrow \mathrm{Sympl}\left(\mathbb{S}^2, \omega\right) \\ e^{i\theta} &\longmapsto \text{ rotation by angle } \theta \text{ around } z\text{-axis} \end{aligned}$$

The flow lines of the action are indicated by the yellow lines with arrows in Figure 1.

Thus, $X^\# = \frac{\partial}{\partial\theta}$. Note that a point p on the sphere has the same vertical height after the horizontal rotation, i.e., $H(p) = H(\Psi_g(p))$. In view of Remark 2.20, this hints us to guess the moment map for this circle action is the height function. Indeed, the Hamiltonian condition is also true:

$$\begin{aligned} \iota_{X^\#}\omega(v) &= \omega(X^\#, v) \\ &= \mathrm{d}\theta \wedge \mathrm{d}h\left(\frac{\partial}{\partial\theta}, v\right) \\ &= \mathrm{d}\theta\left(\frac{\partial}{\partial\theta}\right)\mathrm{d}h(v) - \mathrm{d}\theta(v)\mathrm{d}h\left(\frac{\partial}{\partial\theta}\right) \\ &= \mathrm{d}h(v) = \mathrm{d}H(v). \end{aligned}$$

♣

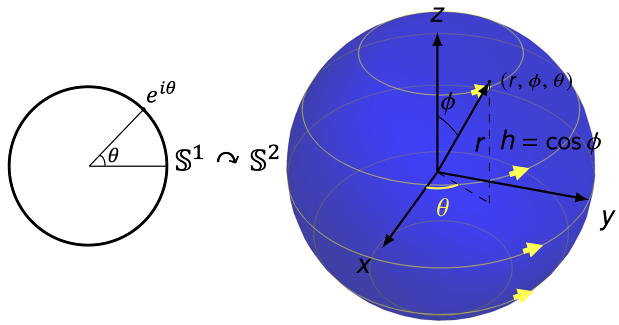


Figure 1: Circle action on sphere.

Here is another circle action, this time on the prototypical symplectic manifold $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Example 2.23 (Circle Action on $\mathbb{C}^n$)**.** In Example 2.1, we see $\mathbb{C}^n$ can be equipped with a symplectic form $\omega = \frac{i}{2}\sum \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j$. Under the transformations

$$\begin{cases} z^j = x^j + iy^j \\ \bar{z}^j = x^j - iy^j \end{cases} \quad \text{and} \quad \begin{cases} x^j = r^j \cos\theta^j \\ y^j = r^j \sin\theta^j \end{cases}$$

we can also write it as $\sum \mathrm{d}x^j \wedge \mathrm{d}y^j$ and $\sum r^j \mathrm{d}r^j \wedge \mathrm{d}\theta^j$. Also recall that

$$\begin{cases} \frac{\partial}{\partial x^j} = \cos\theta^j \frac{\partial}{\partial r^j} - \frac{\sin\theta^j}{r^j}\frac{\partial}{\partial\theta^j} \\ \frac{\partial}{\partial y^j} = \sin\theta^j \frac{\partial}{\partial r^j} + \frac{\cos\theta^j}{r^j}\frac{\partial}{\partial\theta^j} \end{cases}$$

Consider the following $\mathbb{S}^1$-action on $(\mathbb{C}^n, \omega)$:

$$t \in \mathbb{S}^1 \longmapsto \Psi_t = \text{ multiplication by } t$$

We claim that Ψ is Hamiltonian with moment map

$$\begin{aligned} \mu : \mathbb{C}^n &\longrightarrow \mathbb{R} \\ z &\longmapsto -\frac{\|z\|^2}{2} + \text{ constant} \end{aligned}$$

In view of remark 2.19, i.e., when the Lie group G is $\mathbb{R}$ we pick the generator $X = 1$ of $\mathfrak{g}$ to write μ^X and $X^\#$:

$$\mathrm{d}\mu^X = \mathrm{d}\mu = -\frac{1}{2}\mathrm{d}\left(\sum r_j^2\right)$$

and

$$X^\#(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0} \Psi_{\exp\theta\cdot 1}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0} \overbrace{\Psi_{e^{i\theta}}(z)}^{e^{i\theta}z} = iz.$$

Note that $X^{\#}(z) = iz$ should not be misinterpreted (see footnote †). Then,

$$\begin{aligned} X^{\#}(z) = iz = i(x^j + iy^j)_j^n = (-y^j + ix^j)_j^n &\xlongequal{x+yi \leftrightarrow (x,y)} \sum_{j=1}^{n} \left(-y^j \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{j=1}^{n} \left((-r^j \sin\theta^j) \left(\cos\theta^j \frac{\partial}{\partial r^j} - \frac{\sin\theta^j}{r^j} \frac{\partial}{\partial \theta^j} \right) + (r^j \cos\theta^j) \left(\sin\theta^j \frac{\partial}{\partial r^j} + \frac{\cos\theta^j}{r^j} \frac{\partial}{\partial \theta^j} \right) \right) \\ &= \frac{\partial}{\partial \theta^1} + \frac{\partial}{\partial \theta^2} + \cdots + \frac{\partial}{\partial \theta^n} \end{aligned}$$

and

$$\begin{aligned} (\iota_{X^{\#}}\omega)(v) &= \left(\sum r^j \mathrm{d}r^j \wedge \mathrm{d}\theta^j \right) (X^{\#}, v) = \sum \left(r^j \underbrace{\mathrm{d}r^j(X^{\#})}_{=0} \mathrm{d}\theta^j(v) - r^j dr^j(v) \underbrace{\mathrm{d}\theta^j(X^{\#})}_{=1} \right) \\ \Longrightarrow \iota_{X^{\#}}\omega &= -\sum r^j \mathrm{d}r^j = -\frac{1}{2} \sum \mathrm{d}((r^j)^2) \end{aligned}$$

If we choose the constant in definition of μ to be $\frac{1}{2}$, then $\mu^{-1}(0) = \mathbb{S}^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/\mathbb{S}^1 = \mathbb{S}^{2n-1}/\mathbb{S}^1 \cong \mathbb{CP}^{n-1}$$

We will see in the next section that $\mathbb{CP}^{n-1}$ arisen in this way is called a reduced space. Notice also that the image of the moment map is half-space. We use this simple example $(\mathbb{C}^n, \omega, \mathbb{S}^1, \mu)$ as a precursor of the major theorems in this expository paper. Under assumptions,

- [Marsden-Weinstein-Meyer] reduced spaces are symplectic manifolds;
- [Atiyah-Guillemin-Sternberg] the image of the moment map is a convex polytope;
- [Delzant] Hamiltonian $\mathbb{T}^n$-spaces are classified by the image of the moment map.

♣

There are some more examples of Hamiltonian G-spaces. We will use them in the next section. For the proofs showing the moment maps are valid, see appendix.

Example 2.24. Let G be any Lie group and H a closed subgroup of G, with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. The projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Suppose that (M, ω, G, ϕ) is a Hamiltonian G-space. The restriction of the G-action to H is Hamiltonian with moment map

$$i^* \circ \phi : M \longrightarrow \mathfrak{h}^*$$

♣

Example 2.25. Suppose that a Lie group G acts in a Hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. The diagonal action of G on $M_1 \times M_2$ is Hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2), \text{ for } p_j \in M_j$$

♣

† The vector field $X^{\#}(z) = iz$ we computed above should not be interpreted as $i\left(z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + \cdots + z^n \frac{\partial}{\partial z^n}\right)$. Recall $\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$ and $\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$. Thus we instead have $(iz^1, \cdots, iz^n) = (ix^1 - y^1, \cdots, ix^n - y^n) \equiv (-y^1, \cdots, -y^n, x^1, \cdots, x^n) = \sum_{j=1}^n \left(-y^j \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial y^j}\right) = \sum_{j=1}^n \left(iz^j \frac{\partial}{\partial z^j} - iz^j \frac{\partial}{\partial \bar{z}^j}\right)$.

Example 2.26. Let $\mathbb{T}^n = \{(t_1, \cdots, t_n) \in \mathbb{C}^n : |t_j| = 1, \text{ for all } j\}$ be a torus acting on $\mathbb{C}^n$ by

$$(t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n) = \left(t_1^{k_1} z_1, \cdots, t_n^{k_n} z_n\right)$$

where $k_1, \cdots, k_n \in \mathbb{Z}$ are fixed. This action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to (\mathfrak{t}^n)^* \cong \mathbb{R}^n$ given by

$$\mu(z_1, \cdots, z_n) = -\frac{1}{2}\left(k_1 |z_1|^2, \cdots, k_n |z_n|^2\right) (+\text{constant})$$

♣

Now suppose (M, ω, G, μ) is a Hamiltonian G-space. Note that the equivariance of μ and linearity of co-adjoint action (so zero is sent to zero) imply that

Lemma 2.27. *If $\mu(p) = 0$. then for any $g \in G, \mu \circ \Psi_g(p) = 0$. Thus, the G-action on M induces a G-action on $\mu^{-1}(0)$.*

Next lemma computes the image and kernel of differential of μ and we will use this to see $\mu^{-1}(0)$ is a closed submanifold of M with $\mathrm{codim} = \dim G$.

Lemma 2.28. *For any $p \in M$,*

(1) $\ker(\mathrm{d}\mu_p) = (T_p\mathcal{O}_p)^{\omega_p} := \{v \in T_pM \mid \omega_p(v, w) = 0$ *for any* $w \in T_p\mathcal{O}_p\}$*, called the **symplectic orthocomplement** of $T_p\mathcal{O}_p$ in (T_pM, ω_p);*

(2) $\mathrm{Im}(\mathrm{d}\mu_p) = \mathfrak{g}_p^0 := \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0$ *for any* $X \in \mathfrak{g}_p\}$*, called the **annihilator** of $\mathfrak{g}_p$ in $\mathfrak{g}^*$.*

Proof. (1): For any $v \in T_pM$ and any $X \in \mathfrak{g}$ one has

$$\omega_p\left(X^{\#}(p), v\right) = (\iota_{X^{\#}}\omega)_p(v) = \left(\mathrm{d}\mu^X\right)_p(v) = \langle \mathrm{d}\mu_p(v), X\rangle$$

The last step comes from the following observation:

$$\begin{aligned}\left(\mathrm{d}\mu^X\right)_p(v) &\xlongequal[\gamma \text{ w}/\gamma(0)=p,\gamma'(0)=v]{\text{[Lee12] 3.25}} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu^X(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \overline{X}(\mu(\gamma(t))) \\ &\xlongequal{\overline{X}\text{ linear}} \overline{X}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu(\gamma(t))\right) \xlongequal{\text{[Lee12] 3.25}} \overline{X}(\mathrm{d}\mu_p(v)) = \langle \mathrm{d}\mu_p(v), X\rangle\end{aligned}$$

(2): It is easy to see $\mathrm{Im}(\mathrm{d}\mu_p) \subseteq \mathfrak{g}_p^0$, so (2) follows from a dimensionality argument:

$$\dim \mathrm{Im}(\mathrm{d}\mu_p) = \dim T_pM - \dim\ker(\mathrm{d}\mu_p) \xlongequal{(1)} \dim T_p\mathcal{O}_p = \dim \mathrm{Im}(A_p),$$

where A_p is the linear map $A_p : \mathfrak{g} \to T_pM, X \mapsto X^{\#}(p)$. Thus, $\dim \mathrm{Im}(A_p) = \dim \mathfrak{g} - \dim\ker(A_p) = \dim \mathfrak{g} - \dim \mathfrak{g}_p = \dim \mathfrak{g}_p^0$. ■

Corollary 2.29.
The action is locally free at p, i.e., stabilizer subgroup G_p is discrete
$\iff \mathfrak{g}_p = \{0\}$
$\iff \mathrm{d}\mu_p$ *is surjective*
$\iff p$ *is a regular point of* μ.

Proof. For the first $\iff$, just note that (1) [Lee12] Proposition 21.28; (2) $\mathfrak{g}_p = \mathrm{Lie}(G_p) \cong T_e(G_p)$; and (3) zero dimensional vector space is exactly 0.

For the second $\iff$: When $\mathfrak{g}_p = \{X \in \mathfrak{g} | X^{\#}(p) = 0\} = 0$, we see that any annihilator has nothing to annihilate, i.e., $\mathfrak{g}_p^0$ includes the whole $\mathfrak{g}^*$, which as a vector space is identified with the tangent space of itself, i.e., $\mathrm{Im}(\mathrm{d}\mu_p)$. Conversely, suppose $\forall \xi \in \mathfrak{g}^*$, $\xi(X) = 0$, then it has to be the case $X = 0$ (for if not then there is some basis element b_j with nonzero coefficient a_i of which X consists. Then we can construct a linear map ξ that only evaluates b_i nontrivially to see $\xi(X) \neq 0$.)

The last $\iff$ is just the definition. ■

Corollary 2.30.
G acts freely on $\mu^{-1}(0)$
$\implies$ *0 is a regular value of μ by Corollary 2.29*
$\implies$ *$\mu^{-1}(0)$ is a closed submanifold of M with $\operatorname{codim} = \dim G$ by regular level set theorem (see [Lee12] Cor.5.14)*
$\implies$ *$T_p\mu^{-1}(0) = \ker \mathrm{d}\mu_p$ for $p \in \mu^{-1}(0)$ by Proposition 2.13 by Lemma 2.28 (a)*
$\implies$ *$T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements in T_pM.*
The inclusion $\mathcal{O}_p \subseteq \mu^{-1}(0)$ (see Lemma 2.27) implies the inclusion $T_p\mathcal{O}_p \subseteq T_p\mu^{-1}(0) = (T_p\mathcal{O}_p)^{\omega_p}$. Thus, the tangent space to the orbit through $p \in \mu^{-1}(0)$ is an isotropic subspace of T_pM. Hence, orbits in $\mu^{-1}(0)$ are isotropic.

The following lemma is called linear reduction. It will be used for the main reduction theorem in the next section.

Lemma 2.31. *Let (V, ω) be a symplectic vector space. Suppose that I is an isotropic subspace, that is, $\omega|_I \equiv 0$. Then ω induces a canonical symplectic form Ω on I^ω/I.*

Proof. Let $u, v \in I^\omega$, and $[u], [v] \in I^\omega/I$. Define $\Omega([u],[v]) = \omega(u,v)$.

- Ω is well-defined: $\omega(u+i, v+j) = \omega(u,v) + \underbrace{\omega(u,j)}_{0} + \underbrace{\omega(i,v)}_{0} + \underbrace{\omega(i,j)}_{0}, \quad \forall i,j \in I.$

- Ω is nondegenerate: Suppose that $[u] \in I^\omega/I$ has $\omega([u],[v]) = 0$, for all $[v] \in I^\omega/I$. Then $\omega(u,v) = 0$, for all $v \in I^\omega$. Then $u \in (I^\omega)^\omega = I$, i.e., $[u] = 0$. ■

3 Two Classical Theorems of Symplectic Geometry: Reduction and Convexity

We prove two classical theorems of symplectic geometry in this section. They are Marsden-Weinstein-Meyer theorem [MW74; Mey73] and Atiyah-Guillemin-Sternberg theorem [Ati82; GS82a], both of them crucial for constructing a symplectic toric manifold from Delzant's polytope.

3.1 Marsden-Weinstein-Meyer Theorem

Theorem 3.1 (Marsden-Weinstein-Meyer)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

(1) the orbit space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold,

(2) $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ is a principal G-bundle, and

(3) there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{\mathrm{red}}$.

Definition 3.2. *The pair $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ is called the **reduction** of (M, ω) with respect to G, μ, or the **reduced space**.*

Proof.

Special Cases:

The simplest case for a compact Lie group is $G = \mathbb{S}^1$ and the simplest case for M is when $\dim M = 2$. For example, the action $\mathbb{S}^1 \times \mathbb{C}^n \to \mathbb{C}^n$ with $n = 1$ gives $\mu^{-1}(0)/\mathbb{S}^1 = \mathbb{S}^1/\mathbb{S}^1$ a single point with zero dimension.

The next simplest example is when $G = \mathbb{S}^1$ and $\dim M = 4$. In this case the moment map is $\mu : M \to \mathbb{R}$ and the embedded submanifold $\mu^{-1}(0)$ has dimension 3. Let $p \in \mu^{-1}(0)$. Choose local coordinates on M:

- θ along the orbit through p;
- μ given by the moment map; and
- η_1, η_2 pullback of coordinates on two-dimensional $\mu^{-1}(0)/\mathbb{S}^1$.

To see how these four coordinates determine a point p in $\mu^{-1}(0) \subseteq M$, we first observe that $\mu : M \to \mathbb{R}$ cuts M into slices of level sets $\mu^{-1}(c)$, and each of them is a 3-manifold. For $\mu^{-1}(0)$, Lemma 2.27 shows that G acts on $\mu^{-1}(0)$ by the same Ψ. Thus, each point p in $\mu^{-1}(0)$ is classified in different orbits $\mathcal{O}_p = G{\cdot}p \subseteq \mu^{-1}(0)$. Now, note that Lemma 2.11 says that the orbit map $\Psi^{(p)}$ corresponds each value of Lie group G with one point in $\mathcal{O}_p$. In all, μ determines whether or not p is in $\mu^{-1}(0)$; η_1, η_2 determines which orbit $\mathcal{O}_p$ in the orbit space $\mu^{-1}(0)/\mathbb{S}^1$ does p lie; θ determines the location of that p is its orbit $\mathcal{O}_p$.

Using these coordinates, we can write the symplectic form as

$$\omega = A\mathrm{d}\theta \wedge \mathrm{d}\mu + B_j \mathrm{d}\theta \wedge d\eta_j + C_j \mathrm{d}\mu \wedge d\eta_j + D d\eta_1 \wedge d\eta_2$$

Since $\mathrm{d}\mu = \iota_{\left(\frac{\partial}{\partial\theta}\right)}\omega$, we must have $A = 1, B_j = 0$. Hence,

$$\omega = \mathrm{d}\theta \wedge \mathrm{d}\mu + C_j \mathrm{d}\mu \wedge d\eta_j + D d\eta_1 \wedge d\eta_2$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}.

<u>General Cases:</u>

Due to Corollary 2.30, we see $\mu^{-1}(0)$ is a smooth manifold of dimension $\dim M - \dim G$. For the first two parts of the theorem it is enough to apply Theorem 2.16 to the free action of compact Lie group G on $\mu^{-1}(0)$. M_{red} has dimension $\dim M - 2\dim G$. Again Corollary 2.30 tells us that at $p \in \mu^{-1}(0)$ the tangent space to the orbit $T_p\mathcal{O}_p$ is an isotropic subspace of the symplectic vector space (T_pM, ω_p), i.e., $T_p\mathcal{O}_p \subseteq (T_p\mathcal{O}_p)^{\omega_p}$, and $(T_p\mathcal{O}_p)^{\omega_p} = \ker \mathrm{d}\mu_p = T_p\mu^{-1}(0)$. The Lemma 2.31 gives a canonical symplectic structure on the quotient $T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. The point $[p] \in M_{\text{red}} = \mu^{-1}(0)/G$ has tangent space $T_{[p]}M_{\text{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. And π has differential

$$\mathrm{d}\pi_p : T_p\mu^{-1}(0) \ni v \mapsto [v] \in T_{[p]}M_{\text{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p. \tag{7}$$

Thus the Lemma defines a nondegenerate 2-form ω_{red} on M_{red}. This is well-defined because ω is G-invariant. Therefore, $i^*\omega$, which is simply $\omega|_{T_p\mu^{-1}(0)}$, is equal to $\pi^*\omega_{\text{red}}$, where

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{\text{red}} & & \end{array}$$

Hence, using naturality of exterior derivative ([Lee12, Proposition 14.26]) twice and noticing closedness of ω, we see

$$\pi^* d\omega_{\text{red}} = d\pi^*\omega_{\text{red}} = di^*\omega = i^*d\omega = 0.$$

Then the closedness of ω_{red} follows from the injectivity of π^*. ∎

The following proposition shows a way to naturally realize the reduced symplectic manifold as a new Hamiltonian space.

Proposition 3.3. *Let (M, ω, G, μ) be a Hamiltonian G-space and $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be the symplectic reduction. Suppose that another Lie group H acts on (M, ω) in a Hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. If H-action commutes with the G-action and ϕ is G-invariant, then the action of H on M_{red} admits a Hamiltonian action of H with moment map ϕ_{red}.*

Proof. The H-action is defined in a natural way:

$$\begin{aligned} H \times M_{\mathrm{red}} &\to M_{\mathrm{red}} \\ h \cdot \mathcal{O}_p &\mapsto \mathcal{O}_{h \cdot p} \end{aligned}$$

This is well-defined, i.e, whatever representative in the orbit $\mathcal{O}_p$ is used to get $\mathcal{O}_{h\cdot p}$, the result is the same. In symbols, for any $q = g \cdot p \in \mathcal{O}_p = G \cdot p$ for some $g \in G$, we have $h \cdot q = g \cdot (h \cdot p) \in \mathcal{O}_{h\cdot p}$.

Since ϕ is G-invariant and thus ϕ is constant on each orbit, we can define $\phi_{\mathrm{red}} : M_{\mathrm{red}} \to \mathfrak{h}^*$ by $\phi_{\mathrm{red}}(\mathcal{O}_p) = \phi(p)$. That is, $\phi_{\mathrm{red}} \circ \pi = \phi \circ i$. We now show that $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, H, \phi_{\mathrm{red}})$ is a Hamiltonian H-space.

Hamiltonian condition: Starting from the Hamiltonian condition on M, i.e., $\mathrm{d}\phi^X = \iota_{X^\#}\omega$, and restricting this equation to $\mu^{-1}(0)$, we have

$$\mathrm{d}\left(i^*\phi^X\right) = \iota_{X^\#}\left(i^*\omega\right).$$

Observe that $\phi^X_{\mathrm{red}}(\mathcal{O}_p) = \phi^X(p) \implies \phi^X_{\mathrm{red}} \circ \pi = \phi^X \circ i$. Then

$$\mathrm{d}(i^*\phi^X)_p = \mathrm{d}(\phi^X \circ i)_p = \mathrm{d}(\phi^X_{\mathrm{red}} \circ \pi)_p = \mathrm{d}\left(\phi^X_{\mathrm{red}}\right)_{\pi(p)} \circ \mathrm{d}\pi_p(\,\cdot\,)$$

Since $i^*\omega = \pi^*\omega_{\mathrm{red}}$, we have

$$\left[\iota_{X^\#}\left(i^*\omega\right)\right]_p = \left[\iota_{X^\#}\left(\pi^*\omega_{\mathrm{red}}\right)\right]_p = (\omega_{\mathrm{red}})_{\pi(p)}\left(\mathrm{d}\pi_p(X^\#), \mathrm{d}\pi_p(\,\cdot\,)\right)$$

Since π is a submersion, i.e, the map in (7) is surjective, we see every element in T_pM_{red} is of the form $\mathrm{d}\pi_p(\,\cdot\,)$. Thus,

$$\mathrm{d}\phi^X_{\mathrm{red}} = \iota_{X^\#}\omega_{\mathrm{red}}.$$

Equivariance condition: We want to show $\forall h \in H$, $\mathcal{O}_p \in M_{\mathrm{red}}$,

$$\phi_{\mathrm{red}}\left(h \cdot \mathcal{O}_p\right) = \mathrm{Ad}^*_h\left(\phi_{\mathrm{red}}\left(\mathcal{O}_p\right)\right)$$

LHS is $\phi_{\mathrm{red}}(\mathcal{O}_{h\cdot p}) = \phi(h \cdot p)$, and RHS is $\mathrm{Ad}^*_h(\phi(p))$. They are equal because of the equivariance of ϕ. ■

In the next subsection, we will realize $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ as a reduced space and use this proposition to give it Hamiltonian actions.

3.2 Example: Reduction for Product Groups

Let G_1 and G_2 be two Lie groups and let $G = G_1 \times G_2$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$$

Suppose each G_i symplectically acts on (M, ω) with moment map $\psi_i : M \to \mathfrak{g}_i$.

Remark 3.4. Recall that if we let θ be a flow on M, we say a smooth tensor field A on M is **invariant under** θ if for each t, the map θ_t pulls A back to itself wherever it is defined, i.e., $\left(\theta_t^* A\right)_p = \mathrm{d}(\theta_t)_p^*\left(A_{\theta_t(p)}\right) = A_p,\ \forall (t,p) \in \mathcal{D}(\theta)$. For functions, we have $\theta_t^* f(p) = f(\theta_t(p))$. [Lee12, Theorem 12.37] claims that A is invariant under the flow θ of $V \in \mathfrak{X}(M)$ if and only if $\mathcal{L}_V A = 0$. ♠

Remark 3.5. Now note that $\theta(t,p) = \Psi_{\exp tX}(p)$ is a flow on M. The integral curves of θ are trajectories of $X^\#$, which is the Hamiltonian vector field of μ^X. Also,

- f is invariant under $\theta \overset{\text{defn.}}{\iff} \forall(t,p),\ f(p) = (\theta_t^* f)(p) := f(\theta_t(p)) = f(\Psi_{\exp tX}(p)) \overset{12.37}{\iff} \mathcal{L}_{X^\#} f = 0$;
- f is G-invariant $\iff f(p) = f(\Psi_g(p))$.

The second invariance is stronger than the first invariance.

However, for connected Lie group G, we can use the fact that the exponential map is a local diffeomorphism to write any element g of G as a product of elements of the form $\exp(X_1)\cdots\exp(X_k)$. Then

$$\begin{aligned} f(g\cdot p) = f(\exp(X_1)\cdots\exp(X_k)\cdot p) &= f(\exp(X_1)(\exp(X_2)\cdots\exp(X_k)\cdot p)) \\ &\overset{\text{1st condition}}{=\!=\!=} f(\exp(X_2)\cdots\exp(X_k)\cdot p) = \cdots = f(p). \end{aligned}$$

Then in this case, the two kinds of invariance are the same. ♠

We shall see how the following assumptions can help us reduce the action.

- **A1**: The actions of G_1 and G_2 on M commute.
- **A2**: ψ_1 is G_2-invariant and ψ_2 is G_1-invariant.
- **A3**: The action of $G_1 \times G_2$ on M (defined as in the following lemma) is free and proper.

Lemma 3.6. *If we assume **A1**, then there is a well-defined action of $G_1\times G_2$ on M given by $(g_1,g_2)\cdot z = g_1\cdot(g_2\cdot p) = g_2\cdot(g_1\cdot p)$. We claim that*

$$\psi := \psi_1\times\psi_2 : M \to \mathfrak{g} = \mathfrak{g}_1^*\oplus\mathfrak{g}_2^*$$

is an action of $G_1\times G_2$ on M satisfying the Hamiltonian condition.

Proof. For $X\in\mathfrak{g}_1$ and $Y\in\mathfrak{g}_2$, we have $\exp(t(X,Y)) = (\exp(tX),\exp(tY))$ and thus $(X,Y)^\#(p) = X^\#(p) + Y^\#(p)$. Note that $\psi^{(X,Y)}(p) = \langle\psi(p),(X,Y)\rangle = \langle(\psi_1(p),\psi_2(p)),(X,Y)\rangle = \langle\psi_1(p),X\rangle + \langle\psi_2(p),Y\rangle = \psi_1^X(p) + \psi_2^Y(p)$. Therefore, $\mathrm{d}\psi^{(X,Y)} = \mathrm{d}\psi_1^X + \mathrm{d}\psi_2^Y = \iota_{X^\#}\omega + \iota_{Y^\#}\omega = \iota_{(X,Y)^\#}\omega$. ∎

There are some remarks we want to make for assumption **A2**.

Remark 3.7. If the symplectic manifold arises as the cotangent bundle of a manifold, i.e, $M = T^*X$ and the actions are lifted from commuting actions on X, then we assert that the condition **A2** automatically holds:

In the cotangent case, we can use the explicit formula for the equivariant moment maps ψ_1 and ψ_2. Let $g_2\in G, \alpha_p\in T_p^*X$ where $p=(x,v)$ and $\xi\in\mathfrak{g}_1$. Then

$$\langle\psi_1(g_2\cdot\alpha_p),\xi\rangle = \left\langle g_2\cdot\alpha_p, \xi^\#(g_2\cdot p)\right\rangle = \left\langle g_2\cdot\alpha_p, g_2\cdot\xi^\#(p)\right\rangle = \left\langle\alpha_p,\xi^\#(p)\right\rangle = \langle\psi_1(\alpha_p),\xi\rangle.$$

There is a similar argument for ψ_2. This proves our assertion. ♠

Remark 3.8. In a sense, one needs to only assume that "half" of **A2** holds. Namely, we claim that if ψ_2 is G_1-invariant and G_2 is connected, then ψ_1 is G_2-invariant. Indeed, $\mathrm{d}\langle\psi_2,\eta\rangle\cdot\xi^\# = (\mathrm{d}\psi_2^\eta)(\xi^\#) = (\iota_{\eta^\#}\omega)(\xi^\#) = 0$ for all $\xi\in\mathfrak{g}_1$ and $\eta\in\mathfrak{g}_2$ and hence

$$\begin{aligned} 0 &\overset{G_1-\text{inv.}}{=\!=\!=} -\mathcal{L}_{\xi^\#}\psi_2^\eta = -\mathcal{L}_{X_{\psi_1^\xi}}\psi_2^\eta \\ &\overset{(14)}{=\!=} \{\psi_1^\xi,\psi_2^\eta\} = \mathcal{L}_{X_{\psi_2^\eta}}\psi_1^\xi = \mathcal{L}_{\eta^\#}\psi_1^\xi \end{aligned}$$

Remark 3.5 shows that when G_2 is connected, $\mathcal{L}_{\eta^\#}\psi_1^\xi = 0$ gives G_2-invariance of ψ_1^ξ and thus ψ_1, as ξ is arbitrary. ♠

Now we have the ingredients needed to get a moment map.

Proposition 3.9. *Under hypotheses **A1** and **A2**, ψ is a moment map for the action of $G = G_1 \times G_2$ on M.*

Proof. The Hamiltonian condition is already proved in the Lemma.

For all $p \in M$ and $(g_1, g_2) \in G_1 \times G_2$ we have

$$\begin{aligned}(\psi_1 \times \psi_2)\left((g_1, g_2) \cdot p\right) &= (\psi_1(g_1 \cdot g_2 \cdot p), \psi_2(g_1 \cdot g_2 \cdot p)) \\ &= (g_1 \cdot \psi_1(p), g_2 \cdot \psi_2(p)) \\ &= (g_1, g_2) \cdot (\psi_1 \times \psi_2)(p)\end{aligned}$$

where we have used equivariance of each of ψ_1 and ψ_2, the fact that the actions commute (**A1**), and condition **A2**, the invariance of ψ_1 and ψ_2. ∎

We need a stronger version of Proposition 3.3:

Lemma 3.10. *Under hypotheses **A1**, **A2**, and **A3**, the space $(M_{\text{red}} = Z_1/G_1 = \psi_1^{-1}(0)/G_1, \omega_{\text{red}})$, reduced from $(M, \omega, G, \psi_1 \times \psi_2)$, can be realized as a Hamiltonian space with G_2-action $g_2 \cdot \mathcal{O}_p := \mathcal{O}_{g_2 \cdot p}$ and moment map $\mu_2 : M_1 \to \mathfrak{g}_2^*$; $\mathcal{O}_p \mapsto \psi_2(p)$. The action is also free and proper.*

Proof. See [Mar+07, Lemma 4.1.2]. ∎

Now, we can take the (second) reduced space $\mu_2^{-1}(0)/G_2$ from the (first) reduced space $M_{\text{red}} = \psi_1^{-1}(0)/G_1$. The main result is the following:

Theorem 3.11. *Under hypotheses **A1**, **A2**, and **A3**, there is a natural symplectomorphism such that*

$$\mu_2^{-1}(0)/G_2 \simeq (\psi_1 \times \psi_2)^{-1}(0,0)/G_1 \times G_2.$$

Proof. See [Mar+07, Theorem 4.1.3]. ∎

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H and can also be extended to semidirect products; see [Mar+07, Chapter 3 and 4].

3.3 Example: Complex Projective Space and Fubini-Study Form

Consider the real-valued function on $\mathbb{C}^n$

$$f : z \longmapsto \log\left(\|z\|^2 + 1\right) \tag{8}$$

which can be shown to be strictly plurisubharmonic and then the 2-form

$$\omega = \frac{i}{2}\partial\bar{\partial}\log\left(\|z\|^2 + 1\right)$$

is Kähler and thus symplectic, where ∂ and $\overline{\partial}$ are **Dolbeault operators**.

Recall that $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1}\backslash\{0\}$ by making the identifications $(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$; $[z_0 : \cdots : z_n]$ is the equivalence class of $(z_0, \cdots, z_n)$.

For $j = 0, 1, \cdots, n$, let

$$\begin{aligned}&\mathcal{U}_j = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\} \\ &\varphi_j : \mathcal{U}_j \to \mathbb{C}^n \quad \varphi_j\left([z_0 : \cdots : z_n]\right) = \left(\frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j}\right).\end{aligned}$$

$\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \cdots, n\}$ is a complex atlas (i.e., the transition maps are biholomorphic). In particular, it was shown that the transition diagram associated with $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ and $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$ has the form

$$\begin{tikzcd} & \mathcal{U}_0 \cap \mathcal{U}_1 \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\ \mathcal{V}_{0,1} \arrow[rr, "\varphi_{0,1}"] & & \mathcal{V}_{1,0} \end{tikzcd}$$

where $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(x_1, \cdots, x_n) \in \mathbb{C}^n \mid x_1 \neq 0\}$ and $\varphi_{0,1}(x_1, \cdots, x_n) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\right)$. We can easily show that $\varphi_{0,l}$ is biholomorphic and for $x \in \mathbb{C}^n$,

$$\begin{aligned}(\varphi_{0,1}^* f)(x) = f(\varphi_{0,1}(x)) &= \log\left(\frac{1 + \sum_{i=2}^n |x_i|^2}{|x_1|^2} + 1\right) = \log\left(\frac{1 + \sum_{i=1}^n |x_i|^2}{|x_1|^2}\right) \\ &= \log\left(|x|^2 + 1\right) + \log\frac{1}{|x_1|^2} = \log\left(|x|^2+1\right) - \log x_1 - \log \overline{x_1}\end{aligned}$$

which implies

$$\varphi_{0,1}^* \partial\overline{\partial} f = \partial\overline{\partial}\varphi_{0,1}^* f = \partial\overline{\partial}\left(\log\left(|x|^2+1\right) - \log x_1 - \log\overline{x_1}\right) = \partial\overline{\partial}\log\left(|x|^2+1\right)$$

where the sum $\log x_1 + \log \overline{x_1}$ of a holomorphic function and an anti-holomorphic function is killed by $\overline{\partial}$ and ∂ due to [Sil06, Definition 14.4]. Thus,

$$\varphi_{0,1}^* \omega = \varphi_{0,1}^* \frac{i}{2}\partial\bar{\partial} f = \frac{i}{2}\partial\bar{\partial} f = \omega$$

Now for on $\mathcal{U}_0 \cap \mathcal{U}_1$, we have $\varphi_1 = \varphi_{0,1} \circ \varphi_0 \implies \varphi_1^* = \varphi_0^* \circ \varphi_{0,1}^*$ and

$$\omega_1 = \varphi_0^* \circ \varphi_{0,1}^* \omega = \varphi_0^* \omega = \omega_0$$

In general, for $\mathcal{U}_k \cap \mathcal{U}_l$, we have

$$\begin{aligned}\varphi_{k,l}(x_1, \cdots, x_n) &= \varphi_l \circ \varphi_k^{-1}(x_1, \cdots, x_n) \\ &= \varphi_l([x_1 z_k : \cdots : x_k z_k : z_k : x_{k+1} z_k : \cdots : x_l z_k : \cdots : x_n z_k]) \\ &= \left(\frac{x_1 z_k}{x_l z_k}, \cdots, \frac{z_k}{x_l z_k}, \cdots, \frac{x_{l-1} z_k}{x_l z_k}, \frac{x_{l+1} z_k}{x_l z_k}, \cdots, \frac{x_n z_k}{x_l z_k}\right) \\ &= \left(\frac{x_1}{x_l}, \cdots, \frac{1}{x_l}, \cdots, \frac{x_n}{x_l}\right)\end{aligned}$$

and one can show $\omega_k = \omega_l$ on $\mathcal{U}_k \cap \mathcal{U}_l$. Therefore, $\varphi_i^* \omega$'s "glue together" to define a Kähler structure on $\mathbb{CP}^n$. This is called the **Fubini-Study form on complex projective space** and is denoted as ω_{FS}.

Recall the $\mathbb{S}^1$-action on $\left(\mathbb{C}^{n+1}, \omega_0\right)$ by the multiplication by e^{it}. This action is Hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = -\frac{1}{2}\|z\|^2 + \frac{1}{2}$$

We show that the reduction $\mu^{-1}(0)/\mathbb{S}^1 \simeq \mathbb{S}^{2n+1}/\mathbb{S}^1 \simeq \mathbb{CP}^n$ has the Fubini-Study symplectic form ω_{FS} as its reduced symplectic form ω_{red}.

Let $\mathrm{pr} : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n \simeq \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$ denote the standard projection. For every $z \in \mathbb{C}^{n+1} \setminus \{0\}$, the point $\mathrm{pr}(z) = [z]$ is in some chart $(\mathcal{U}_k, \varphi_k)$. Note that the composition $\varphi_k \circ \mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^n$ is given by

$$(z_0, \cdots, z_n) \mapsto \left(\frac{z_0}{z_k}, \cdots, \frac{\widehat{z_k}}{z_k}, \cdots, \frac{z_n}{z_k}\right)$$

which is clearly holomorphic. Thus,

$$\mathrm{pr}^*\,\omega_{\mathrm{FS}}(z)=\mathrm{pr}^*\,\varphi_k^*\omega(z)=(\varphi_k\circ\mathrm{pr})^*\omega(z)=\frac{i}{2}\partial\bar\partial f\circ\varphi_k\circ\mathrm{pr}(z)=\frac{i}{2}\partial\bar\partial\log\left(\sum_{j\neq k}\frac{|z_j|^2}{|z_k|^2}+1\right)=\frac{i}{2}\partial\bar\partial\log\|z\|^2.$$

We prove that this form has the same restriction to $\mathbb{S}^{2n+1}$ as ω_0.

It is not hard to compute that

$$\partial f=\sum_{j=1}^n\frac{\partial f}{\partial z_j}\mathrm{d}z_j,\quad \bar\partial f=\sum_{j=1}^n\frac{\partial f}{\partial\bar z_j}\mathrm{d}\bar z_j,\quad \partial\bar\partial f=\sum_{j,k=1}^n\frac{\partial^2 f}{\partial z_j\partial\bar z_k}\mathrm{d}z_j\wedge\mathrm{d}\bar z_k.$$

Let f be $\log\|z\|^2=\log\left(\sum_{k=0}^n z_k\bar z_k\right)$ (note that this is different from (8)). Then

$$\frac{\partial f}{\partial z_j}=\frac{1}{\|z\|^2}\frac{\partial}{\partial z_j}\left(\sum_{k=0}^n z_k\bar z_k\right)=\frac{\bar z_j}{\|z\|^2},\quad \frac{\partial^2 f}{\partial z_j\partial\bar z_k}=\frac{\partial}{\partial\bar z_k}\left(\frac{\bar z_j}{\|z\|^2}\right)=\frac{\|z\|^2\delta_{jk}-\bar z_j z_k}{\|z\|^4}$$

Thus,

$$\partial\bar\partial f=\sum_{j,k=0}^n\left(\frac{1}{\|z\|^2}\delta_{jk}-\frac{\bar z_j z_k}{\|z\|^4}\right)\mathrm{d}z_j\wedge\mathrm{d}\bar z_k.$$

When $\|z\|=1$, we see

$$\frac{i}{2}\partial\bar\partial f=\frac{i}{2}\sum_{j,k=0}^n\left(\delta_{jk}-\bar z_j z_k\right)\mathrm{d}z_j\wedge\mathrm{d}\bar z_k=\omega_0+\underbrace{\frac{1}{2i}\sum_{j,k=0}^n\bar z_j z_k\mathrm{d}z_j\wedge\mathrm{d}\bar z_k}_{R}.$$

Thus we just need to show that the 2-form R is 0 on $\mathbb{S}^{2n+1}$. But suppose that $w=(w_0,\cdots,w_n)$ is a tangent vector at $z=(z_0,\cdots,z_k)$. Then we have

$$(*):\qquad w\cdot z=\sum_j\bar w_j z_j=0.$$

We compute that

$$\begin{aligned}(\iota_w R)_z&=\frac{1}{2i}\sum_{j,k=0}^n\bar z_j z_k\left(\mathrm{d}z_j(w)\mathrm{d}\bar z_k-\mathrm{d}\bar z_k(w)\mathrm{d}\bar z_j\right)\\&=\frac{1}{2i}\sum_{j,k=0}^n\bar z_j z_k(w_j\mathrm{d}\bar z_k-\bar w_k\mathrm{d}z_j)\\&=\frac{1}{2i}\sum_{k=0}^n\sum_{j=0}^n z_k\underbrace{\bar z_j w_j}\mathrm{d}\bar z_k-\frac{1}{2i}\sum_{j=0}^n\sum_{k=0}^n\bar z_j\underbrace{z_k\bar w_k}\mathrm{d}z_j\\&\overset{(*)}{=\!=}0.\end{aligned}$$

This shows R is zero when $\|z\|=1$. We have shown $(\mathrm{pr}\circ i)^*\omega_{\mathrm{FS}}=i^*\,\mathrm{pr}^*\,\omega_{\mathrm{FS}}=i^*\omega_0$. To show $(\mathbb{CP}^n,\omega_{\mathrm{FS}})$ is symplectomorphic to $(M_{\mathrm{red}},\omega_{\mathrm{red}})=(\mu^{-1}(0)/\mathbb{S}^1,\omega_{\mathrm{red}})=(\mathbb{S}^{2n+1}/\mathbb{S}^1,\omega_{\mathrm{red}})$, we first note the following bijection ℓ

$$\begin{aligned}\mathbb{S}^{2n+1}/\mathbb{S}^1=\{\mathcal{O}_{(z_0,\cdots,z_n)}\,|\,|z_0|^2+\cdots+|z_n|^2=1\}&\longrightarrow\{[z_0:\cdots:z_n]\,|\,z_j\text{ not all }0\}=\mathbb{CP}^n\\ \mathcal{O}_{(z_0,\cdots,z_n)}&\longmapsto[z_0:\cdots:z_n]\\ \mathcal{O}_{(w_0,\cdots,w_n)}\text{ where }w_j=\frac{z_j}{\sqrt{|z_0|^2+\cdots+|z_n|^2}}&\longleftarrow[z_0:\cdots:z_n]\end{aligned}$$

is a diffeomorphism such that $\ell \circ \pi = \operatorname{pr} \circ i$ where $i : \mathbb{S}^{2n+1} \to \mathbb{C}^{n+1} \setminus \{0\}$, $\operatorname{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$, $\pi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\mathbb{S}^1$. ℓ is then the desired symplectomorphism:

$$\begin{aligned} \pi^*\ell^*\omega_{\mathrm{FS}} = (\ell \circ \pi)^*\omega_{\mathrm{FS}} = (\operatorname{pr} \circ i)^*\omega_{\mathrm{FS}} = i^*\omega_0 \xlongequal{\text{Marsden-Weinstein-Meyer}} \pi^*\omega_{\mathrm{red}} \\ \Longrightarrow \ell^*\omega_{\mathrm{FS}} = \omega_{\mathrm{red}} \qquad \text{using the lemma below plus surjective submersion } \pi \end{aligned}$$

Lemma: Consider a smooth surjective map $\pi : M \to N$ and a smooth submersion $\mathrm{d}\pi_p : T_pM \to T_pN$. Then $\pi^*\omega = \omega^*\eta \implies \omega = \eta$.
proof: $\forall a, b \in T_qN$, $\exists u, v \in T_pM$ s.t. $\pi(p) = q$ and $\mathrm{d}\pi_p u = a, \mathrm{d}\pi_p v = b$. Then

$$\omega_q(a, b) = \omega_q(\mathrm{d}\pi_p u, \mathrm{d}\pi_p v) = \eta_q(\mathrm{d}\pi_p u, \mathrm{d}\pi_p v) = \eta_q(a, b) \implies \omega = \eta.$$

Proposition 3.12. *The natural actions of $\mathbb{T}^{n+1}$ and $U(n+1)$ on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ are Hamiltonian.*

Proof. We have the symplectomorphism $\tau : (\mathbb{CP}^n, \omega_{\mathrm{FS}}) \to (M_{\mathrm{red}} = \mathbb{S}^{2n+1}/\mathbb{S}^1, \omega_{\mathrm{red}})$, the reduced space from Hamiltonian G-space $(M = \mathbb{C}^{n+1}, \omega, G = \mathbb{S}^1, \mu)$. Now Proposition 3.3 says $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ can be realized as Hamiltonian H-spaces $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, H, \phi_{\mathrm{red}})$ for $(M, \omega, H = \mathbb{T}^{n+1}, \phi)$ and $(M, \omega, H = U(n+1), \phi)$ if we can verify two conditions required in the proposition for each of these two H-actions.

1. The action of $H = \mathbb{T}^{n+1} = \{(t_0, \cdots, t_n) \in \mathbb{C}^{n+1} : |t_j| = 1\}$ on $M = \mathbb{C}^{n+1}$ and the moment map ϕ are given by Example 2.26:

$$h \cdot p = (t_0, \cdots, t_n) \cdot (z_0, \cdots, z_n) = (t_0 z_0, \cdots, t_n z_n)$$

$$\phi(z_0, \cdots, z_n) = -\frac{1}{2}\left(|z_0|^2, \cdots, |z_n|^2\right) (+ \text{ constant }).$$

Two conditions to verify are:

- commutativity of G and H actions: this is due to commutativity of complex multiplication:

$$\begin{aligned} h \cdot (g \cdot p) &= (t_0, \cdots, t_n) \cdot (e^{i\theta} \cdot (z_0, \cdots, z_n)) = (t_0, \cdots, t_n) \cdot (e^{i\theta} z_0, \cdots, e^{i\theta} z_n) \\ &= (t_0 e^{i\theta} z_0, \cdots, t_n e^{i\theta} z_n) = (e^{i\theta} t_0 z_0, \cdots, e^{i\theta} t_n z_n) \\ &= e^{i\theta} \cdot ((t_0, \cdots, t_n) \cdot (z_0, \cdots, z_n)) = g \cdot (h \cdot p). \end{aligned}$$

- ϕ is G-invariant:

$$\phi(g \cdot p) = -\frac{1}{2}\left(\left|e^{i\theta} z_0\right|^2, \cdots, \left|e^{i\theta} z_n\right|^2\right) (+ \text{ constant }) = -\frac{1}{2}\left(|z_0|^2, \cdots, |z_n|^2\right) (+ \text{ constant }) = \phi(p).$$

The conclusion of the proposition is that $(\mathbb{S}^{2n+1}/\mathbb{S}^1, \omega_{\mathrm{red}}, \mathbb{T}^{n+1}, \phi_{\mathrm{red}})$ is a Hamiltonian H-space, i.e., the action is given by

$$\begin{aligned} \Phi : \mathbb{T}^{n+1} \times \mathbb{S}^{2n+1}/\mathbb{S}^1 \to \mathbb{S}^{2n+1}/\mathbb{S}^1 \\ (t_0, \cdots, t_n) \cdot \mathcal{O}_{(z_0, \cdots, z_n)} \mapsto \mathcal{O}_{(t_0 z_0, \cdots, t_n z_n)} \end{aligned}$$

and ϕ_{red} below is its moment map,

$$\begin{aligned} \phi_{\mathrm{red}} : \mathbb{S}^{2n+1}/\mathbb{S}^1 = \{\mathcal{O}_{(z_0, \cdots, z_n)} \,|\, |z_0|^2 + \cdots + |z_n|^2 = 1\} \longrightarrow \mathfrak{h}^* = \mathbb{R}^{n+1} \\ \mathcal{O}_{(z_0, \cdots, z_n)} \longmapsto \phi(z_0, \cdots, z_n) = -\frac{1}{2}\left(|z_0|^2, \cdots, |z_n|^2\right) (+ \text{ constant }). \end{aligned}$$

We claim $(\mathbb{CP}^n, \omega_{\text{FS}}, \mathbb{T}^{n+1}, \phi_{\text{red}} \circ \tau)$ is a Hamiltonian H-space with action

$$\begin{aligned}\Psi : \mathbb{T}^{n+1} \times \mathbb{CP}^n &\to \mathbb{CP}^n \\ (t_0, \cdots, t_n) \cdot [z_0 : \cdots : z_n] &\mapsto [t_0 z_0 : \cdots : t_n z_n]\end{aligned}$$

and moment map

$$\begin{aligned}\phi_{\text{red}} \circ f : \mathbb{CP}^n &\longrightarrow \mathbb{S}^{2n+1}/\mathbb{S}^1 \longrightarrow \mathfrak{h}^* = \mathbb{R}^{n+1} \\ [z_0 : \cdots : z_n] &\longmapsto \mathcal{O}_{(w_0, \cdots, w_n)} \longmapsto \phi(w_1, \cdots, w_n) \\ &= -\frac{1}{2} \frac{\left(|z_0|^2, \cdots, |z_n|^2\right)}{|z_0|^2 + \cdots + |z_n|^2} (+\text{ constant}) = -\frac{1}{2\|z\|^2} \left(|z_0|^2, \cdots, |z_n|^2\right) (+\text{ constant}).\end{aligned}$$

In fact, Hamiltonian condition is satisfied as $\omega_{\text{FS}}(X, \cdot) = f^*\omega_{\text{red}}(X, \cdot) = f^*(\mathrm{d}(\phi_{\text{red}}^X)) = \mathrm{d}(\phi_{\text{red}}^X \circ f) = \mathrm{d}(\phi_{\text{red}} \circ f)^X$; we verify the equivariance condition: $\ell \circ \Phi_h \circ f = \Psi_h \implies (\phi_{\text{red}} \circ f)(\Psi_h) = \phi_{\text{red}} \circ f \circ \Psi_h = \phi_{\text{red}} \circ f \circ \ell \circ \Phi_h \circ f = \phi_{\text{red}} \circ \Phi_h \circ f = \mathrm{Ad}_h^* \circ \phi_{\text{red}} \circ f = \mathrm{Ad}_h^*(\phi_{\text{red}} \circ f)$.

2. The action of $U(n+1)$ on $(\mathbb{CP}^n, \omega_{\text{FS}})$ is

$$U \cdot [z_0 : \cdots : z_n] = [U(z_0 : \cdots : z_n)^T]$$

The moment map is

$$[z_0 : \cdots : z_n] \mapsto \frac{i}{2\|z\|^2} z z^*$$

where we identify the Lie algebra $\mathfrak{u}(n+1)$ with its dual via the inner product $(A, B) = \mathrm{tr}(A^*B)$. One uses a similar process as that of $\mathbb{T}^{n+1} \curvearrowright M_{\text{red}}$ descending to $\mathbb{T}^{n+1} \curvearrowright \mathbb{CP}^n$ to show above action with the moment map form a Hamiltonian H-sapce. Solution 5.21 from here gives an attempt to show this directly; without evoking Proposition 3.3, checking Hamiltonian and equivariance conditions can be rather tedious.

■

3.4 Atiyah-Guillemin-Sternberg Theorem

From now on, we will concentrate on actions of a torus $G = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. Recall that any compact connected abelian Lie group must be a torus.

Theorem 3.13 (Atiyah-Guillemin-Sternberg Theorem)**.** *Let (M, ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M, \omega)$ is a Hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(1) the levels of μ are connected;

(2) the image of μ is convex;

(3) the image of μ is the convex hull of the images of the fixed points of the action.

The image $\mu(M)$ of the moment map is hence called the ***moment polytope****.*

In the following writeup, we will follow [MS17] to present Atiyah's approach. Guillemin-Sternberg approach [GS82a] is sketched in this note. Some of the proofs of the same lemmas in the following sections are more concisely done in this note.

The following proposition is adapted from [MS17, Lemma 5.53].

Proposition 3.14. *Consider the symplectic action of $\mathbb{T}^m$ on (M,ω). For any subgroup $G \subseteq \mathbb{T}^m$, the fixed-point set for G,*

$$\mathrm{Fix}(G) = \bigcap_{\theta \in G} \mathrm{Fix}\,(\psi_\theta)$$

is a symplectic submanifold of M.

Proof. Let $x \in \mathrm{Fix}(G)$ and, for $\theta \in \mathrm{G}$, denote the differential of the symplectomorphism ψ_θ at x by

$$\Psi_\theta = \mathrm{d}(\psi_\theta)_p : T_pM \to T_{\psi_\theta(p)}M = T_pM.$$

Because of Proposition 2.17, these maps commute with J_x. Now consider the exponential map $\exp_x : T_xM \to M$ on the Riemannian manifold (M,g) with respect to the invariant metric $g(v,w) = \omega(v,Jw)$ (obtained by eq.(6)). Thus, $\psi_\theta^* g = g$, which means ψ_θ is an isometry. By the naturality of exponential map ([Lee18, Proposition 5.20]), we have

$$\exp_x(\Psi_\theta \xi) = \psi_\theta(\exp_x(\xi))$$

for $\theta \in \mathrm{G}$ and $\xi \in T_xM$. Hence the fixed points of ψ_θ near x correspond to the fixed points of Ψ_θ on the tangent space T_xM: if $y = \exp_x \xi$ for some ξ is fixed by ψ_θ, then $\exp_x(\xi) = y = \psi_\theta(y) = \exp_x(\Psi_\theta\xi)$. By uniqueness of geodesic, $\Psi_\theta\xi = \xi$. Conversely, if $\Psi_\theta\xi = \xi$, then $y = \exp_x \xi$ is a fixed point of ψ_θ.

In other words,

$$T_x\,\mathrm{Fix(G)} = \bigcap_{\theta \in \mathrm{G}} \ker(\mathbb{1} - \Psi_\theta)$$

Now, let $x \in \mathrm{Fix}(G)$ and $\xi \in T_x\,\mathrm{Fix}(G)$. Since $\Psi_\theta J_x = J_x\Psi_\theta$, we have $\Psi_\theta J_x\xi = J_x\Psi_\theta\xi = J_x\xi$. Thus, $J_x\xi \in \bigcap_{\theta\in G}\ker(1-\Psi_\theta) = T_x\,\mathrm{Fix}(G)$. Therefore, $J_x : T_x\,\mathrm{Fix}(G) \to T_x\,\mathrm{Fix}(G)$ and $T_x\,\mathrm{Fix}(G)$ is a symplectic vector space (see lemma below), and we conclude that $\mathrm{Fix}(G)$ is a symplectic submanifold. ■

Lemma 3.15. *Given (V,Ω,g,J) and subspace $W \leq V$ such that $JW \subseteq W$, show that W is a symplectic vector space.*

Proof. We aim to show $\Omega|_{W\times W}$ is nondegenerate. That is, if $u \in W$ is some vector such that $\Omega(u,v) = 0$ for every $v \in W$, then $u = 0$. Notice that $Jw = v$ with $w = J^3v$ shows that $J : W \to W$ is surjective. Thus,

$$\forall v \in W, 0 = \Omega(u,v) = g(u,Jv) = g(u,w) \implies \forall w \in W, g(u,w) = 0 \implies u \perp W \implies u \in W^\perp$$

Since $u \in W$, we have $u \in W \cap W^\perp = \{0\} \implies u = 0$. ■

Atiyah's proof of theorem 3.13 uses induction over $m = \dim \mathbb{T}^m$. Consider the statements:
A_m: "the levels of μ are connected, for any $\mathbb{T}^m$-action;"
B_m: "the image of μ is convex, for any $\mathbb{T}^m$-action."

Then
The connectedness statement (1) $\iff \mathrm{A}_m$ holds for all m,
The convexity statement (2) $\iff \mathrm{B}_m$ holds for all m.

The proof of the induction $\mathrm{A}_{m-1} \implies \mathrm{A}_m$ is from this note, which is the same as the proof from [MS17]. We will show in this paper A_1, B_1, and the induction $\mathrm{B}_{m-1} \implies \mathrm{B}_m$.

Connectedness

Lemma 3.16. *Let $\mathbb{T}^m$ acts on (M,ω) by ψ in symplectomorphism with moment map $\mu : M \to \mathbb{R}^m$. If $X \in \mathbb{R}^m$, then μ^X is a Morse-Bott function with even-dimensional critical manifolds of even index and coindex. Also, the critical set*

$$\mathrm{Crit}(\mu^X) = \bigcap_{\theta \in \mathbb{T}^X} \mathrm{Fix}(\psi_\theta) \tag{9}$$

is a symplectic manifold. Here, $\mathbb{T}^X$ is the closure of the one-parameter subgroup $\exp(tX)$ generated by X in $\mathbb{T}^m$.

Proof. Observe that

$$\begin{aligned} p \in \mathrm{Crit}(\mu^X) &\iff \mathrm{d}\mu^X(p) = \iota_{X^\#(p)}\omega = 0 \\ &\iff X^\#(p) = 0 \quad \text{by nondegeneracy of } \omega \\ &\iff \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{\exp tX}(p) = 0. \end{aligned}$$

Note that the last condition shows that $\psi_{\exp tX}(p) = p$. Thus, $p \in \mathrm{Fix}(\psi_\theta)$ for any $\theta = \exp tX$. By continuity, this is also true for any $\theta \in \mathbb{T}^X$. Conversely, $p \in \bigcap_{\theta \in \mathbb{T}^X} \mathrm{Fix}(\psi_\theta) \implies p \in \mathrm{Fix}(\psi_{\exp tX})$ for some t, i.e., $\psi_{\exp tX}(p) = p$. This gives $p \in \mathrm{Crit}(\mu^X)$ by above displayed equations.

Now, since $\mathbb{T}^X \subseteq \mathbb{T}^m$ is a subgroup, and the RHS of the eq.(9) is just $\mathrm{Fix}(\mathbb{T}^X)$, we see Proposition 3.14 concludes that $\mathrm{Crit}(\mu^X)$ is a symplectic manifold.

Now, we show μ^X is a Morse-Bott function with even-dimensional critical manifolds of even index and coindex. We can assume that X has components independent over $\mathbb{Q}$, so that $\mathbb{T}^X = \mathbb{T}^m$ and $\mathrm{Crit}(\mu^X) = \mathrm{Fix}(\mathbb{T}^m)$. We aim to show $\ker \widehat{\mathrm{Hess}_x(\mu^X)} = \bigcap_{\theta \in \mathbb{T}^m} \ker(\mathrm{id} - \mathrm{d}\psi_\theta(x))$. From the solution of Proposition 3.14, this will show $T_x\mathrm{Crit}(\mu^X) = \ker \widehat{\mathrm{Hess}_x(\mu^X)}$. First, denote that $\mathcal{H}_x = \widehat{\mathrm{Hess}_x(\mu^X)}$.

Notice that $X^\# = X_{\mu^X} = -J \operatorname{grad} \mu^X$. [†] This is a smooth vector field on M and thus a smooth function $M \to TM$, where $\operatorname{grad}\mu^X : M \to TM$ and $J : TM \to TM$ are also smooth functions. Let ∇ be the Levi-Civita connection on M. Then for $x \in \mathrm{Crit}(\mu^X)$, for any vector field Z on M, $\mathrm{d}(\operatorname{grad}\mu^X) \overset{\text{eq.(1)}}{=\!=} \nabla_Z(\operatorname{grad}\mu^X) = \widehat{\mathrm{Hess}(\mu^X)}(Z) = \mathcal{H}(Z)$ on x, where the last step is from the observation 2.2 and Remark 2.3 (d). Thus, for $x \in \mathrm{Crit}(\mu^X)$, the differential $dX^\#$ of the function $X^\# : M \to TM$ is $\mathrm{d}(-J \operatorname{grad}\mu^X) = -J\mathrm{d}(\operatorname{grad}\mu^X) = -J\mathcal{H}$.

We know $\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\exp tX}(x) = X_{\mu^X}(\psi_{\exp tX}(x))$.[†] Differentiating this flow equation with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\psi_{\exp tX}(x) = dX_{\mu^X}(\psi_{\exp tX}(x)) \circ \mathrm{d}\psi_{\exp tX}(x)$$

At the critical point x of μ^X, we from eq.(9) know that $\psi_\theta(x) = x$ for $\theta = \exp tX$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\psi_{\exp tX}(x) = -J_x\mathcal{H}_x \cdot \mathrm{d}\psi_{\exp tX}(x)$$

This is a linear ODE of the form $\frac{\mathrm{d}}{\mathrm{d}t}A(t) = MA(t)$ with constant matrix $M = -J_x\mathcal{H}_x$ and initial condition $A(0) = \mathrm{d}\psi_{\exp(0)X}(x) = \mathrm{d}\psi_e(x) = \mathrm{id}_{T_xM}$. The unique solution to such a matrix ODE is given by the matrix exponential $A(t) = \exp(tM) = \exp(-tJ_x\mathcal{H}_x)$. Thus, we conclude

$$\mathrm{d}\psi_{\exp tX}(x) = \exp(-tJ_x\mathcal{H}_x).$$

[†] This is because $\omega(-J\operatorname{grad}\mu^X, \cdot) = \omega(\cdot, J\operatorname{grad}\mu^X) = \langle \cdot, \operatorname{grad}\mu^X\rangle = \mathrm{d}\mu^X(\cdot) \implies$ the Hamiltonian vector field for function μ^X is $-J\operatorname{grad}\mu^X$. That is, $X^\# = X_{\mu^X} = -J\operatorname{grad}\mu^X$.

[†] This is because $p = \exp tX$ and $X^\#(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{s=0}\psi_{\exp sX}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{s=0}\psi_{\exp sX}(\psi_{\exp tX}(p)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{s=0}\psi_{\exp(s+t)X}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{s=t}\psi_{\exp sX}(p)$.

Now, note that the fixed set of $\mathrm{d}\psi_{\exp tX}(x) = \exp(-tJ_x\mathcal{H}_x)$ is $\ker J_x\mathcal{H}_x$: if we let $B = J_x\mathcal{H}_x$, then

$$\exp(-tB) = I - tB + \frac{t^2}{2}B^2 + \cdots, \tag{10}$$

so $\{v \in T_xM | \exp(-tB)v = v\} = \{v \in TxM | Bv = 0\} = \ker B$. Since J_x is invertible (its inverse is just $-J_x$), so $\ker J_x\mathcal{H}_x = \ker \mathcal{H}_x$. Since X has rationally independent components, these are the common fixed points of all $\mathrm{d}\psi_\theta(p)$, $\theta \in \mathbb{T}^m$. Thus,

$$\ker \widehat{\mathrm{Hess}_x(\mu^X)} = \bigcap_{\theta\in\mathbb{T}^m} \ker(\mathrm{id} - \mathrm{d}\psi_\theta(x)).$$

This shows μ^X is a Morse-Bott function.

From Proposition 2.17 and equation (10), we see that $\mathrm{d}\psi_{\exp tX}(x) = \exp(-tJ_x\mathcal{H}_x) = \exp(-tB)$ commutes with J and this commutativity implies $BJ_x = J_xB$ and thus $\mathcal{H}_xJ_x = J_x\mathcal{H}_x$. Hence by Lemma 3.17 below, all the eigenspaces of $\mathcal{H}_x$ are invariant under J_x and are therefore even-dimensional. Thus, the critical manifolds of μ^X, as symplectic manifolds, have even dimensions, with even index and coindex. ■

Lemma 3.17. *Let (M, ω, J) be a symplectic manifold with a compatible almost complex structure J, and let $\mathcal{H}_x : T_xM \to T_xM$ be a self-adjoint operator. If $J_x\mathcal{H}_x = \mathcal{H}_xJ_x$, then every eigenspace of $\mathcal{H}_x$ is invariant under J_x and has even dimension.*

Proof. Since $J_x\mathcal{H}_x = \mathcal{H}_xJ_x$, the operator J_x preserves each eigenspace of $\mathcal{H}_x$. If v is an eigenvector of $\mathcal{H}_x$ with eigenvalue λ, then

$$\mathcal{H}_x(J_xv) = J_x(\mathcal{H}_xv) = J_x(\lambda v) = \lambda J_xv$$

Thus, J_xv is also an eigenvector with the same eigenvalue λ. Since $J_x^2 = -\mathrm{id}$, it follows that v and J_xv are linearly independent (if $Jv = kv$, then $JJv = J(kv) = kJv = k^2v$. But $JJv = -v$ and $k^2 \geq 0$). Hence, the eigenspace corresponding to λ contains such pairs and is even-dimensional. ■

We still need another concept to begin our proof of connectedness. We continue with our Hamiltonian $\mathbb{T}^m$-space setting.

Definition 3.18. *We denote the components of the moment map $\mu : M \to \mathbb{R}^m$ as $\mu = (\mu_1, \cdots, \mu_m)$. We say that μ is* ***irreducible*** *if the 1-forms $\mathrm{d}\mu_1, \cdots, \mathrm{d}\mu_m$ are linearly independent, i.e., given a scalar $(\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m$, then*

$$\alpha_1\mathrm{d}\mu_1(x)(\xi) + \cdots + \alpha_m\mathrm{d}\mu_m(x)(\xi) = 0$$

at all points $x \in M$ and all vectors $\xi \in T_xM$ if and only if $\alpha_1 = \cdots = \alpha_m = 0$. We say the μ is ***reducible*** *otherwise.*

Definition 3.19. *We say that a set of real numbers $\{\theta_i \mid 1 \leq i \leq s, \theta_i \in \mathbb{R}\}$ is* ***rationally dependent*** *if $\frac{\theta_i}{\theta_j}$ is rational for all nonzero $\theta_{i,j}$ with $1 \leq i, j \leq s$.*

Statement A_1.

A_1: for $m = 1$, the level sets $\mu^{-1}(c)$ of the moment map are connected.

Proof. For $m = 1$, $\mathbb{T}^m = \mathbb{S}^1$. Let $X \in \mathfrak{g} \cong \mathfrak{g}^* = \mathbb{R}$. Lemma 3.16 and Remark 2.19 says $\mu = \mu^1$ is Morse-Bott with critical manifolds of even index and coindex. Lemma 2.8 then concludes. ■

Induction: $\mathrm{A}_{m-1} \implies \mathrm{A}_m$.

Assuming every level set of μ for an $(m-1)$-dimensional toral action is connected, aim to show the level sets of μ of an m-dimensional toral action is connected.

Proof. As aforementioned, see this note or [MS17]. ■

Atiyah's Approach: Convexity

In this section, we show part (2) (B_1 and $\mathrm{B}_{m-1} \implies \mathrm{B}_m$) and part (3) of the theorem.

Statement B_1.

B_1: for $m = 1$, the image of the moment map μ is convex.

Proof. For $m = 1$, $\mathbb{T}^m = \mathbb{S}^1$ and $\mathfrak{g}^* = \mathbb{R}$. Since M is connected, $\mu(M)$ is also connected. In $\mathbb{R}$, connectedness implies convexity. ■

Induction: $\mathrm{B}_{m-1} \implies \mathrm{B}_m$**.**

Assuming the image of μ for an $(m-1)$-dimensional toral action is convex, aim to show that the image of μ of an m-dimensional toral action is convex.

Proof. Denote $H = \mathbb{T}^{m-1}$ and $G = \mathbb{T}^m$, so $\mathrm{Lie}(H) = \mathfrak{h}^*$ and $\mathrm{Lie}(G) = \mathfrak{g}^*$. Choose an injective matrix $A \in \mathbb{Z}^{m \times (m-1)}$, so it can be either seen as a map $A : \mathbb{R}^{m-1} \cong \mathfrak{h} \to \mathfrak{g} \cong \mathbb{R}^m$ (so $A^t : \mathbb{R}^m \cong \mathfrak{g}^* \to \mathfrak{h}^* \cong \mathbb{R}^{m-1}$) or as a map

$$\begin{aligned} A : \mathbb{T}^{m-1} &\longrightarrow \mathbb{T}^m \\ \left(e^{2\pi i \theta_1}, \cdots, e^{2\pi i \theta_{m-1}}\right) &\longmapsto \left(e^{2\pi i \sum_{j=1}^{m-1} a_{1j}\theta_j}, \cdots, e^{2\pi i \sum_{j=1}^{m-1} a_{mj}\theta_j}\right). \end{aligned}$$

Consider the action of an $(m-1)$ subtorus

$$\begin{aligned} \psi_A : \mathbb{T}^{m-1} &\longrightarrow \mathrm{Sympl}(M, \omega) \\ \theta &\longmapsto \psi_{A\theta} \end{aligned}$$

It is a Hamiltonian action with moment map $\mu_A = A^t \mu$ by Lemma 3.20 below.

Given any $p_0 \in \mu_A^{-1}(\xi)$, $p \in \mu_A^{-1}(\xi) \iff A^t \mu(p) = \xi = A^t \mu(p_0)$, so

$$(*) : \qquad \mu_A^{-1}(\xi) = \{p \in M \mid \mu(p) - \mu(p_0) \in \ker A^t\}.$$

By A_{m-1}, this level set of μ_A is connected. Also note that $\dim \operatorname{Im} A = \operatorname{rank} A = (m-1) - \dim\ker A = m-1$, so $\dim \ker A^t = m - \dim \operatorname{Im} A = 1$. Now, if we connect p_0 to p_1 by a path p_t in $\mu_A^{-1}(\xi)$, we obtain a path $\mu(p_t) - \mu(p_0)$ in $\ker A^t$. Since $\ker A^t$ is 1-dimensional, we see $\mu(p_t)$ must go through any convex combination of $\mu(p_0)$ and $\mu(p_1)$, which shows that any point on the line segment from $\mu(p_0)$ to $\mu(p_1)$ must be in $\mu(M)$:

$$(1-t)\mu(p_0) + t\mu(p_1) \in \mu(M), \quad 0 \le t \le 1$$

Now suppose p, q are arbitrary two points in M. Since $\mu(M) \subseteq \mathbb{R}^m$ is compact and thus closed, we consider two sequences in $\mu(M)$, $\mu(p_i) \to \mu(p)$, $\mu(q_i) \to \mu(q)$. These points can be approximated arbitrarily closely by rational points, so we assume these two sequences are in $\mathbb{Q}^m \cap \mu(M)$. Each $v_i = \mu(q_i) - \mu(p_i)$ is a rational vector and can be extended to a basis for $\mathbb{Q}^m$. It is not hard to see we can construct a matrix $A \in \mathbb{Z}^{m \times (m-1)}$ such that $\ker A^t = \mathbb{R} v_i$. Thus, $(*)$ implies $p_i, q_i \in \mu_A^{-1}(v_i)$. Thus, $\forall t \in [0,1]$, $(1-t)\mu(p_i) + t\mu(q_i) \in \mu(M)$. Thus, $\forall t \in [0,1]$, $(1-t)\mu(p) + t\mu(q) = \lim_{i\to\infty}[(1-t)\mu(p_i) + t\mu(q_i)] \in \mu(M)$. The image $\mu(M)$ is therefore convex. ■

Lemma 3.20. *The action ψ_A is Hamiltonian with moment map $\mu_A = A^t\mu : M \to \mathbb{R}^{m-1}$.*

Proof. Note that $A^t : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual mapping of $A : \mathfrak{h} \to \mathfrak{g}$. Thus, for $f \in \mathfrak{g}^*$ and $\xi \in \mathfrak{h}$, we have $\langle A^t(f), \xi \rangle = \langle f, A\xi \rangle$. This gives $\mu^{A\xi}(p) = \mu_A^{\xi}(p)$. Notice that the exponential map $\exp$ of the torus of dimension ℓ sends $(\xi_1, \cdots, \xi_\ell)$ to $(e^{2\pi\xi_1}, \cdots, e^{2\pi\xi_\ell})$. $\exp$ thus coincides with the projection $\pi : \mathbb{R}^\ell \to \mathbb{R}^\ell / \mathbb{Z}^\ell$. It is easy to see the following diagram is commutative (with $\exp$ and A interpreted as above):

$$\begin{tikzcd}
\mathbb{R}^{m-1} \arrow[r, "A"] \arrow[d, "\exp"'] & \mathbb{R}^m \arrow[d, "\exp"] \\
\mathbb{T}^{m-1} \arrow[r, "A"'] & \mathbb{T}^m
\end{tikzcd}$$

Thus, $A \exp t\xi = \exp tA\xi$. † For $\xi \in \mathfrak{h}$,

$$\xi^{\#}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\psi_A)_{\exp t\xi}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{A \exp t\xi}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_{\exp(tA\xi)}(p) = (A\xi)^{\#}(p)$$

Then, the Hamiltonian condition for μ_A is satisfied:

$$\omega(\xi^{\#}, \cdot) = \omega((A\xi)^{\#}, \cdot) = \mathrm{d}\mu^{A\xi} = \mathrm{d}\mu_A^{\xi}.$$

The equivariance condition becomes invariance condition for the torus being abelian (see Remark 2.20). It is satisfied as well: for $\theta \in \mathbb{T}^{m-1}$,

$$\mu_A \circ \psi'_\theta(p) = A^t \mu \circ (\psi_{A\theta}(p)) = A^t(\mu \circ \psi_{A\theta}(p)) = A^t \mu(p) = \mu_A(p).$$

■

The remaining is to show statement (3) of Atiyah-Guillemin-Sternberg Theorem 3.13.

Vertices of the Convex Hull $\mathrm{Im}(\mu)$

The image of the moment map is also a convex hull supported by the images of the points fixed by all the actions.

Proof. Proposition 3.14 shows that $\mathrm{Fix}(\mathbb{T}^m) = \bigcap_{\theta \in \mathbb{T}^m} \mathrm{Fix}(\psi_\theta)$ is a symplectic submanifold of M. Due to the compactness of M, $\mathrm{Fix}(\mathbb{T}^m)$ decomposes into finitely many connected symplectic submanifolds, $C_1, \cdots, C_N$.

Notice that each component μ_i of the moment map μ is a Hamiltonian function of vector field $E_i^{\#}$. † Therefore, we can apply Lemma 3.16 to get

$$\bigcup_j^N C_j \xlongequal{\text{Proposition 3.14}} \bigcap_{\theta \in \mathbb{T}^m} \mathrm{Fix}(\psi_\theta) \overset{\mathbb{T}^{E_i} \subseteq \mathbb{T}^m}{\subseteq} \bigcap_{\theta \in \mathbb{T}^{E_i}} \mathrm{Fix}(\psi_\theta) \xlongequal{\text{Lemma 3.16}} \mathrm{Crit}(\mu^{E_i}) = \mathrm{Crit}(\mu_i).$$

Thus, C_j's are in $\mathrm{Crit}(\mu_i) \implies \mathrm{d}\mu_i = 0$ on C_j's $\implies \mu_i$ constant on C_j's $\implies \mu$ constant on C_j's. In symbols,

$$\mu(C_j) = \eta_j \in \mathbb{R}^m, \quad 1 \le j \le N.$$

Since $\mu(M)$ is convex, the convex hull of $\{\eta_1, \cdots, \eta_N\} \subseteq \mu(M)$, which is the smallest convex set containing these points, has to be contained inside $\mu(M)$.

†More explicitly, $A \exp t\xi = A(e^{2\pi it\xi_1}, \cdots, e^{2\pi it\xi_{m-1}}) = \left(e^{2\pi i \sum_{j=1}^{m-1} a_{1j} t\xi_j}, \cdots, e^{2\pi i \sum_{j=1}^{m-1} a_{mj} t\xi_j}\right)$, which is the same as $\exp(tA\xi) = \exp(t\sum_{j=1}^{m-1} a_{1j}\xi_j, \cdots, t\sum_{j=1}^{m-1} a_{mj}\xi_j)$.

†let $X = E_i = (0, \cdots, 0, 1, 0, \cdots, 0)$. Then $\mu^{E_i} = \langle \mu(p), E_i \rangle = \langle (\mu_1(p), \cdots, \mu_m(p)), (0, \cdots, 0, 1, 0, \cdots, 0) \rangle = \mu_i(p)$. Thus, $\mu_i = \mu^{E_i}$ is a Hamiltonian function of the vector field $E_i^{\#}$.

We show the converse, i.e., $\mu(M) \subseteq \operatorname{conv}\{\eta_1, \cdots, \eta_N\} =: K$: we want to show that the points in $\mathbb{R}^m \setminus K$ is also in $\mathbb{R}^m \setminus \mu(M)$.

Let $\alpha \in \mathbb{R}^m \setminus K$, so there is a hyperplane $\langle n, x\rangle = c$ for some vector n and some real number c such that α is on one side, say the side with all x such that $\langle n, x\rangle > c$; while the whole K is on the other side, $\{x|\langle n, x\rangle\} < c$. Thus, $\forall x \in K$, $\langle n, \alpha\rangle > \langle n, x\rangle$, which is equivalent of saying $\forall j = 1, \cdots, N$, $\langle n, \alpha\rangle > \langle n, \eta_j\rangle$. [†] Now, the vector $n \in \mathbb{R}^m$ can be approximated by a vector in $\mathbb{R}^m$ whose components are rationally independent (see defn. 3.19) and we will use θ to denote it, so $\langle \theta, \alpha\rangle > \langle \theta, \eta_j\rangle$ for $j = 1, \cdots, N$. Rational independency implies $\mathbb{T}^\theta = \mathbb{T}^m$. Proposition 3.14 and Lemma 3.16 again give

$$\bigcup_{j=1}^{N} C_j = \bigcap_{\tau \in \mathbb{T}^n} \operatorname{Fix}(\psi_\tau) = \operatorname{Crit}(\mu^\theta).$$

Since M is compact, μ^θ must have maximum on some $q \in M$. In that case, the gradient vanishes at q. But gradient vanishes exactly at critical set, so $q \in C_j$ for some j. Since $\mu(C_j) = \eta_j$, we see the maximum is $\mu^\theta(q) = \langle \mu(p), \theta\rangle = \langle \eta_j, \theta\rangle$, which is smaller than $\langle \theta, \alpha\rangle$. Thus,

$$\langle \theta, \alpha\rangle > \mu^\theta(q) = \max_{p \in M} \mu^\theta(p) \geq \mu^\theta(p) = \langle \mu(p), \theta\rangle \text{ for any } p \in M.$$

This strict inequality over all $p \in M$ shows that α cannot be any one of $\mu(p)$ (otherwise violating this inequality). That is, $\alpha \notin \mu(M)$, or $\alpha \in \mathbb{R}^m \setminus \mu(M)$. ■

3.5 Example: Complex Projective Space Continued

We go back to $(\mathbb{CP}^n, \omega_{\mathrm{FS}}, \mathbb{T}^{n+1}, \phi_{\mathrm{red}} \circ f)$. It is good to see $(\mathbb{CP}^n, \omega_{\mathrm{FS}}, \mathbb{T}^{n+1}, \phi_{\mathrm{red}} \circ f)$ is a reduced space and is equipped with a Hamiltonian torus action. We now compute the fixed set $\operatorname{Fix}(\psi_g)$ for all possible cases of $g = (e^{i\theta_0}, \cdots, e^{i\theta_n})$ in $\mathbb{T}^{n+1}$. Specifically, we examine the cases when:

(1) All θ_i are distinct.

(2) Exactly two θ_i are equal.

(3) Exactly l of the θ_i are equal.

(4) All θ_i are equal (i.e., $g \in$ the diagonal subgroup).

(1): If $g = (e^{i\theta_0}, \cdots, e^{i\theta_n})$ and all θ_i are distinct, then there is no nonzero solution to the fixed point condition:

$$[e^{i\theta_0} z_0, \cdots, e^{i\theta_n} z_n] = [z_0, \cdots, z_n]$$

This is because no two coordinates can be scaled by the same complex phase unless the corresponding θ_i are equal. Therefore:

$$\operatorname{Fix}(\psi_g) = \{[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], \cdots, [0 : 0 : \cdots : 1]\}$$

These are the coordinate points corresponding to each basis vector being nonzero while all others are zero.

(2) Suppose $\theta_i = \theta_j$ for exactly one pair $i \neq j$, and all other θ_k are distinct. Then the fixed set consists of:

- $\mathbb{CP}^1$, corresponding to the projectivization of the 2-plane spanned by the i-th and j-th coordinates:

$$\mathbb{CP}^1 = \{[0 : \cdots : 0 : z_i : 0 : \cdots : 0 : z_j : 0 : \cdots : 0] \mid [z_i : z_j] \in \mathbb{CP}^1\}$$

- All coordinate points, because each coordinate point has only one nonzero entry and is thus fixed under any torus action:

$$\{[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], \cdots, [0 : 0 : \cdots : 1]\}$$

[†] $\Longrightarrow$ is clear. The other direction is by noticing that $\lambda_j\langle n, \alpha\rangle > \lambda_j\langle n, \eta_j\rangle \implies \sum_j \lambda_j\langle n, \alpha\rangle > \sum_j \lambda_j\langle n, \eta_j\rangle \implies \langle n, \alpha\rangle > \sum_j \lambda_j\langle n, \eta_j\rangle$ as $\sum \lambda_j = 1$.

Therefore:

$$\mathrm{Fix}(\psi_g) = \mathbb{CP}^1 \cup \{[1:0:\cdots:0],[0:1:\cdots:0],\cdots,[0:0:\cdots:1]\}.$$

(3) If exactly l of the θ_i are equal and the remaining $n+1-l$ are distinct, then similarly,

$$\mathrm{Fix}(\psi_g) = \mathbb{CP}^{l-1} \cup \{[1:0:\cdots:0],[0:1:\cdots:0],\cdots,[0:0:\cdots:1]\}$$

(4) If $g = (e^{i\theta},\cdots,e^{i\theta})$, then the action is trivial on all of $\mathbb{CP}^n$. Therefore:

$$\mathrm{Fix}(\psi_g) = \mathbb{CP}^n$$

Combining all of these, we see

$$\mathrm{Fix}(\mathbb{T}^{n+1}) = \bigcap_{g\in\mathbb{T}^{n+1}} \mathrm{Fix}(\psi_g) = \{[1:0:\cdots:0],[0:1:\cdots:0],\cdots,[0:0:\cdots:1]\}$$

Recall the moment map is given by

$$\phi_{\mathrm{red}} \circ f([z_0:z_1:\cdots:z_n]) = -\frac{1}{2\|z\|^2}\left(|z_0|^2,\cdots,|z_n|^2\right)(+\overbrace{\text{constant}}^{C}) = -\frac{1}{2}(x_0,x_1,\cdots,x_n)\,(+C)$$

where we let $x_i = |z_i|^2/\|z\|^2$. Note that these x_i's have the properties that $x_i \geq 0$ and $x_0+x_1+\cdots+x_n = 1$. This shows that $(x_0,x_1,\cdots,x_n)$ lies on the standard n-simplex in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{(x_0,x_1,\cdots,x_n)\,\middle|\, x_i \geq 0, \sum_{i=0}^n x_i = 1\right\}$$

Then,

$$\mathrm{Im}(\phi_{\mathrm{red}}\circ f) = -\frac{1}{2}\Delta^n(+C) = \mathrm{Conv}\left\{\left(-\frac{1}{2},0,\cdots,0\right)+C,\cdots,\left(0,0,\cdots,-\frac{1}{2}\right)+C\right\}$$

which is indeed supported by the images of fixed points in $\mathrm{Fix}(\mathbb{T}^{n+1})$.

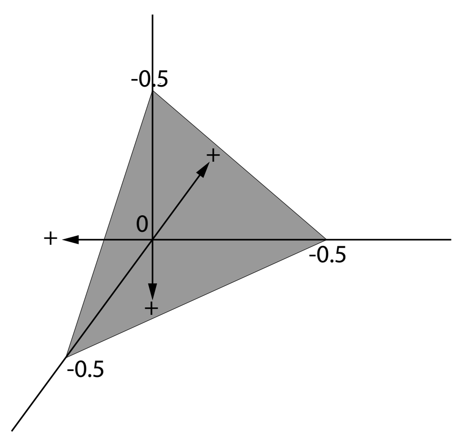


Figure 2: Moment polytope $\mathrm{Im}(\phi_{\mathrm{red}}\circ f)$ when $n=2$.

In contrast, if we consider the action η of $\mathbb{T}^n$ over $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ by

$$\left(e^{i\theta_1}, \cdots, e^{i\theta_n}\right) \cdot [z_0 : z_1 : \cdots : z_n] = \left[z_0 : e^{i\theta_1} z_1 : \cdots : e^{i\theta_n} z_n\right]$$

with moment map

$$\nu\,[z_0 : z_1 : \cdots : z_n] = -\frac{1}{2\|z\|^2}\left(|z_1|^2, \cdots, |z_n|^2\right) = -\frac{1}{2}\left(x_1, \cdots, x_n\right),$$

then the $n+1$ fixed points are

$$[1, 0, \cdots, 0], \cdots, [0, 0, \cdots, 1]$$

and the image of ν is

$$\mathrm{Im}(\nu) = \mathrm{Conv}\left\{(0, \cdots, 0), \left(-\frac{1}{2}, 0, \cdots, 0\right), \cdots, \left(0, 0, \cdots, -\frac{1}{2}\right)\right\},$$

a pyramid with apex at origin and bottom a dilated standard $(n-1)$-simplex in $\mathbb{R}^n$.

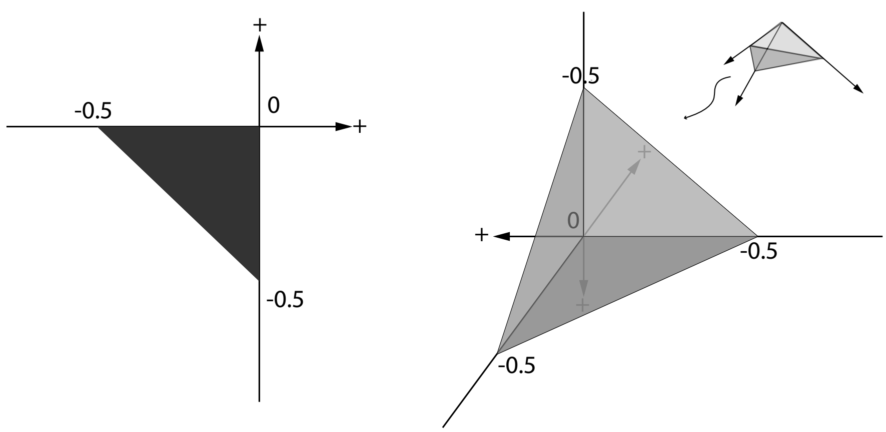


Figure 3: Moment polytopes $\mathrm{Im}(\nu)$ when $n = 2, 3$.

4 Symplectic Toric Manifolds and Delzant Polytopes

4.1 Symplectic Toric Manifolds

An action of a group G on a manifold M is called **effective** if each group element $g \neq e$ moves at least one $p \in M$, that is,

$$\bigcap_{p \in M} G_p = \{e\}$$

where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

Theorem 4.1. *Effectiveness of the action implies that the set of regular points of $\mu : M \to \mathbb{R}^m$, i.e., the set of points where $\mathrm{d}\mu_p$ is surjective, is open and dense in M.*

Proof. We note from Proposition 2.29 that $\mathrm{d}\mu_p$ surjective $\iff$ the action is locally free at p, i.e., the stabilizer subgroup G_p is discrete and thus zero dimensional.

We then consider the set of $p \in M$ whose stabilizer is zero-dimensional, that is the complement of those points $q \in M$ whose stabilizer is at least one-dimensional. As every connected component of a Lie group has the same dimension, we can only consider below connected subgroups of dimension at least one. Notice also that a stabilizer subgroup is closed.

Let $K < G = \mathbb{T}^m$ be a connected closed subgroup of dimension at least 1. It is thus comppact, Lie, and abelian, so a torus $\mathbb{T}^k$ of some periods. If we choose $X \in \mathrm{Lie}(K) \subseteq \mathrm{Lie}(G)$ with components which are incommensurable with those periods, then the one-parameter subgroup $\mathbb{T}^X = \{\exp t\xi | t \in \mathbb{R}\}$ is dense in K (so by density and continuity, $\mathrm{Fix}(\mathbb{T}^X) = \mathrm{Fix}(K)$). Thus, by Lemma 3.16 we have

- $\mathrm{Crit}(\mu^X) = \mathrm{Fix}(\mathbb{T}^X) = \mathrm{Fix}(K)$; and
- μ^X is a Morse-Bott function on M.

If μ^X is constant, the critical set $\mathrm{Crit}\left(\mu^X\right)$ is the whole M; if not constant, then $\mathrm{Crit}(\mu^X)$ is a smooth submanifold of lower dimension. In the latter case, at any $p \in M$, $\mathrm{d}\mu^X(p)$ has rank 0 (where p is critical) or 1 (where p is regular as 1 is full rank), meaning that the regular points, where $\mathrm{d}\mu^X$ is surjective, are exactly $M \backslash \mathrm{Crit}\left(\mu^X\right)$. The complement of a lower-dimensional submanifold is always open (open submanifold is the same dimension as the ambient manifold) and dense (measure of a lower-dimensional set is zero), so the set of regular points is open and dense in M. In the former case that μ^X is constant, we have $0 = \mathrm{d}\mu^X = \omega(\xi^\#, \cdot) \implies \xi^\# = 0 \implies \gamma(t) = \psi_{\exp t\xi}(p)$ is constant on p. $\xi^\#(p)$ is zero for each $p \in M$, so for any t, $\psi_{\exp t\xi}$ is an action fixing the whole M. Therefore, if we assume the effectiveness of the action, this former case will be eliminated.

In all, G effective $\implies \{p \in M \mid K \subsetneq G_p\} = M \setminus \mathrm{Fix}(K) = M \setminus \mathrm{Crit}(\mu^X) = \mathrm{Reg}(\mu^X)$ is an open dense subset of M. Now, if we denote $\mathcal{K}$ the set of all stabilizer subgroups K of G with dimension at least one (so in particular they are all closed connected subgroup with dimension at least one, and the above argument can apply), then we can consider the set

$$\bigcap_{K \in \mathcal{K}} \{p \in M \mid K \subsetneq G_p\} = \{p \in M \mid G_p \text{ contains no } K \text{ in } \mathcal{K}\} \subseteq \{p \in M \mid \dim(G_p) = 0\}$$

which is shown to be open dense in M if we show $\mathcal{K}$ is a countable collection and use Baire's Category theorem.

Suggested by Jordan Payette's reply on stackexchange, to show that this collection is countable, there is two instructive ways:

(1) Recall that stabilizers are compact Lie subgroups and have as such finitely many connected components. The number of closed connected subgroups $K < G$ is countable; this follows from arguing on the possible closure of one-parameter subgroups of G. Thus stabilizers subgroups belong to a countable family.

(2) A result related to what Guillemin-Sternberg call the Koszul-Mostow theorem in their book *Symplectic Techniques in Physics* [GS90] states that for a smooth group action by a compact Lie group on a compact manifold, the number of conjugacy classes of stabilizer subgroups is finite. When the group is abelian, it follows that the number of stabilizers is finite.

■

Corollary 4.2. *Under the conditions of the convexity theorem, if the $\mathbb{T}^m$-action is effective, then there must be at least $m + 1$ fixed points.*

Proof. If the $\mathbb{T}^m$-action is effective, the theorem above implies that there must be a point p where the moment map is a submersion, i.e., $(\mathrm{d}\mu_1)_p, \cdots, (\mathrm{d}\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate convex polytope. Any nondegenerate convex polytope in $\mathbb{R}^m$ must have at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. ■

Theorem 4.3. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then $\dim M \geq 2m$.*

Proof. On an orbit $\mathcal{O}$, the moment map $\mu(\mathcal{O}) = \xi$ is constant by equivariance condition (or invariance condition in light of Remark 2.20). For $p \in \mathcal{O}$, the exterior derivative

$$\mathrm{d}\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker \mathrm{d}\mu_p \xlongequal{\text{Lemma 2.28}} (T_p\mathcal{O})^\omega$$

which shows that orbits $\mathcal{O}$ of a Hamiltonian torus action are always isotropic submanifolds of M. Since isotropic subspace has dimension less than half of that of the vector space (Corollary 4.7; the proof is at the end of this subsection), we see $\dim \mathcal{O} = \dim T_p\mathcal{O} \leq \frac{1}{2}\dim T_pM = \frac{1}{2}\dim M$. ■

Definition 4.4. *A* ***(symplectic) toric manifold*** *is a compact connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold:*

$$\dim \mathbb{T} = \frac{1}{2}\dim M$$

and with a choice of a corresponding moment map μ.

Example 4.5. The action of $\mathbb{T}^{n+1}$ over the complex projective space satisfies the conditions of the convexity theorem. However, the action is not effective, as we have seen that elements in the diagonal subgroup of $\mathbb{T}^{n+1}$ acts trivially over the whole space, i.e., fixing the whole space. The action of $\mathbb{T}^n$ over the complex projective space is effective because the only element in $\mathbb{T}^n$ that acts trivially is the identity. Besides, $\dim \mathbb{T}^n = \frac{1}{2}\dim \mathbb{CP}^n$. Thus, there exists an effective action of a torus over $\mathbb{CP}^n$ and $\mathbb{CP}^n$ is a symplectic toric manifold. ♣

Lemma 4.6. *Given a linear subspace Y of a symplectic vector space (V, Ω), its symplectic complement Y^Ω is the linear subspace defined by $Y^\Omega := \{v \in V \mid \Omega(v, u) = 0$ for all $u \in Y\}$. Then, $\dim Y + \dim Y^\Omega = \dim V$.*

Proof. Let $Y \subseteq V$ be a subspace, and consider the map

$$\begin{array}{rcl} \Phi : V & \longrightarrow & Y^* = \mathrm{Hom}(Y, \mathbb{R}) \\ v & \longmapsto & \Omega(v, \cdot)|_Y \end{array}$$

That is, $\Phi(v) = (v \lrcorner \omega)\,|_Y$. Suppose φ is an arbitrary element of Y^*, and let $\tilde{\varphi} \in V^*$ be any extension of φ to a linear functional on all of V. Since the map $\widetilde{\Omega} : V \to V^*$ defined by $v \mapsto v \lrcorner \Omega$ is an isomorphism, there exists $v \in V$ such that $v \lrcorner \Omega = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the rank-nullity law, $Y^\Omega = \operatorname{Ker}\Phi$ has dimension equal to $\dim V - \dim Y^* = \dim V - \dim Y$. ■

Corollary 4.7. *Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2}\dim V$.*

Proof. Since $Y \subseteq Y^\Omega$ and thus $\dim Y \leq \dim Y^\Omega$, Lemma 4.6 implies that $\dim V = \dim Y + \dim Y^\Omega \geq \dim Y + \dim Y \implies \frac{1}{2}\dim V \geq \dim Y$. ■

4.2 Unimodular Polytopes

Native to algebraic geometry, toric manifolds have been studied by symplectic geometers as examples of extremely symmetric Hamiltonian spaces, and as guinea pigs for new theorems. Delzant showed that symplectic toric manifolds are classified (as Hamiltonian spaces) by a set of special polytopes.

Definition 4.8 (Unimodular Polytope)**.** *A convex polytope $\Delta \subset \mathbb{R}^n$ is called **Delzant**, or **unimodular** if it satisfies*

- *(Simplicity) there are n edges meeting at each vertex,*
- *(Rationality) the edges meeting at the vertex p are rational in the sense that every edge E_k is of the form $p + tu_k$ where $t \in [0, T]$ and $u_k \in \mathbb{Z}^n$,*
- *(Smoothness) for each vertex with edges $E_1, \ldots, E_n$ the corresponding vectors $u_1, \ldots, u_n$ spanning the edges can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

The following lemma will prove very useful for proving that a given set of vectors $u_1, \ldots, u_n$ is indeed a $\mathbb{Z}$-basis:

Lemma 4.9. *The vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if*

$$\det \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} = \pm 1$$

Proof. Since $u_1, \ldots, u_n \in \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if the matrix is invertible, and the matrix is invertible if and only if its determinant is a unit, which in $\mathbb{Z}$ are exactly ± 1, the result follows. ∎

Remark 4.10. The name unimodular comes from the fact that a square integer matrix having determinant $+1$ or -1 is called a **unimodular matrix**. Unimodular matrices form a subgroup of the general linear group under matrix multiplication. Pascal matrices and permutation matrices are unimodular. ♠

Example 4.11 (Unimodular matrices)**.** Permutation matrices are unimodular, although there are only two elements in S_2:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Pascal matrices are unimodular too. Recall that Pascal triangle can be put into a lower-triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

That is, $L_{ij} = \binom{i}{j} = \frac{i!}{j!(i-j)!}, j \leq i$. We use L_n to denote its $n \times n$ truncated version. Then observe that the determinant of a triangular matrix is the product of its diagonal. In this case, the determinant is then just 1. The matrix $A_n = L_n L_n^T$ has $\binom{i+j}{i} = \binom{i+j}{j} = \frac{(i+j)!}{i!j!}$ and $|A_n| = 1$. Consider

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and see Figure 4 for the lattice generated by $(1, 1)$ and $(1, 2)$. ♣

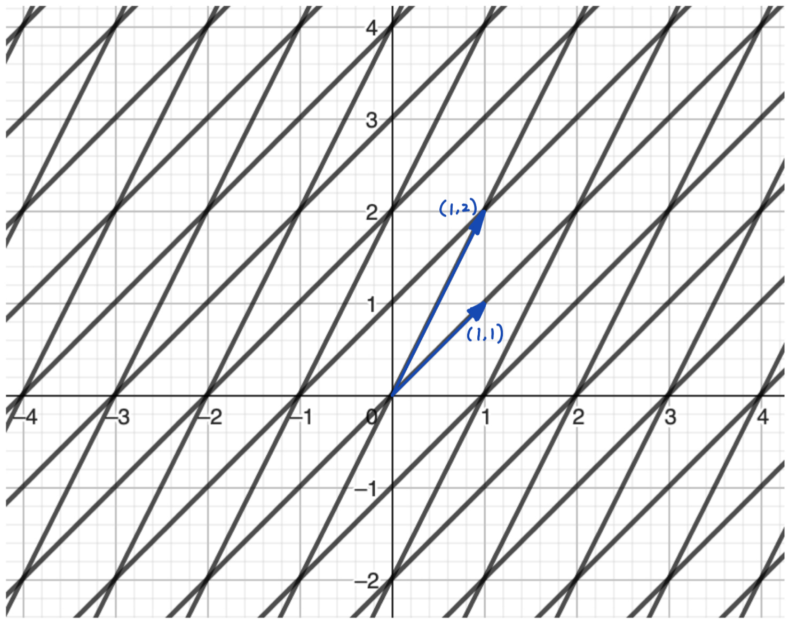
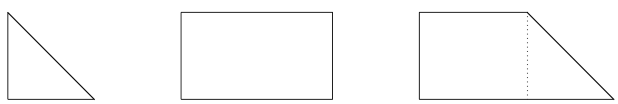
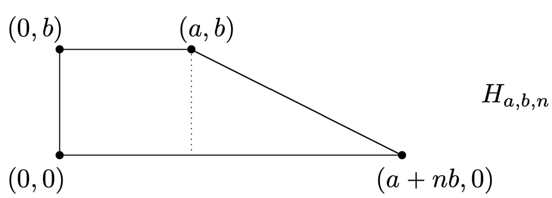


Figure 4: Lattice generated by $(1,1)$ and $(1,2)$

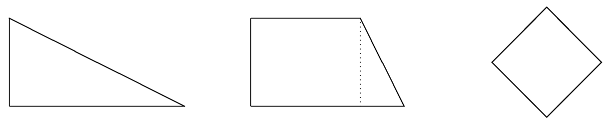
Example 4.12 (Examples of unimodular polytope).



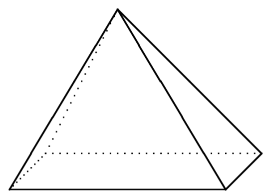
The pictures above represent polytopes in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$, i.e., standard horizontal and vertical cartesian axes with same scale. The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an isosceles triangle. For "taller" triangles, smoothness would be violated. "Wider" triangles may still be unimodular as in the examples below, denoted $H_{a,b,n}$, as long as the slope of the hypothenuse satisfies an integrality condition given by $n = 0, 1, 2, \ldots$ The positive real parameters a and b are the width and height of the left rectangle. We call these examples **Hirzebruch trapezoids**. In particular, $H_{a,b,0}$ is just a rectangle. ♣



Example 4.13 (Non-examples of unimodular polytope).



Once again, the pictures above represent polytopes in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. The picture on the left fails the smoothness condition on the upper vertex (see Figure 5), whereas the one in the middle fails the smoothness condition on the two right vertices, and the one on the right fails the smoothness condition on all vertices. Moreover, the following pyramid in $\mathbb{R}^3$ fails the simplicity condition.



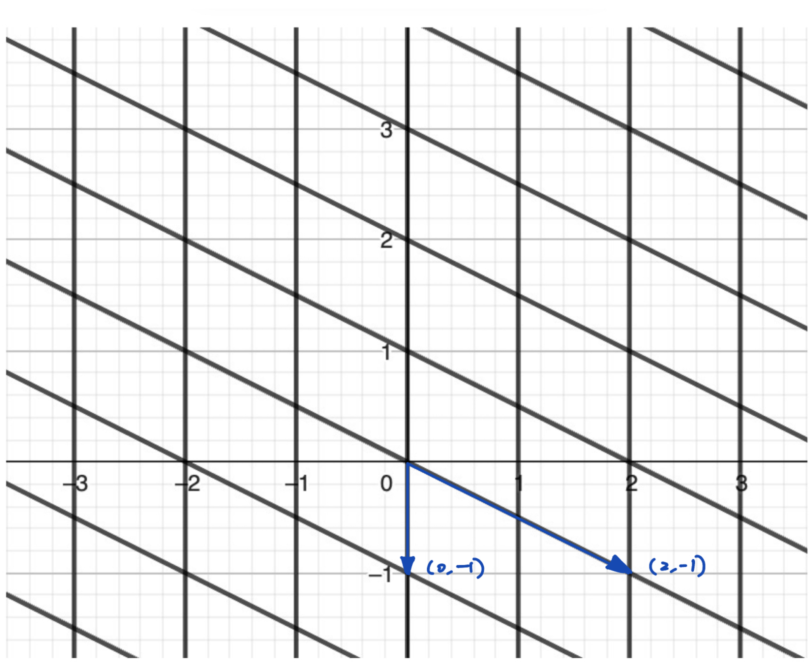


Figure 5: Lattice generated by $(0, -1)$ and $(2, -1)$.

♣

Definition 4.14. *A **facet** of a polytope is a $(n-1)$-dimensional face. Let Δ be a unimodular polytope with $n = \dim \Delta$ and $d =$ number of facets. A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n, k \in \mathbb{Z}$ and $|k| > 1$, or equivalently, if the gcd of entries is equal to one; for instance, $(1, 1), (4, 3), (1, 0)$ are primitive, but $(2, 2), (3, 6)$ are not.*

Let $v_i \in \mathbb{Z}^n, i = 1, \ldots, d$, be the primitive inward-pointing normal vectors to the facets. Then we can describe Δ as an intersection of half-spaces

$$\Delta = \left\{x \in (\mathbb{R}^n)^* \mid \langle x, v_i\rangle \geqslant \lambda_i, i = 1, \ldots, d\right\}$$

for some $\lambda_i \in \mathbb{R}$.

Proposition 4.15. *Let $\Delta \subseteq \mathbb{R}^n$ be a unimodular polytope and v some vertex of Δ. Then there are n facets meeting at v and the primitive inward-pointing normal vectors to these facets form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof. After a translation, we can assume that $v = 0$. Let $u_1, \ldots, u_n$ be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ arising from the unimodular conditions. Then there exists a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ such that $Au_i = e_i$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. Denote by $\Delta' := A\Delta$ the transformed polytope.

One can show that Δ' lies inside the cone based at $v = 0$ spanned by the vectors e_i, see [Zie07, Lemma 3.6]. Now let

$$H_i := \{x \in \mathbb{R}^n \mid \langle e_i, x\rangle = 0\}$$

be the hyperplane at $v = 0$ with primitive inner normal e_i. Then by the above statement, Δ' lies in the positive half-space bounded by H_i. Consider $F_i := \Delta' \cap H_i$. By convexity of Δ' and by definition of H_i, $v + \sum_{j \neq i} t_j e_j \in F_i$ for small enough $t_j > 0$. Moreover, the affine hull of F_i is given by $\left\{v + \sum_{j\neq i} t_j e_j \mid t_j \in \mathbb{R}\right\}$. Thus F_i is $n-1$-dimensional and therefore a facet of Δ'. The primitive inner unit normal at the facet F_i is e_i.

Since A is invertible, there are n facets of Δ meeting at v with outer or inner unit normals $A^{-1}e_i =: v_i$. Since $A \in \mathrm{GL}(n, \mathbb{Z})$ these v_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Let $v_i = lu_i$ for some $u_i \in \mathbb{Z}^n$. Then $e_i = Av_i = lAu_i$ and thus $l = \pm 1$ and we conclude that v_i is primitive. ■

Example 4.16. For an illustrative purpose, we use outward-pointing normals to draw an example:

$$\begin{aligned}
\Delta &= \left\{x \in \left(\mathbb{R}^2\right)^* \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\right\} \\
&= \left\{x \in \left(\mathbb{R}^2\right)^* \mid \langle x, (-1, 0)\rangle \leq 0, \langle x, (0, -1)\rangle \leq 0, \langle x, (1, 1)\rangle \leq 1\right\} \\
&= \left\{x \in \left(\mathbb{R}^2\right)^* \mid \langle x, (1, 0)\rangle \geq 0, \langle x, (0, 1)\rangle \geq 0, \langle x, (-1, -1)\rangle \geq 1\right\}.
\end{aligned}$$

♣

4.3 Delzant Classification Theorem

Moment maps are unique up to elements of the dual of the Lie algebra which annihilate the commutator ideal.

Remark 4.17. The two extreme cases are:

- G semisimple:
  any symplectic action is Hamiltonian,
  moment maps are unique.
- G commutative:
  symplectic actions may not be Hamiltonian,
  moment maps are unique up to any constant $c \in \mathfrak{g}^*$.

♠

We do not have a classification of symplectic manifolds, but we do have a classification of toric manifolds in terms of combinatorial data. This is the content of the Delzant theorem.

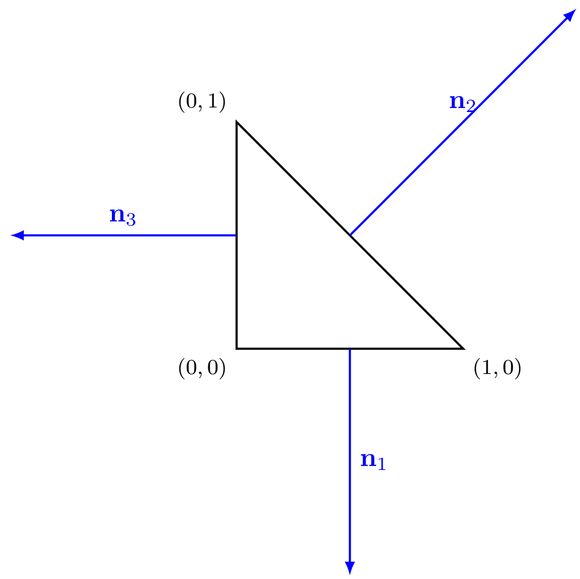


Figure 6: Outward-pointing normals.

Definition 4.18. *Two symplectic toric manifolds,* $(M_k, \omega_k, \mathbb{T}^n, \mu_k)\,, k = 1, 2$*, are* ***isomorphic*** *if there exists an equivariant symplectomorphism* $\varphi : M_1 \to M_2$*, i.e., a symplectomorphism* φ *such that* $\varphi([\theta]\cdot p) = [\theta]\cdot\varphi(p)$*.*

Remark 4.19. Let the actions be denoted by ϕ^i, $i = 1, 2$. We show that $\mu_2 \circ \varphi$ also serves as a moment map for the action ϕ_1. The equivariance condition is easy: in view of Remark 2.20, we see that

$$\mu_x \circ \varphi \circ \phi^1_g = \mu_2 \circ \phi^2_g \circ \varphi = \mu_2 \circ \varphi.$$

We show the Hamiltonian condition. For $q \in M_2$, we have $p \in M_1$ such that $\varphi(p) = q$. Thus, $\varphi(\phi^1_g(p)) = \phi^2_g(\varphi(p)) = \phi^2_g(q)$. Then $\varphi \circ \phi^1_g \circ \varphi^{-1}(q) = \varphi \circ \phi^1_g(p) = \phi^2_g(p)$ and thus $\phi^1_g \circ \varphi^{-1}(q) = \varphi^{-1} \circ \phi^2_g(p)$. Let $\gamma(t) = \phi^2_{\exp tX}(q)$ to see

$$\mathrm{d}\varphi^{-1}(X^{\#}_{\phi^2}(q)) = \mathrm{d}(\varphi^{-1}\circ\gamma)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi^{-1}\phi^2_{\exp tX}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi^1_{\exp tX}(\varphi^{-1}(q)) = X^{\#}_{\phi^1}(\varphi^{-1}(q)).$$

We have,

$$\omega_1(X^{\#}_{\phi^1}(p), \cdot\,) = \omega_1(\mathrm{d}\varphi^{-1}(X^{\#}_{\phi^2}(q)), \cdot\,) = \omega_2(\mathrm{d}\varphi(\mathrm{d}\varphi^{-1}(X^{\#}_{\phi^2}(q))), \mathrm{d}\varphi(\,\cdot\,)) = \omega_2(X^{\#}_{\phi^2}(q), \mathrm{d}\varphi(\,\cdot\,)),$$

Since the moment map satisfies the defining condition $\mathrm{d}\mu^X_i = \iota_{X^{\#}_{\phi^i}}\omega_i$, it follows that

$$\mathrm{d}(\mu^X_1)|_p = \mathrm{d}(\mu^X_2)|_q \circ \mathrm{d}\varphi|_p = \mathrm{d}(\mu^X_2 \circ \varphi)_p = \mathrm{d}(\mu_2 \circ \varphi)^X|_p.$$

Thus, $\mu_2 \circ \varphi$ also serve as a moment map for the action ϕ_1. Since $\mathbb{T}^n$ is commutative, Remark 4.17 (when G is abelian, moment maps are unique up to a constant in $\mathfrak{g}^*$) concludes that μ_1 and $\mu_2 \circ \varphi$ only differ by a constant:

$$\mu_1 = \mu_2 \circ \varphi + c, \quad \text{for some } c \in (\mathrm{Lie}(\mathbb{T}^n))^* \cong \mathbb{R}^n.$$

Notice that we will then get two moment polytopes differed by a translation:

$$\mu_1(M_1) = \mu_2 \circ \varphi(M_2) + c = \mu_2(M_2) + c.$$

♠

The main result in this chapter is the following Delzant's theorem.

Theorem 4.20 (Delzant, [Del88]). *Symplectic toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\frac{\{\textit{symplectic toric manifolds}\}}{\{\textit{isomorphisms}\}} \longleftrightarrow \frac{\{\textit{Delzant polytopes}\}}{\{\textit{translations}\}}$$
$$\left(M^{2n}, \omega, \mathbb{T}^n, \mu\right) \longmapsto \mu(M).$$

Steps of the proof.

1. The map is well-defined: M is toric $\Longrightarrow$ $\mu(M)$ is Delzant. This is a consequence of the equivariant Darboux theorem (see [ACL12] for example).
2. The map is surjective: let $M \mapsto \mu(M)$ be denoted by f and define $g : \Delta \to M_\Delta$ where Δ is Delzant and M_Δ is toric with $\omega_\Delta, \mathbb{T}^n, \mu_\Delta$. The constant involved in μ_Δ can be chosen such that $\mu(M_\Delta) = \Delta$, i.e., $f \circ g = \mathrm{id}$. This will prove the surjectivity of f. We will show this using Delzant's construction of M_Δ in subsection 4.4.
3. The map is injective: we also need to show $g \circ f = \mathrm{id}$. [ACL12, Sections 2.4 and 2.5] show how Lerman did a different construction, i.e., a symplectic toric manifold E^Δ from a given Δ such that $\mu(E^\Delta) = \Delta$ (so surjectivity is fulfilled); but also that if we start with M and let $\Delta = \mu(M)$, then E^Δ is isomorphic to M (this shows injectivity). We will not include Lerman's construction here.

4.4 Delzant's Construction

Let Δ be a Delzant polytope with d facets. Let $v_i \in Z^n, i = 1, \ldots, d$, be the primitive inward-pointing normal vectors to the facets. For some $\lambda_i \in \mathbb{R}$,

$$\Delta = \left\{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geqslant \lambda_i, i = 1, \ldots, d\right\}.$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{aligned} \Pi : \quad \mathbb{R}^d &\longrightarrow \mathbb{R}^n \\ e_i &\longmapsto v_i. \end{aligned}$$

It follows from Proposition 4.15 reformulating the Delzant conditions that Π is surjective and maps $\mathbb{Z}^d$ onto Z^n. Therefore, Π induces a surjective map, still called Π, between tori:

$$\begin{tikzcd} \mathbb{R}^d/(2\pi\mathbb{Z}^d) \arrow[r, "\Pi"] \arrow[d, "="'] & \mathbb{R}^n/(2\pi\mathbb{Z}^n) \arrow[d, "="] & \\ \mathbb{T}^d \arrow[r] & \mathbb{T}^n \arrow[r] & 0 \end{tikzcd}$$

Let

$$\begin{aligned} N &= \text{kernel of } \Pi \ \left(N \text{ is a Lie subgroup of } \mathbb{T}^d\right) \\ \mathfrak{n} &= \text{Lie algebra of } N \\ \mathbb{R}^d &= \text{Lie algebra of } \mathbb{T}^d \\ \mathbb{R}^n &= \text{Lie algebra of } \mathbb{T}^n. \end{aligned}$$

The exact sequence of tori

$$0 \longrightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\Pi} \mathbb{T}^n \longrightarrow 0$$

induces an exact sequence of Lie algebras (by noticing that Lie homomorphisms have constant rank ([Lee12, Theorem 7.5]) and then using global rank theorem ([Lee12, Theorem 4.14])):

$$0 \longrightarrow \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\Pi} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\Pi^*} \left(\mathbb{R}^d\right)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2}\sum \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$, and standard Hamiltonian action of $\mathbb{T}^d$ as in Example 2.26 given by

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_d}\right) \cdot (z_1, \ldots, z_d) = \left(e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d\right)$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow \left(\mathbb{R}^d\right)^*$ defined by

$$\phi\left(z_1, \ldots, z_d\right) = \frac{1}{2}\left(\left|z_1\right|^2, \ldots, \left|z_d\right|^2\right) + \text{ constant}$$

where we will choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. By Example 2.24, the subtorus N acts on $\mathbb{C}^d$ in a Hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^*$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set. Note that Z is connected, because $(i^*)^{-1}(0)$ is a linear subspace of $\mathbb{R}^d$ and the fibers $\phi^{-1}(x)$ are path-connected.

Lemma 4.21. *The set Z is compact and N acts freely on Z.*

We will give the proof of this shortly after we finish getting the reduced space from this using Marsden-Weinstein-Meyer theorem.

$0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \phi$ (free implies locally free at all $z \in Z$, then use Proposition 2.29). Hence, Z is a compact submanifold of $\mathbb{C}^d$ of dimension

$$\dim_{\mathbb{R}} Z = 2d - \underbrace{(d-n)}_{\dim n^*} = d + n$$

The orbit space $M_\Delta = Z/N$ is a compact manifold of dimension

$$\dim_{\mathbb{R}} M_\Delta = d + n - \underbrace{(d-n)}_{\dim N} = 2n$$

The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ.

Consider the diagram

$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathbb{C}^d \\ M_\Delta & \end{tikzcd}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^*\omega_\Delta = j^*\omega_0$$

To show Lemma 4.21, we first observe $\phi(Z) = \Pi^*(\Delta)$, i.e., $y \in \phi(Z) \iff y \in \Pi^*(\Delta)$.

We compute that

$$\operatorname{Im}(\phi) = \phi(\mathbb{C}^d) = \left\{ \frac{1}{2}\left(\left|z_1\right|^2, \ldots, \left|z_d\right|^2\right) + (\lambda_1, \cdots, \lambda_d) \,\middle|\, (z_1, \cdots, z_d) \in \mathbb{C}^d \right\} = \prod_{i=1}^{d} [\lambda_i, \infty)$$

Let $y \in \left(\mathbb{R}^d\right)^*$. Then

$$\begin{aligned} y \in \phi(Z) = \phi((i^* \circ \phi)^{-1}(0)) &\iff \exists z \in (i^* \circ \phi)^{-1}(0) \text{ s.t. } \phi(z) = y \\ &\iff \exists z \in \mathbb{C}^d \text{ s.t. } \phi(z) = y \text{ and } i^*(y) = 0 \\ &\iff y \in \phi(\mathbb{C}^d) = \prod_{k=1}^{d} [\lambda_k, \infty) \text{ and } y \in \ker(i^*) \\ &\iff \langle y, e_k \rangle \geqslant \lambda_k, \forall k \text{ and } y \in \operatorname{Im}(\Pi^*) \end{aligned}$$

Suppose that the second condition holds, so that $y = \Pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$. Then

$$\begin{aligned} \langle y, e_k \rangle \geqslant \lambda_k, \forall k &\iff \langle \Pi^*(x), e_k \rangle \geqslant \lambda_k, \forall k \\ &\iff \langle x, \Pi\left(e_k\right) \rangle \geqslant \lambda_k, \forall k \\ &\iff \langle x, v_k \rangle \geqslant \lambda_k, \forall k \\ &\iff x \in \Delta \end{aligned}$$

Thus,

$$y \in \phi(Z) \iff y \in \Pi^*(\Delta) \tag{11}$$

proof of the lemma 4.21. The set Z is clearly closed, hence in order to show that it is compact it suffices (by the Heine-Borel theorem) to show that Z is bounded. Let $\Delta' = \Pi^*(\Delta)$. Since we have that Δ' is compact, that ϕ is a proper map [†] and that $\phi(Z) = \Delta'$, we conclude that $Z \subseteq \mu^{-1}(\mu(Z))$ must be bounded, and hence compact.

It remains to show that N acts freely on Z. Pick a vertex τ of Δ, and let $I = \{k_1, \ldots, k_n\}$ be the set of indices for the n facets meeting at τ. Pick $z \in Z$ such that $\phi(z) = \Pi^*(\tau)$. Then τ is characterized by n equations $\langle \tau, v_k \rangle = \lambda_k$ where k ranges in I:

$$\begin{aligned} \langle \tau, v_k \rangle = \lambda_k &\iff \langle \tau, \Pi\left(e_k\right) \rangle = \lambda_k \\ &\iff \langle \Pi^*(\tau), e_k \rangle = \lambda_k \\ &\iff \langle \phi(z), e_k \rangle = \lambda_k \\ &\iff k\text{-th coordinate of } \phi(z) \text{ is equal to } \lambda_k \\ &\iff \frac{1}{2}\left|z_k\right|^2 + \lambda_k = \lambda_k \\ &\iff z_k = 0. \end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \ldots, n\}$. The stabilizer of such z for group $\mathbb{T}^d$ is

$$\left(\mathbb{T}^d\right)_z = \left\{\left(e^{i\theta_1}, \ldots, e^{i\theta_n}, 1, \ldots, 1\right) \in \mathbb{T}^d\right\} \supseteq N_z \tag{12}$$

By Proposition 4.15, the restriction of Π to the indices in I is still surjective and thus, for dimensional reasons, bijective. [†] Therefore on the torus level, the restriction to $\left(\mathbb{T}^d\right)_z$ is still bijective. Since $N = \ker(\Pi)$, we have

$$N_z = N \cap \left(\mathbb{T}^d\right)_z = \{e\}.$$

Thus N acts freely at each z that corresponds to a vertex. But for all the points in Z, the stabilizer subgroups of such z's are the largest. [†] Therefore, N acts freely over the whole Z. ■

[†]A map between topological spaces is called **proper** if inverse images of compact subsets are compact. One can use Heine-Borel theorem again to show ϕ is proper.

[†]This is the crucial step where we used that the polytope is Delzant.

[†]Note that for a point $z \in Z$ that does not correspond to a vertex, there is at least one equation $\langle p, v_k \rangle = \lambda_k$ violated, i.e., $z_k \neq 0$ where $\phi(z) = \Pi^*(p)$. Then $(\mathbb{T}^d)_z$ will be larger for this nonzero z_k forces another $e^{i\theta_k}$ to be 1 in eq.(12).

Given a Delzant polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta = Z/N$ is a compact $2n$-dimensional manifold and ω_Δ is the reduced symplectic form. We show the following:

Proposition 4.22. *The manifold $(M_\Delta, \omega_\Delta)$ inherits a Hamiltonian $\mathbb{T}^n$-action with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.*

Proof. Let z be such that $\phi(z) = \Pi^*(\tau)$ where τ is a vertex of Δ, as in the proof of Lemma 4.21. Let $\sigma : \mathbb{T}^n \to \left(\mathbb{T}^d\right)_z$ be the inverse for the earlier bijection $\Pi : \left(\mathbb{T}^d\right)_z \to \mathbb{T}^n$. Since we have found a section in the exact sequence

$$0 \longrightarrow N \xrightarrow{\ i\ } \mathbb{T}^d \underset{\xleftarrow{\sigma}}{\xrightarrow{\Pi}} \mathbb{T}^n \longrightarrow 0$$

the exact sequence splits, i.e., $N \oplus \mathbb{T}^n \cong \mathbb{T}^d$. Then $k^* : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$ induced from $k : \mathbb{R}^n \to \mathbb{R}^d$ is the same as $\sigma^* = \mathrm{pr}_2 : (\mathbb{R}^d)^* \cong \mathfrak{n}^* \oplus (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$. By Lemma 2.24, $(\mathbb{C}^d, \omega_0, \mathbb{T}^n, \sigma^* \circ \phi)$ is a Hamiltonian $\mathbb{T}^n$-space. $\sigma^* \circ \phi$ is clearly constant on $\mathbb{T}^d$-orbits as is ϕ, and commutativity of the group gives commutativity between $\mathbb{T}^d$ and $\mathbb{T}^n$ actions. Thus, we conclude using Proposition 3.3 that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, (\sigma^* \circ \phi)_{\mathrm{red}})$ is a Hamiltonian $\mathbb{T}^n$-space with

$$(\sigma^* \circ \phi)_{\mathrm{red}} \circ \pi = \sigma^* \circ \phi \circ j$$

where $j : Z \to \mathbb{C}^d$ is the inclusion and $\pi : Z \to Z/N$ is the projection to the orbit space. We denote $\mu_\Delta = \sigma^* \circ \phi$. Due to eq.(11), the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ \pi)(Z) = (\sigma^* \circ \phi \circ j)(Z) = \sigma^* \circ \phi(Z) = \sigma^* \circ \Pi^*(\Delta) = (\Pi \circ \sigma)^*(\Delta) = \mathrm{id}(\Delta) = \Delta$$

The above $\mathbb{T}^n$-action is effective because $\mathbb{T}^d$, and hence $\mathbb{T}^n$, acts freely on the open dense subset

$$\phi^{-1}\left(\Pi^*\left(\Delta^o\right)\right) \subset Z$$

where Δ° denotes the interior of Δ. We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required toric manifold corresponding to Δ. ∎

5 Applications

5.1 Motivation/Example: Hermitian Spectra

The following theorem on specturm of Hermitian matrices is known prior to the Atiyah-Guillemin-Sternberg convexity theorem and in fact served as a main motivation for the convexity theorem.

Theorem 5.1 (Schur-Horn Theorem, [Hor54])**.** *Let $d_1, \ldots, d_n$ and $\lambda_1, \ldots, \lambda_n$ be real numbers. There is an $n \times n$ Hermitian matrix with diagonal entries $d_1, \ldots, d_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if the vector $(d_1, \ldots, d_n)$ lies in the convex hull of the set of vectors whose coordinates are all possible permutations of $(\lambda_1, \ldots, \lambda_n)$.*

To see how the convexity theorem implies this result, we need to find appropriate spaces and equip them with symplectic forms and actions.

Let $\mathcal{H}$ be the vector space of $n \times n$ complex Hermitian matrices ξ, i.e., $\xi = \xi^* = \overline{\xi^T} = \bar{\xi}^T$. Consider the unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid AA^* = I\}$. Its Lie algebra, denoted by $\mathfrak{u}(n)$, is the set of skew-Hermitian matrices ξ s.t. $\xi^* = -\xi$. $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation, $\Psi(A)(\xi) = A\xi A^{-1}$, and acts on the dual space $\mathfrak{u}(n)^*$ by coadjoint action $\mathrm{Ad}^* : \mathrm{U}(n) \to \mathrm{GL}(\mathfrak{u}(n)^*)$. We can identify the dual space $\mathfrak{u}(n)^*$ with Hermitian matrices $\mathcal{H}$ via the linear isomorphism $T : \mathcal{H} \to \mathfrak{u}(n)^*; \xi \mapsto \mathrm{tr}(i\xi\, \cdot)$. If we regard Ψ as a representation, it is easy to see T is an intertwining operator, i.e., $\forall A \in \mathrm{U}(n)$, $\mathrm{Ad}^*(A) \circ T = T \circ \Psi(A)$. [†] In particular, it takes orbits in $\mathcal{H}$ to coadjoint orbits in $\mathfrak{u}(n)^*$.

[†] The coadjoint action Ad_g^* is defined by the duality condition: $\langle \mathrm{Ad}_g^*(\xi), X\rangle = \left\langle \xi, \mathrm{Ad}_{g^{-1}}(X)\right\rangle$. Substituting the expressions for $\mathrm{Ad}_{g^{-1}}(X) = g^{-1}X(g^{-1})^{-1}$ (true for any matrix lie group), we get $\mathrm{tr}\left(i\,\mathrm{Ad}_g^*(\xi)X\right) = \mathrm{tr}\left(i\xi g^{-1}Xg\right)$. By the cyclic property of the trace, $\mathrm{tr}\left(i\xi g^{-1}Xg\right) = \mathrm{tr}\left(ig\xi g^{-1}X\right)$, which by the uniqueness of the trace pairing gives $\mathrm{Ad}_g^*(\xi) = g\xi g^{-1}$.

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ complex Hermitian matrices whose spectrum is λ. Recall a linear algebra fact (see for example [RAG05, Theorem 10.18 or 10.19]) that a complex square matrix A is Hermitian if and only if it is unitarily diagonalizable with real eigenvalues. Since conjugation preserves spectrum, the orbit of the action $\mathcal{O}_\xi = \{A\xi A^{-1} \mid A \in (n)\}$ is $\mathcal{H}_\lambda$ with $\lambda = \lambda(\xi)$. By proposition 2.9 (1), $\mathcal{H}_\lambda$ is equipped with a manifold structure. In fact, when all of the eigenvalues in the spectrum λ are the same, the orbit $\mathcal{H}_\lambda$ is a singleton; when all of them are distinct, the orbit $\mathcal{H}_\lambda$ is a complete flag, constructed by n independent eigenvectors; when λ has some identical occurrences of eigenvalues, we get partial flags.

The canonical Kostant-Kirillov symplectic form Ω^{KKS} on the coadjoint orbit (see Appendix) corresponding to $\mathcal{H}_\lambda$ gives rise to a symplectic form Ω on $\mathcal{H}_\lambda$ via T: for $X_\xi^{\#}, Y_\xi^{\#} \in T_\xi \mathcal{H}_\lambda = [\mathfrak{u}(n), \xi]$, $\Omega_\xi(X_\xi^{\#}, Y_\xi^{\#}) = i\,\mathrm{tr}(\xi[Y, Y])$. $\mathcal{H}_\lambda$ is a Hamiltonian $\mathrm{U}(n)$-space.

proof of the Schur-Horn theorem. Let $\mathbb{T}$ denote the Cartan subgroup of $\mathrm{U}(n)$ which consists of diagonal complex matrices with diagonal entries of modulus 1. The Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ consists of diagonal skew-Hermitian matrices and the dual space $\mathfrak{t}^*$ consists of diagonal Hermitian matrices, under the isomorphism T. In other words, $\mathfrak{t}$ consists of diagonal matrices with purely imaginary entries and $\mathfrak{t}^*$ consists of diagonal matrices with real entries. The inclusion map $\mathfrak{t} \hookrightarrow \mathfrak{u}(n)$ induces a map $S : \mathcal{H} \cong \mathfrak{u}(n)^* \to \mathfrak{t}^*$, which projects a matrix ξ to the diagonal matrix with the same diagonal entries as ξ. By Example 2.24, the set $\mathcal{H}_\lambda$ is a Hamiltonian $\mathbb{T}$-space, and the restriction of S to this set is a moment map for this action.

By the Atiyah-Guillemin-Sternberg theorem, $S(\mathcal{H}_\lambda)$ is a convex polytope. A matrix $\xi \in \mathcal{H}$ is fixed under conjugation by every element of $\mathbb{T}$ if and only if ξ is diagonal. The only diagonal matrices in $\mathcal{H}_\lambda$ are the ones with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ in some order. Thus, these matrices generate the convex polytope $S(\mathcal{H}_\lambda)$. This is exactly the statement of the Schur-Horn theorem. ■

The Schur-Horn theorem, which characterizes the relationship between the diagonal entries and eigenvalues of Hermitian matrices, extends naturally within symplectic geometry. [Knu00] reformulates Horn's problem—the description of possible eigenvalues of sums of Hermitian matrices—via moment maps and geometric invariant theory. The Atiyah-Guillemin-Sternberg convexity theorem generalizes Schur-Horn by associating eigenvalue problems with moment polytope descriptions in Hamiltonian spaces.

More precisely, [Knu00] identifies the sum of Hermitian matrices as a moment map for the diagonal conjugation action of $\mathrm{U}(n)$, encoding the admissible eigenvalue spectra in the Horn polytope. Using Schubert calculus and intersection theory on flag varieties, [Knu00] derives eigenvalue inequalities, recasting classical combinatorial results such as Horn's conjecture within a symplectic and representation-theoretic framework. Combinatorially, [KT01] introduces honeycomb models, providing a graphical interpretation of eigenvalue constraints through hexagonal tilings. These structures encode spectral inequalities and generalize the Littlewood-Richardson rule, extending Schur-Horn from individual matrix spectra to eigenvalue inequalities for sums. Together, these developments connect symplectic geometry, algebraic geometry, and combinatorics, offering a unified perspective on spectral constraints.

5.2 Generalization of the Convexity Theorem

Knutson used in his paper [Knu00] the following Kirwan's generalization of the Atiyah-Guillemin-Sternberg convexity theorem for nonabelian Lie groups.

Theorem 5.2 (Kirwan, [Kir84])**.** *Let (M, ω, G, ϕ) be a compact Hamiltonian G-manifold with G a compact Lie group. Then the intersection of the image $\phi(M)$ with the positive Weyl chamber $\mathfrak{t}_+^*$ let $\mathfrak{t}_+^*$ be a positive Weyl chamber for a maximal compact subgroup K of G is a convex polytope.*

Weinstein in [Wei01] further generalized the convexity theorem that to noncompact cases. The paper also includes a brief survey on generalizations of the convexity theorem.

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $X \in \mathfrak{g}$, let T^X be the 1-parameter subgroup of G generated by X and let G^X be the adjoint isotropy group $\{g \in G \mid \mathrm{Ad}_g(X) = X\}$. We shall say that μ is **stable** if T^X is compact and **strongly stable** if G^X is compact. Let $\mathcal{D}$ denote the set of all strongly stable elements of $\mathfrak{g}$.

Let $\mathcal{U} \subseteq \mathfrak{g}^*$ be a coadjoint-invariant open subset. We define a **Hamiltonian** $(G,\mathcal{U})$**-space** (M,μ) to be a symplectic manifold M with a symplectic G-action and a coadjoint-equivariant momentum map $\mu : M \to \mathcal{U} \subseteq \mathfrak{g}^*$. We shall consider μ as a map to $\mathcal{U}$ rather than to $\mathfrak{g}^*$ and will call the $(G,\mathcal{U})$ space **proper** if μ is a proper mapping and if the action of G on M is proper. By [Wei01, Lemma 2.12], the second condition follows from the first if the coadjoint action of G on $\mathcal{U}$ is proper, which happens when G is semisimple and $\mathcal{U}$ consists of strongly stable elements.

Theorem 5.3 (Weinstein, [Wei01])**.** *Let G be a semisimple Lie group, let $\mathfrak{t}^*_+$ be a positive Weyl chamber for a maximal compact subgroup K of G, and let $\mathcal{U}$ be a coadjoint-invariant open subset of the set $\mathcal{D} \subset \mathfrak{g}^*$ such that $\mathcal{U} \cap \mathfrak{t}^*_+$ is convex. If (M,μ) is a connected, proper, Hamiltonian $(G,\mathcal{U})$-space, then $\mu(M) \cap \mathfrak{t}^*_+$ is a closed, convex, locally polyhedral subset of $t^*_+ \cap \mathcal{U}$, and $\mu^{-1}(\xi)$ is connected for each $\xi \in \mathcal{U}$.*

We do not include here the proof of this theorem for noncompact semisimple Lie group G, which leverages a reduction to the known compact case on Hamiltonian $(K,\mathcal{U} \cap \mathcal{E})$-space $N := \mu^{-1}(\mathcal{U} \cap \mathcal{E})$ where $\mathcal{E} = \mathcal{D} \cap \mathfrak{t}$, adopting the identification betwen $\mathfrak{t}$ and $\mathfrak{t}^*$ using bi-invariant metric on K.

6 Conclusion: Reduction, Convexity, and Unimodularity in Symplectic Geometry

This thesis has explored the role of reduction, convexity, and unimodularity in symplectic geometry, focusing on classical results in Hamiltonian group actions. The Marsden-Weinstein-Meyer theorem and the Atiyah-Guillemin-Sternberg convexity theorem provide foundational tools for understanding the geometry of moment maps and symplectic quotients. A key application of these ideas is the classification of symplectic toric manifolds via Delzant polytopes, which serve as a geometric realization of unimodular polytopes in this setting. By reviewing these results and their interplay, this work offers a structured exposition of well-established theorems rather than original contributions. However, organizing these ideas cohesively helps clarify their significance and the underlying connections between symplectic geometry and combinatorial structures.

While the main results are classical, the discussion of Hamiltonian actions by semisimple Lie groups extends the context beyond toric geometry, touching on the challenges that arise in noncompact settings. The structure of coadjoint orbits and momentum convexity remains an active area of research, with potential links to geometric quantization and representation theory. Future work could further explore how these methods apply to broader classes of symplectic manifolds, including settings where the torus action is replaced by more general group actions, or where moment convexity properties extend beyond compact cases.

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Appendix

We record two formulas about Lie derivatives and brackets.

If X is a vector field and $f \in C^\infty(M)$, $\mathrm{d}f$ being the corresponding 1-form, then

$$Xf \xlongequal{\text{[Lee12, Thm.4.24(iv)]}} \overbrace{\mathrm{d}f(X)}^{=\iota_X \mathrm{d}f} \xlongequal{\text{[Lee12, Prop.12.32(a)]}} \mathcal{L}_X f \tag{13}$$

eq.(13) gives $\mathcal{L}_{X_f} g = \iota_{X_f} \mathrm{d}g$ (one could also see this by Cartan's magic formula, noticing the convention at [Lee12, pp.358] that $\iota_V \eta = 0$ for any zero-covector field η, i.e., a function). Then by definition of Hamiltonian function, $\mathrm{d}g = \iota_{X_g}\omega$,

$$\begin{aligned} \mathcal{L}_{X_f} g &= \iota_{X_f} \mathrm{d}g \\ &= \iota_{X_f}\iota_{X_g}\omega \\ &= \omega(X_g, X_f) \\ &= -\{f, g\} \end{aligned} \tag{14}$$

proof of equivalence between equivariance and lie algebra homomorphism.
First assume the G-action is Hamiltonian. Then for any $X, Y \in \mathfrak{g}$,

$$\begin{aligned} \left\{\mu^X, \mu^Y\right\}(p) &\xlongequal{\text{eq.(13)}} X_{\mu^Y}(\mu^X)(p) \xlongequal{\mathrm{d}\mu^Y = \iota_{Y^\#}\omega} [Y^\#(\mu^X)](p) \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Psi_{\exp tY}(p)\right](\mu^X) \\ &\xlongequal{\text{let } f: t\mapsto \Psi_{\exp tY}(p)} \mathrm{d}f_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\right)(\mu^X) \\ &\xlongequal{\text{defn. of differential; see [Lee12, pp.55]}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu^X \circ f \\ &\xlongequal{\text{defn. of } \mu^X: M\to\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle \mu(\Psi_{\exp tY}(p)), X\rangle \\ &\xlongequal{\text{equivariance}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left\langle \mathrm{Ad}^*_{\exp tY}(\mu(p)), X\right\rangle \\ &\xlongequal{\text{defn. of coadjoint}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle \mu(p), \mathrm{Ad}_{\exp -tY} X\rangle \\ &\xlongequal{\text{by [Lee12, Prop.20.8(g), Thm.20.27]}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left\langle \mu(p), \left(\overbrace{\exp(\underbrace{\mathrm{ad}(-tY)}_{\in\mathfrak{gl}(\mathfrak{g})})}^{\in GL(\mathfrak{g})}\right) X \right\rangle \\ &= \left\langle \mu(p), \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(X - t\,\mathrm{ad}(Y)(X) + \frac{t^2}{2!}\mathrm{ad}(Y)^2(X) - \cdots\right)\right\rangle \\ &= \langle \mu(p), -[Y, X]\rangle = \langle \mu(p), [X, Y]\rangle \\ &= \mu^{[X,Y]}(p) \end{aligned} \tag{15}$$

Conversely, suppose μ^* is a Lie algebra homomorphism. Since G is connected and the exponential map $\exp$ is a local diffeomorphism ([Lee12, Proposition 20.8(f)]), any element g of G can be written as a product of elements of the form $\exp(X)$. As a result, to prove G-equivariance it is enough to prove

$$\mu(\Psi_{\exp tX}(p)) = \mathrm{Ad}^*_{\exp(tX)}\,\mu(p)$$

We will need a result analogous to [Lee12, Proposition 9.13]:

Lemma 6.1. *Let M and N be two smooth manifolds and $F : M \to N$ be a smooth map. Let G be a Lie group. Consider the smooth actions $\theta : G \times M \to M$ and $\eta : G \times N \to N$. Let $X, Y \in \mathfrak{g}$. Define*

$$X^{\#}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta_{\exp tX}(p), \quad Y^{\#}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta_{\exp tY}(q).$$

If $X^{\#}$ and $Y^{\#}$ are F-related, i.e., $dF_p(X^{\#}_p) = Y^{\#}_{F(p)}$, $\forall p \in M$, then $\eta_g \circ F = F \circ \theta_g$.

sketch of proof: We note that

$$X^{\#}((\theta_{\exp t_0 X})(p)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta_{\exp(t+t_0)X}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \theta_{\exp(t+t_0)X}(p)$$

which shows that the curve $\gamma(t) : t \mapsto \theta_{\exp tX}(p)$ has the property that $\gamma(0) = p$ and $(d/dt)\gamma(t) = X^{\#}(\gamma(t))$. By F-relatedness, it is easy to see $\sigma = F \circ \gamma$ is the curve with the property that $\sigma(0) = F(p)$ and $(d/dt)\sigma(t) = Y^{\#}(\sigma(t))$. Then $t \mapsto \eta_{\exp tY}(p)$ coincide with $\sigma(t)$ by uniquness from ODE theory.

Now consider the action $\Psi : G \times M \to M$ and $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. Let X^* be the vector field generated by $\mathrm{Ad}^*_{\exp tX}$, i.e.,

$$X^*(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp tX}(\xi)$$

Notice that in eq.(15), we have shown that

$$\underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp tX}(\xi), Y \right\rangle}_{\langle X^*(\xi), Y\rangle} = \langle \xi, [Y, X]\rangle$$

Using this plus the first line of eq.(15) and the preservation of bracket of μ^*, we see that for $Y \in \mathfrak{g}$,

$$\langle X^*(\mu(p)), Y\rangle = \langle \mu(p), [Y, X]\rangle = \mu^{[Y,X]}(p) = \left\{\mu^Y, \mu^X\right\}(p) = [X^{\#}(\mu^Y)](p) = [X^{\#}\langle \mu, Y\rangle](p)$$

With the evaluation map $\overline{Y} : \xi \mapsto \xi(Y)$, we see

$$\langle \mathrm{d}\mu_p(X^{\#}(p)), Y\rangle = \left(\mathrm{d}\mu_p(X^{\#}(p))\right)(Y) = \overline{Y}\left(\mathrm{d}\mu_p(X^{\#}(p))\right) = \mathrm{d}\overline{Y}_{\mu(p)}\mathrm{d}\mu_p X^{\#}_p = \mathrm{d}(\overline{Y} \circ \mu)_p X^{\#}_p$$

where the second-to-last equality is because $\overline{Y}$ is a linear map. Since $\overline{Y} \circ \mu : M \to \mathbb{R}$ is a smooth function, eq.(13) equates the RHS of the last two equations:

$$\mathrm{d}(\overline{Y} \circ \mu)_p X^{\#}_p = [X^{\#}(\overline{Y} \circ \mu)](p) = [X^{\#}\langle \mu, Y\rangle](p)$$

Thus,

$$\mathrm{d}\mu_p(X^{\#}(p)) = X^*(\mu(p))$$

Due to the lemma above, the $\Longleftarrow$ direction is proved. ∎

Example 2.24

First, by Cartan's Closed Subgroup Theorem, H is a Lie subgroup. The map $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual to $i : \mathfrak{h} \to \mathfrak{g}$, so for $f \in \mathfrak{g}^*$ and $X \in \mathfrak{h}$, we have $\langle i^* f, X\rangle = \langle f, iX\rangle = \langle f, X\rangle$. Thus,

$$\mu^X(p) = \langle \mu(p), X\rangle = \langle i^*(\phi(p)), X\rangle = \langle \phi(p), X\rangle = \phi^X(p) \implies \mu^X = \phi^X.$$

The Hamiltonian condition then follows. For the equivariance condition, it can be mostly easily seen from the comoment map characterization (see Proposition 2.21): $\forall X \in \mathfrak{h}$, $\mu^X = \phi^X$, so $\phi^*|_{\mathfrak{h}} = \mu^*$. Now, $\phi^* : (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\})$ is a Lie homomorphism, so does its restriction to $\mathfrak{h}$. This gives the equivariance condition.

Example 2.25

Let $\omega = (\mathrm{pr}_1)^* \omega_1 + (\mathrm{pr}_2)^* \omega_2$ be the symplectic form on $M_1 \times M_2$ (it is easy to verify that it is a symplectic form by computation; see [Sil06, Chapter 4])

Hamiltonian condition:

Let $X \in \mathfrak{g}^*$. $\langle \mu(p_1,p_2), X\rangle = \langle \mu_1(p_1), X\rangle + \langle \mu_2(p_2), X\rangle \implies \mu^X(p_1,p_2) = \mu_1^X(\mathrm{pr}_1(p_1,p_2)) + \mu_2^X(\mathrm{pr}_2(p_1,p_2))$. Then,

$$\begin{aligned}
&\omega\left(X^{\#}, v\right)\\
=&\omega_1\left(\mathrm{d}(\mathrm{pr}_1)X^{\#}, \mathrm{d}(\mathrm{pr}_1)v\right) + \omega_2\left(\mathrm{d}(\mathrm{pr}_2)X^{\#}, \mathrm{d}(\mathrm{pr}_2)v\right)\\
=&\omega_1\left(X_{M_1}(p_1), \mathrm{d}(\mathrm{pr}_1)v\right) + \omega_2\left(X_{M_2}(p_2), \mathrm{d}(\mathrm{pr}_2)v\right)\\
=&\left(\iota_{X_{M_1}}\omega_1\right)(\mathrm{d}(\mathrm{pr}_1)v) + \left(\iota_{X_{M_2}}\omega_2\right)(\mathrm{d}(\mathrm{pr}_2)v)\\
=&(\mathrm{d}\mu_1^X)(\mathrm{d}(\mathrm{pr}_1)v) + (\mathrm{d}\mu_2^X)(\mathrm{d}(\mathrm{pr}_2)v)\\
&= \mathrm{d}\mu^X(v)
\end{aligned}$$

Equivariant condition:

$$\begin{aligned}
\mathrm{Ad}^*_{g^{-1}}\left(\mu\left(p_1,p_2\right)\right) &= \mathrm{Ad}^*_{g^{-1}}\left(\mu_1\left(p_1\right) + \mu_2\left(p_2\right)\right)\\
&= \mathrm{Ad}^*_{g^{-1}}\left(\mu_1\left(p_1\right)\right) + \mathrm{Ad}^*_{g^{-1}}\left(\mu_2\left(p_2\right)\right)\\
&= \mu_1\left(g\cdot p_1\right) + \mu_2\left(g \cdot p_2\right)\\
&= \mu(g\cdot(p_1,p_2))
\end{aligned}$$

Example 2.26

Let $(E_1,\cdots,E_n)$ be the basis of $\mathfrak{t}^n \cong \mathbb{R}^n$ and $(\varepsilon^1,\cdots,\varepsilon^n)$ be the basis of $(\mathfrak{t}^n)^* \cong \mathbb{R}^n$. For each component z_j in $\mathbb{C}^n$, write:

$$z_j = r_j e^{i\theta_j} = r_j\cos\theta_j + ir_j\sin\theta_j,$$

where $r_j = |z_j|$ represents the modulus and θ_j the argument of z_j. In terms of these coordinates, we have

$$\omega = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j = \sum_{j=1}^n r_j \mathrm{d}r_j \wedge \mathrm{d}\theta_j.$$

The proposed moment map becomes

$$\mu(z_j) = \mu(r_j,\theta_j) = -\frac{1}{2}\left(k_1 r_1^2,\cdots,k_n r_n^2\right) + \text{ constant } = -\frac{1}{2}\sum_{i=1}^n k_i r_i^2 \varepsilon^i + \text{ constant.}$$

Hamiltonian condition:

Let $X = \sum_j X_j E_j \in \mathfrak{t}^n \cong \mathbb{R}^n$ and $z = (z_1,\cdots,z_n) \in \mathbb{C}^n$.

$$\langle \mu(z), X\rangle = -\frac{1}{2}\sum_{j=1}^n k_j r_j^2 \varepsilon^j(X) + \text{constant} = -\frac{1}{2}\sum_{j=1}^n k_j r_j^2 X_j + \text{constant}.$$

Then

$$\mathrm{d}\mu^X = -\frac{1}{2}\sum_{j=1}^n k_j X_j \mathrm{d}\left(r_j^2\right) = -\sum_{j=1}^n k_j X_j r_j \mathrm{d}r_j.$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp tX_j \cdot z_j = \frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\theta=0} e^{itX_jk_j} z_j = iX_jk_jz_j$ and by paying attention to remark **??**, we see

$$X^{\#}(z_j) = X^{\#}(r_j, \theta_j) = \sum_{j=1}^{n} ik_jX_jz_j\frac{\partial}{\partial z_j} - ik_jX_j\bar{z}_j\frac{\partial}{\partial \bar{z}_j} = \sum_{j=1}^{n} k_jX_j\frac{\partial}{\partial \theta_j}.$$

Then

$$\iota_{X^{\#}}\omega = \sum_{j=1}^{n} r_j\left(0 - \mathrm{d}\theta_j(X^{\#})\mathrm{d}r_j\right) = -\sum_{j=1}^{n} r_j\mathrm{d}r_j\,(k_jX_j) = -\sum_{j=1}^{n} k_jX_jr_j\mathrm{d}r_j = \mathrm{d}\mu^X.$$

<u>Equivariant condition</u>:

Note that $\mathbb{T}^n$ is abelian so by Remark 2.20, we need to check $\mu((t_1, \cdots, t_n)\cdot(z_1, \cdots, z_n)) = \mu(z_1, \cdots, z_n)$. This is true because $|t_j| = 1 \implies |t_j^{k_j}z_j|^2 = |z_j|^2$.

Coadjoint Orbits

Let $X^{\#}$ be the vector field generated by $X \in \mathfrak{g}$ for the coadjoint representation of G on $\mathfrak{g}^*$. Then for any $Y \in \mathfrak{g}$, $\left\langle X^{\#}_\xi, Y\right\rangle = \langle \xi, [Y, X]\rangle$. For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by $\omega_\xi(X, Y) := \langle \xi, [X, Y]\rangle$. By eq.(5), we have $\mathrm{Lie}(\mathfrak{g}_\xi) = \{X \in \mathfrak{g} | X^{\#}_\xi = 0\} = \{X \in \mathfrak{g} | \langle \xi, [Y, X]\rangle = 0, \forall Y \in \mathfrak{g}\} = \ker \widetilde{\omega}_\xi$. From eq.(4) we see the tangent space of an orbit is $T_\xi\mathcal{O}_\xi = \{X^{\#}_\xi | X \in \mathfrak{g}\}$. Then the 2-form $\Omega_\xi(X^{\#}_\xi, Y^{\#}_\xi) = \omega_\xi(X, Y) = \langle \xi, [X, Y]\rangle$ is nondegenerate. We will show that ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^*$ so that Ω is a symplectic form on the coadjoint orbit $\mathcal{O}_\xi$ in $\mathfrak{g}^*$. This is known as the **Lie-Poisson** or **Kostant-Kirillov symplectic structure**.

We still use $X^{\#}, Y^{\#}, Z^{\#}$ to denote their restrictions on the orbit. Then, by [Lee12, Proposition 14.32], we have

$$\begin{aligned}(\mathrm{d}\Omega)(X^{\#}, Y^{\#}, Z^{\#}) =& X^{\#}(\Omega(Y^{\#}, Z^{\#})) - Y^{\#}(\Omega(X^{\#}, Z^{\#})) + Z^{\#}(\Omega(X^{\#}, Y^{\#})) \\ &- \Omega([X^{\#}, Y^{\#}], Z^{\#}) - \Omega([Z^{\#}, X^{\#}], Y^{\#}) - \Omega([Y^{\#}, Z^{\#}], X^{\#}) \\ =& \langle X^{\#}_\xi, [Y, Z]\rangle - \langle Y^{\#}_\xi, [X, Z]\rangle + \langle Z^{\#}_\xi, [X, Y]\rangle \\ &- \langle \xi, [[X, Y], Z]\rangle - \langle \xi, [[Z, X], Y]\rangle - \langle \xi, [[Y, Z], X]\rangle = 0\end{aligned}$$

We evaluate $X^{\#}(\Omega(Y^{\#}, Z^{\#}))$ pointwise: for every ξ on the coadjoint orbit,

$$\begin{aligned}X^{\#}_\xi(\overbrace{\Omega_\xi(Y^{\#}_\xi, Z^{\#}_\xi)}^{f:\mathcal{O}_\xi\to\mathbb{R};\xi\mapsto\Omega_\xi(Y^{\#}_\xi,Z^{\#}_\xi)}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\overbrace{\mathrm{Ad}^*_{\exp tX}(\xi)}^{\gamma(t)}) \quad \text{because } Xf = \mathrm{d}fX = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\circ\gamma \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\xi(\mathrm{Ad}_{\exp -tX}(\,\cdot\,))) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \xi(\mathrm{Ad}_{\exp -tX}([Y, Z])) \\ &= \xi\left(-[X, [Y, Z]]\right) \quad \text{pass by linearity and use eq.(15)}\end{aligned}$$

The first line vanishes by writing all three components in this way and apply Jacobi identity. The second line also vanishes by Jacobi identity as

$$\begin{aligned}&-\Omega_\xi([X^{\#}_\xi, Y^{\#}_\xi], Z^{\#}_\xi) - \Omega([Z^{\#}_\xi, X^{\#}_\xi], Y^{\#}_\xi) - \Omega([Y^{\#}_\xi, Z^{\#}_\xi], X^{\#}_\xi) \\ =& -\langle \xi, [[X, Y], Z]\rangle - \langle \xi, [[Z, X], Y]\rangle - \langle \xi, [[Y, Z], X]\rangle\end{aligned}$$

This shows the 2-form is closed. For other proofs other than this algebraic one, see [Kir04, pp.6].