Riemannian Geometry

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October 21, 2024

*Lecture note for Prof. Renato Feres's Math 5047 (FL24); mostly based on [5].

Contents

Chapter 1

Basics

1.1 Tensor Algebra

1.1.1 Tensors and Their Products

We generalize from linear functionals to multilinear ones. If $V_1, \ldots, V_k$ and W are vector spaces, a map $F : V_1 \times \cdots \times V_k \to W$ is said to be **multilinear** if it is linear as a function of each variable separately, when all the others are held fixed:

$$F\left(v_1, \ldots, a v_i + a' v_i', \ldots, v_k\right) = a F\left(v_1, \ldots, v_i, \ldots, v_k\right) + a' F\left(v_1, \ldots, v_i', \ldots, v_k\right).$$

Given a finite-dimensional vector space V, a **covariant** k**-tensor on** V is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Similarly, a **contravariant** k**-tensor on** V is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R}.$$

We often need to consider tensors of mixed types as well. A **mixed tensor of type** (k, l), also called a k**-contravariant,** l**-covariant tensor**, is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R}.$$

Actually, in many cases it is necessary to consider real-valued multilinear functions whose arguments consist of k covectors and l vectors, but not necessarily in the order implied by the definition above; such an object is still called a tensor of type (k, l). For any given tensor, we will make it clear which arguments are vectors and which are covectors. The spaces of tensors on V of various types are denoted by

$$\begin{aligned} T^k\left(V^*\right) &= \{\text{ covariant } k\text{-tensors on } V\}; \\ T^k(V) &= \{\text{ contravariant } k\text{-tensors on } V\}; \\ T^{(k,l)}(V) = T^k_l(V) &= \{\text{ mixed } (k,l)\text{-tensors on } V\}. \end{aligned}$$

The **rank** of a tensor is the number of arguments (vectors and/or covectors) it takes. By convention, a 0-tensor is just a real number.

There is a natural product, called the **tensor product**, linking the various tensor spaces over V: if $F \in T^{(k,l)}(V)$ and $G \in T^{(p,q)}(V)$, the tensor $F \otimes G \in T^{(k+p,l+q)}(V)$ is defined by

$$\begin{aligned} F \otimes G \left(\omega^1, \ldots, \right. & \left. \omega^{k+p}, v_1, \ldots, v_{l+q}\right) \\ & = F\left(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l\right) G\left(\omega^{k+1}, \ldots, \omega^{k+p}, v_{l+1}, \ldots, v_{l+q}\right) \end{aligned}$$

The tensor product is associative, so we can unambiguously form tensor products of three or more tensors on V. If (b_i) is a basis for V and $\left(\beta^j\right)$ is the associated dual basis, then a basis for $T^{(k,l)}(V)$ is given by the set of all tensors of the form

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l},$$

as the indices i_p, j_q range from 1 to n. These tensors act on basis elements by

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l}\left(\beta^{s_1}, \ldots, \beta^{s_k}, b_{r_1}, \ldots, b_{r_l}\right) = \delta_{i_1}^{s_1} \cdots \delta_{i_k}^{s_k} \delta_{r_1}^{j_1} \cdots \delta_{r_l}^{j_l}.$$

It follows that $T^{(k,l)}(V)$ has dimension n^{k+l}, where $n = \dim V$. Every tensor $F \in T^{(k,l)}(V)$ can be written in terms of this basis (using the summation convention) as

$$F = F_{j_1 \cdots j_l}^{i_1 \cdots j_k} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l} \tag{1.1}$$

where

$$F_{j_1 \cdots j_l}^{i_1 \cdots i_k} = F\left(\beta^{i_1}, \ldots, \beta^{i_k}, b_{j_1}, \ldots, b_{j_l}\right).$$

If the arguments of a mixed tensor F occur in a nonstandard order, then the horizontal as well as vertical positions of the indices are significant and reflect which arguments are vectors and which are covectors. For example, if A is a $(1,2)$-tensor whose first argument is a vector, second is a covector, and third is a vector, its basis expression would be written

$$A = A_i{}^j{}_k \beta^i \otimes b_j \otimes \beta^k,$$

where

$$A_i{}^j{}_k = A\left(b_i, \beta^j, b_k\right)$$

There are obvious identifications among some of these tensor spaces:

$$\begin{aligned} & T^{(0,0)}(V) = T^0(V) = T^0\left(V^*\right) = \mathbb{R}, \\ & T^{(1,0)}(V) = T^1(V) = V, \\ & T^{(0,1)}(V) = T^1\left(V^*\right) = V^*, \\ & T^{(k,0)}(V) = T^k(V), \\ & T^{(0,k)}(V) = T^k\left(V^*\right). \end{aligned}$$

Due to [4] prop.12.10, we also write

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}},$$

defined as $F(V^{\times k} \times V^{*\times l})/R$, where F is the free vector space on basis $V^{\times k} \times V^{*\times l}$, or the set of all finite formal linear combinations of (k,l)-tuples, and R is the subspace of F spanned by all elements of the following forms

$$\begin{aligned}
(v_1,\cdots,av_i,\cdots,v_k,\omega_1,\cdots,\omega_l)-a(v_1,\cdots,v_i,\cdots,v_k,\omega_1,\cdots,\omega_l)\\
(v_1,\cdots,v_k,\omega_1,\cdots,a\omega_i\cdots,\omega_l)-a(v_1,\cdots,v_k,\omega_1,\cdots,\omega_i\cdots,\omega_l)\\
(v_1,\cdots,v_i+v_i',\cdots,v_k,\omega_1,\cdots,\omega_l)-(v_1,\cdots,v_i,\cdots,v_k,\omega_1,\cdots,\omega_l)-(v_1,\cdots,v_i',\cdots,v_k,\omega_1,\cdots,\omega_l)\\
(v_1,\cdots,v_k,\omega_1,\cdots,\omega_i+\omega_i'\cdots,\omega_l)-(v_1,\cdots,v_k,\omega_1,\cdots,\omega_i\cdots,\omega_l)-(v_1,\cdots,v_k,\omega_1,\cdots,\omega_i'\cdots,\omega_l)
\end{aligned}$$

Let $\Pi : F(V^{\times k}\times V^{*\times l})\to T^{(k,l)}=F(V^{\times k}\times V^{*\times l})/R$ be the natural projection. The equivalence class of an element $(v_1,\cdots,v_k,\omega_1,\cdots,\omega_l)$ in $T^{(k,l)}(V)$ is denoted by

$$v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l=\Pi(v_1,\cdots,v_k,\omega_1,\cdots,\omega_l)=(v_1,\cdots,v_k,\omega_1,\cdots,\omega_l)+R.$$

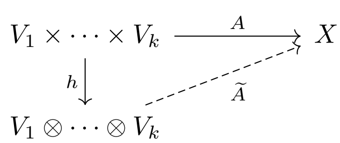
and is called **(abstract) tensor product of** $v_1,\cdots,v_k,\omega_1,\cdots,\omega_l$. We note that $(v_1,\cdots,v_k,\omega_1,\cdots,\omega_l)$

It folllws from the definition that abstract tensor product satisfy

$$\begin{aligned}
v_1\otimes\cdots\otimes av_i\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l&=a(v_1\otimes\cdots\otimes v_i\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l)\\
v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes a\omega_i\cdots\otimes\omega_l&=a(v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_i\cdots\otimes\omega_l)\\
v_1\otimes\cdots\otimes(v_i+v_i')\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l&=v_1\otimes\cdots\otimes v_i\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l\\
&\quad+v_1\otimes\cdots\otimes v_i'\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l\\
v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes(\omega_i+\omega_i')\cdots\otimes\omega_l&=v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_i\cdots\otimes\omega_l\\
&\quad+v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_i'\cdots\otimes\omega_l
\end{aligned}$$

Note that the definition implies that every element of $T^{(k,l)}(V)$ can be expressed as a linear combination of elements of the form $v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l$; but it is not true in general that every element of the tensor product space is of the form $v_1\otimes\cdots\otimes v_k\otimes\omega_1\otimes\cdots\otimes\omega_l$.

Proposition 1.1.1 (Characteristic Property of the Tensor Product Space)**.** *Let $V_1,\cdots,V_k$ be finite-dimensional real vector spaces. If $A : V_1\times\cdots\times V_k\to X$ is any multilinear map into a vector space X, then there is a unique linear map $\widetilde{A} : V_1\otimes\cdots\otimes V_k\to X$ such that the following diagram commutes:*



where h is the composition $h=\Pi\circ i$ of the maps $\Pi : F\to F/R$ and $i : V_1\times\cdots\times V_k\hookrightarrow F$. Explicitly,

$$h(v_1,\cdots,v_k)=v_1\otimes\cdots\otimes v_k$$

Proof. See [4] Proposition 12.7. ■

Proposition 1.1.2. *Above characterization of tensor product is unique up to isomorphism.*

Proof. See Rotman's *An Introduction to Homological Algebra* (e2) Proposition 2.44. ■

Proposition 1.1.3 (Abstract vs. Concrete Tensor Products)**.** *If $V_1,\cdots,V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^*\otimes\cdots\otimes V_k^*\cong\mathcal{L}(V_1,\cdots,V_k;\mathbb{R})$$

under which the abstract tensor product defined by

$$v_1\otimes\cdots\otimes v_k=\Pi(v_1,\cdots,v_k)=(v_1,\cdots,v_k)+R$$

corresponds to the tensor product of covectors defined by

$$\omega^1 \otimes \cdots \otimes \omega^k(v_1, \cdots, v_k) = \omega^1(v_1)\cdots\omega^k(v_k).$$

The isomorphism $\widetilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ *is the map induced by* $\Phi : V_1^* \times \cdots \times V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ *defined by* $\Phi(\omega^1, \cdots, \omega^k)(v_1, \cdots, v_k) = \omega^1(v_1)\cdots\omega^k(v_k)$ *through the universal property 1.1.1.*

Proof. See [4] Proposition 12.10. ∎

Proposition 1.1.4 (Second Dual Space)**.** *There is a canonical isomorphism between* $V^{**} := (V^*)^*$ *and* V*, namely, the isomorphism sending* v *to its* ***evaluation map*** $\bar{v}$*, defined by*

$$\begin{aligned} \bar{v} : V^* &\to \mathbb{R} \\ \omega &\mapsto \omega(v) \end{aligned}$$

Proof. See [4] Proposition 11.8. ∎

We introduce an extremely important identification

$$T^{(1,1)}(V) \cong \mathrm{End}(V),$$

where $\mathrm{End}(V)$ denotes the pace of linear maps from V to itself (also called the **endomorphisms of** V). This is a special case of the following proposition.

Proposition 1.1.5. *Let* V *be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between* $T^{(k+1,l)}(V)$ *and the space of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to V.$$

Lemma 1.1.6. *Let* $\dim V_j = n_j$ *and* $\dim W = n$ *then*

$$\dim \mathcal{L}\left(V_1, \cdots, V_k; W\right) = \sum_{i=1}^{n} \prod_{j=1}^{k} n_j = n n_1 n_2 \cdots n_k$$

Proof. That's because

$$\mathcal{L}\left(V_1, \cdots, V_k; W\right) \cong \mathcal{L}\left(V_1, \cdots, V_k; \mathbb{R}^n\right) \cong \bigoplus_{i=1}^{n} \mathcal{L}\left(V_1, \cdots, V_k; \mathbb{R}\right)$$

and the fact that $\bigoplus_{i=1}^n X_i$ has dimension $\sum \dim X_i$ and

$$\dim \mathcal{L}\left(V_1, \cdots, V_k; \mathbb{R}\right) = \dim V_1 \cdot \cdots \cdot \dim V_k = n_1 \cdots n_k$$

∎

Lemma 1.1.7. *Let* V *be a vector space and* $v \neq 0$ *be a vector in it. There exists a linear mapping* $f : V \to \mathbb{R}$ *such that* $f(v) \neq 0$*.*

Proof. Suppose $V = \mathrm{span}(x_1, \cdots, x_n)$. Let $M = \mathrm{span}(v)$ as in [2] Theorem 1.10.20. Then there is a subspace $H = \mathrm{span}(x_{i_1}, \cdots, x_{i_k})$ such that $V = M \oplus H$. Now define $f(v) = 1$ and $f(x_{i_1}) = \cdots = f(x_{i_k}) = 0$ and extend them linearly to be defined on other vectors in V. ∎

First proof of the proposition.
(1) Case $k=0, l=1$:

Proposition 1.1.3 gives $T^{(1,1)}(V)=V\otimes V^*\cong\mathcal{L}(V^*,V;\mathbb{R})$. We then define the mapping

$$\begin{aligned}\Phi:\operatorname{End}(V)&\to\mathcal{L}(V^*,V;\mathbb{R})\\ A\mapsto\Phi A&:=\begin{pmatrix} V^*\times V & \to\mathbb{R}\\ (\omega,v) & \mapsto\omega(Av)\end{pmatrix}\end{aligned}$$

Since there is a canonical isomorphism between V and V^{**} by Proposition 1.1.4, we let the isomorphism be denoted by $\tau:V^{**}\to V$. We then define the inverse of Φ as below.

$$\begin{aligned}\Psi:\mathcal{L}(V^*,V;\mathbb{R})&\to\operatorname{End}(V)\\ f\mapsto\Psi f&:=\begin{pmatrix} V & \to V\\ v & \mapsto\tau(f(\cdot,v))\end{pmatrix}\end{aligned}$$

where we note that $f(\cdot,v)$ is a map from V^* to $\mathbb{R}$ and thus belongs to V^{**}.

We show that $\Phi(\Psi f)=f$ and $\Psi(\Phi A)=A$:

- $\Phi(\Psi f)=f$. That is, we need to show $\Phi(\Psi f)(\omega,v)=f(\omega,v)$. We compute that

$$\begin{aligned}\Phi(\Psi f)(\omega,v)=\omega((\Psi f)(v))&=\omega(\underbrace{\tau(f(\cdot,v))}_{\xi})\\ &=\overline{\xi}(\omega)=[f(\cdot,v)](\omega)=f(\omega,v)\end{aligned}$$

  where we note that $\tau:V^{**}\to V$ and the evaluation $\overline{\cdot}:V\to V^{**}$ are inverse of each other.

- $\Psi(\Phi A)=A$. That is, we need to show that $\Psi(\Phi A)(v)=A(v)$. We compute that

$$\begin{aligned}\Psi(\Phi A)(v)&=\tau((\Phi A)(\cdot,v))\\ &=\tau(\cdot(Av))\end{aligned}$$

  Note that $\cdot(Av)$ sends every ω to $\omega(Av)$ and therefore equals to $\overline{Av}$. Thus,

$$\begin{aligned}\Psi(\Phi A)(v)&=\tau(\overline{Av})\\ &=Av\end{aligned}$$

  where we again notice that τ and $\overline{\cdot}$ are inverses of each other.

(2) General case:

We similarly consider

$$\begin{aligned}\Phi:\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)&\to\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k+1\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};\mathbb{R})\\ A\mapsto\Phi A&:=\begin{pmatrix} V^*\times\cdots\times V^*\times V\times\cdots\times V\to\mathbb{R}\\ (\omega^1,\cdots,\omega^{k+1},v_1,\cdots,v_l)\mapsto\omega^{k+1}\left(A\left(\omega^1,\cdots,\omega^k,v_1,\cdots,v_l\right)\right)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\Psi:\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k+1\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};\mathbb{R})&\to\mathcal{L}(\underbrace{V^*,\cdots,V^*}_{k\text{ copies}},\underbrace{V,\cdots,V}_{l\text{ copies}};V)\\ f\mapsto\Psi f&:=\begin{pmatrix} V^*\times\cdots\times V^*\times V\times\cdots\times V\to V\\ (\omega^1,\cdots,\omega^k,v_1,\cdots,v_l)\mapsto\tau(f\left(\omega^1,\cdots,\omega^k,\cdot,v_1,\cdots,v_l\right))\end{pmatrix}\end{aligned}$$

■

Second proof of the proposition.
(1) Case $k = 0, l = 1$:

Proposition 1.1.3 gives $T^{(1,1)}(V) = V \otimes V^* \cong \mathcal{L}(V^*, V; \mathbb{R})$. We then define the mapping

$$\begin{aligned} \Phi : \operatorname{End}(V) &\to \mathcal{L}(V^*, V; \mathbb{R}) \\ A \mapsto \Phi A &:= \begin{pmatrix} V^* \times V & \to \mathbb{R} \\ (\omega, v) & \mapsto \omega(Av) \end{pmatrix} \end{aligned}$$

It is easy to see that the map Φ is well-defined and linear. Let $\dim V = n$. Notice that $\dim V = \dim V^* = n$. Then by lemma 1.1.6 we see $\dim T^{(1,1)}(V) = \dim \mathcal{L}(V^*, V; \mathbb{R}) = n^2$ and $\dim \operatorname{End}(V) = \dim \mathcal{L}(V, V) = n^2$. Thus, it suffices to show that Φ is injective: for $A, B \in \operatorname{End}(V)$ we want to show that $\Phi A = \Phi B \Rightarrow A =$ B. $\Phi A = \Phi B$ implies that for any fixed $v \in V$, the following is true:

$$\begin{aligned} \forall \omega \in V^*, \qquad \Phi A(\omega, v) &= \Phi B(\omega, v) \\ \omega(Av) &= \omega(Bv) \\ \omega(Av - Bv) &= 0 \end{aligned}$$

Now, $\forall \omega \in V^*$, $\omega(Av - Bv) = 0$ implies that $Av - Bv$ cannot be a nonzero vector, because for if it is a nonzero vector, then lemma 1.1.7 implies that we can find some $\omega \in V^*$ such that ω sends it elsewhere. Therefore, for any fixed v, $Av - Bv = (A - B)v = 0 \implies A - B$, which sends every vector to zero, is a zero mapping. Thus, $A = B$.

(2) General case: consider the mapping

$$\begin{aligned} \Phi : \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; V) &\to \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k+1 \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R}) \\ A \mapsto \Phi A &:= \begin{pmatrix} V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R} \\ (\omega^1, \cdots, \omega^{k+1}, v_1, \cdots, v_l) \mapsto \omega^{k+1}\left(A\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right)\right) \end{pmatrix} \end{aligned}$$

We similarly only need to show injectivity: suppose

$$\omega^{k+1}\left(A\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right)\right) = \omega^{k+1}\left(B\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right)\right)$$

Then by the same argument, $\forall \omega^1, \cdots, \omega^k, v_1, \cdots, v_l$,

$$\begin{aligned} A\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right) &= B\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right) \\ (A - B)\left(\omega^1, \cdots, \omega^k, v_1, \cdots, v_l\right) &= 0 \end{aligned}$$

$A - B$ is then a zero mapping in $\mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; V)$, so $A = B$. ■

Third proof of the proposition. We cite [6] to give another argument:

[6] Theorem 2.11: There is a natural isomorphism between $\mathcal{L}(V_1, V_2; W)$ and $\mathcal{L}(V_1, \mathcal{L}(V_2, W))$.

[6] Theorem 4.1:

(i) $V_0^2 = V \otimes V \cong \mathcal{L}(V^*, V)$

(ii) $V_1^1 = V \otimes V^* \cong V^* \otimes V \cong \mathcal{L}(V, V) \cong \mathcal{L}(V^*, V^*)$

(iii) $V_2^0 = V^* \otimes V^* \cong \mathcal{L}(V, V^*)$

[6] Theorem 2.12: There is a natural isomorphism between $\mathcal{L}(V_1, V_2, \cdots, V_p; W)$ and $\mathcal{L}(V_i, \mathcal{L}(V_1, \cdots, \widehat{V}_i, \cdots, V_p; W))$.

For the special case $k = 0, l = 1$, let $V_1 = V^*, V_2 = V, W = \mathbb{R}$ in [6] Theorem 2.11 to get

$$\mathcal{L}(V^*, V; \mathbb{R}) \cong \mathcal{L}(V^*, \mathcal{L}(V, \mathbb{R})) = \mathcal{L}(V^*, V^*) \overset{[6]\ 4.1}{\cong} \mathcal{L}(V, V) = \operatorname{End}(V)$$

For the general case, observe the following **corollary**:

$$\begin{aligned}\mathcal{L}(V_1, \cdots, V_p; W) &\overset{[6]\ 2.12}{\cong} \mathcal{L}(V_i, \mathcal{L}(V_1, \cdots, \widehat{V}_i, \cdots, V_p; W)) \\ &\overset{[6]\ 2.12}{\cong} \mathcal{L}(V_i, \mathcal{L}(V_j, \mathcal{L}(V_1, \cdots, \widehat{V}_i, \cdots, \widehat{V}_j \cdots, V_p; W))) \\ &\overset{\text{use backwards [6] 2.12}}{\cong} \mathcal{L}(V_i, V_j; \mathcal{L}(V_1, \cdots, \widehat{V}_i, \cdots, \widehat{V}_j, \cdots, V_p; W)).\end{aligned}$$

Then

$$\begin{aligned}T^{(k+1,l)}(V) &= \underbrace{V \otimes \cdots \otimes V}_{k+1 \text{ copies}} \times \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}} \\ &\overset{[4]\ 12.10}{\cong} \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k+1 \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R}) \\ &\overset{[6]\ 2.12}{\cong} \mathcal{L}(\underbrace{V^*}_{i\text{-th}}, \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R})) \\ &\overset{\text{cor}}{\cong} \cdots \overset{\text{cor}}{\cong} \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}; \mathcal{L}(V^*, \underbrace{V, \cdots, V}_{l \text{ copies}}; \mathbb{R})) \\ &\cong \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; \underbrace{\mathcal{L}(V^*, \mathbb{R})}_{= V^{**} \cong V}) \\ &\cong \mathcal{L}(\underbrace{V^*, \cdots, V^*}_{k \text{ copies}}, \underbrace{V, \cdots, V}_{l \text{ copies}}; V)\end{aligned}$$

■

1.1.2 Contractions

We can use the result of proposition 1.1.5 to define a natural operation called **trace** or **contraction**, which lowers the rank of a tensor by 2. In one special case, it is easy to describe: the operator $\operatorname{tr} : T^{(1,1)}(V) \to \mathbb{R}$ is just the trace of f when it is regarded as an endomorphism of V, or in other words the sum of the diagonal entries of any matrix representation of F.

Recall the following results from basic linear algebra.

Definition 1.1.8. *If T is any linear transformation which maps vector space V of dimension n to vector space W of dimension m, there is always an $m \times n$ matrix A with the property that*

$$Tx = Ax, \qquad \forall x \in V$$

*Let $(E_1, \cdots, E_n)$ be a basis for V and $(\varepsilon^1, \cdot, \varepsilon^m)$ be a basis for W, then the **matrix of linear transformation** A is*

$$A = \begin{bmatrix} | & & | \\ T(E_1) & \cdots & T(E_n) \\ | & & | \end{bmatrix}$$

Proposition 1.1.9. *The sum of the eigenvalues λ_i of the matrix $A \in M_n(\mathbb{R})$ is equal to its trace, i.e., $\sum_{i=1}^n \lambda_i = \operatorname{tr} A$. Besides, $\prod_{i=1}^n \lambda_i = \det A$.*

Proposition 1.1.10. *Let $\mathcal{B}$ and $\mathcal{C}$ be any two bases of the vector space V, and let $\tau \in \mathcal{L}(V, V) = \operatorname{End}(V)$ be a linear endomorphism. Then the eigenvalues and eigenvectors are invariant under change of basis:*

$$[\tau]_{\mathcal{B}}[v]_{\mathcal{B}} = \lambda[v]_{\mathcal{B}} \Rightarrow [\tau]_{\mathcal{C}}[v]_{\mathcal{C}} = \lambda[v]_{\mathcal{C}}$$

Proof. Recall the following change of basis formula (see [7] Corollary 2.17 for (2) below for instance):

(1) $[v]_{\mathcal{C}} = \mathcal{M}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}}$;

(2) $[\tau]_{\mathcal{C}} = \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}\mathcal{M}_{\mathcal{B},\mathcal{C}}^{-1}$.

Then the assertion directly follows from the computation:

$$\begin{aligned}
[\tau]_{\mathcal{C}}[v]_{\mathcal{C}} &= \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}\mathcal{M}_{\mathcal{B},\mathcal{C}}^{-1}\mathcal{M}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \\
&= \mathcal{M}_{\mathcal{B},\mathcal{C}}[\tau]_{\mathcal{B}}[v]_{\mathcal{B}} \\
&= \mathcal{M}_{\mathcal{B},\mathcal{C}}\lambda[v]_{\mathcal{B}} \\
&= \lambda\mathcal{M}\mathcal{B}_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \\
&= \lambda[v]_{\mathcal{C}}
\end{aligned}$$

■

In fact, this invariance can also be seen from the fact that an eigenvalue λ of a linear endomorphism $\tau \in \mathcal{L}(V, V) = \operatorname{End}(V)$ is defined by $\tau x = \lambda x$ for some non-zero vector x and the definition does not involve basis. Now the above proposition combined with the formula of the sum of eigenvalues gives the invariance of trace of a linear endomorphism under change of basis.

Corollary 1.1.11. *The trace of a linear endomorphism is well-defined.*

Proposition 1.1.12. *Let $f \in T^{(1,1)}(V)$. Then under the definition of trace given at the beginning, $\operatorname{tr}(f) := \operatorname{tr}(\Psi f) = \sum f_i^i$, where $f_j^i = f(\varepsilon^i, E_j)$ with respect to the basis $(E_1, \cdots, E_n)$ of V and dual basis $(\varepsilon^1, \cdots, \varepsilon^n)$ of V^*.*

Proof. The linear operator here is

$$\begin{aligned}
\Psi f : V &\to V \\
v &\mapsto \tau(f(\cdot, v))
\end{aligned}$$

where Ψ is defined in the second proof of the proposition 1.1.5. We will show that the matrix $[\Psi f]_{(E_k)}$ of Ψf under basis (E_k) is the following, from which we can obtain that the sum of the diagonal elements is $\sum f_i^i$, proving the statement.

$$[\Psi f]_{(E_k)} = \begin{pmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & \ddots & \vdots \\ f_1^n & \cdots & f_n^n \end{pmatrix}$$

By definition 1.1.8, we want to show that

$$\forall 1 \leq k \leq n : \qquad (\Psi f)(E_k) = \sum_i f_k^i E_i. \tag{1.2}$$

To figure out how Ψf acts on E_k, we need to know what vector ξ has its evaluation map $\bar{\xi}$ equal to $f(\cdot, v)$. Observe that

$$\begin{aligned}
\bar{v} : V^* &\to \mathbb{R} \\
\omega &\mapsto \omega(v) = \omega^i \varepsilon^i(v_j E_j) = \omega^i v_i
\end{aligned}$$

and that

$$\begin{aligned} f(\cdot,v): V^* &\to \mathbb{R} \\ \omega \mapsto f(\omega,v) &\overset{(1.1)}{=\!=} f^i_j E_i\otimes\varepsilon^j(\omega,v) \\ &= f^i_j E_i(\omega)\varepsilon^j(v) = f^i_j\omega_i v^j = \omega_i(f^i_j v^j) \end{aligned}$$

Comparing the above two equations to see $\xi_i = \sum_j f^i_j v^j$ and thus $\xi = \sum_i\left(\sum_j f^i_j v^j\right)E_i$. Then, if we let $v = E_k$, we will get

$$\xi = \sum_i(\sum_j f^i_j\delta_{kj})E_i = \sum_i f^i_k E_i$$

which is just (1.2). ∎

More generally, we can contract a given tensor on any pair of indices as long as one is contravariant, say λ-th $(1\le\lambda\le k+1)$, and one is covariant, say μ-th $(1\le\mu\le l+1)$, and it can be denoted as C^λ_μ, adopted from [6] p.42:

Definition 1.1.13. *Consider the mapping $f: V^{\times(k+1)}\times V^{(l+1)}\to T^{(k,l)}(V)$ defined by*

$$(v_1,\cdots,v_{k+1},\omega_1,\cdots,\omega_{l+1})\mapsto\langle\omega_\mu,v_\lambda\rangle\, v_1\otimes\cdots\otimes\widehat{v}_\lambda\otimes\cdots\otimes v_{k+1}\otimes\omega_1\otimes\cdots\otimes\widehat{\omega}_\mu\otimes\cdots\otimes\omega_{l+1}$$

*The **contraction**, C^λ_μ, is then the unique linear mapping $\widehat{f}: T^{(k+1,l+1)}(V)\to T^{(k,l)}(V)$ with the property*

$$v_1\otimes\cdots\otimes v_{k+1}\otimes\omega_1\otimes\cdots\otimes\omega_{l+1}\mapsto\langle\omega_\mu,v_\lambda\rangle\, v_1\otimes\cdots\otimes\widehat{v}_\lambda\otimes\cdots\otimes v_{k+1}\otimes\omega_1\otimes\cdots\otimes\widehat{\omega}_\mu\otimes\cdots\otimes\omega_{l+1}$$

induced by f through the universal property 1.1.1.

As an example $C^2_1: V^2_1\to V$ is given by $v\otimes w\otimes\sigma\mapsto\langle\sigma,w\rangle v$, and, in particular, $e_i\otimes e_j\otimes\varepsilon^k\mapsto\langle\varepsilon^k,e_j\rangle e_i=\delta^k_j e_i$. Hence

$$A^{ij}_k e_i\otimes e_j\otimes\varepsilon^k\mapsto A^{ij}_k\delta^k_j e_i = A^{ik}_k e_i.$$

In fact, definition 1.1.13 is equivalent to the following definition.

Definition 1.1.14. *The **contraction** C^λ_μ can be also defined by*

$$T^{(k+1,l+1)}(V)\to T^{(k,l)}(V)\cong\mathcal{L}(\underbrace{V^*\times\cdots\times V^*}_{k\text{ copies}}\times\underbrace{V\times\cdots\times V}_{l\text{ copies}};\mathbb{R})$$

$$f\mapsto\begin{pmatrix} V^*\times\cdots\times V^*\times V\times\cdots\times V\to\mathbb{R} \\ (\omega^1,\cdots,\omega^k,v_1,\cdots,v_l)\mapsto\sum_{j=1}^n f\left(\omega^1,\cdots,\omega^{\lambda-1},\varepsilon^j,\omega^{\lambda+1},\cdots,\omega^k,v_1,\cdots,v_{\mu-1},E_j,v_{\mu+1},v_l\right)\end{pmatrix}$$

We also have the following useful result.

Proposition 1.1.15. *For vector space V of dimension n, if $F\in T^{(k+1.l+1)}(V)$ has components $F^{i_1\cdots i_{k+1}}_{j_1\cdots j_{l+1}}$, then $C^\lambda_\mu F$ has components $F^{i_1\cdots i_{\lambda-1}mi_{\lambda+1}\cdots i_{k+1}}_{j_1\cdots j_{\mu-1}mj_{\mu+1}\cdots j_{l+1}}$ (summation on m). Namely,*

$$\left(C^\lambda_\mu F\right)^{i_1\cdots i_k}_{j_1\cdots j_l} = \sum_{m=1}^n F^{i_1\cdots i_{\lambda-1}mi_{\lambda+1}\cdots i_{k+1}}_{j_1\cdots j_{\mu-1}mj_{\mu+1}\cdots j_{l+1}} \tag{1.3}$$

1.1.3 Tensor Bundles and Tensor Fields

On a smooth manifold M with or without boundary, we can perform the same linearalgebraic constructions on each tangent space T_pM that we perform on any vector space, yielding tensors at p. The disjoint union of tensor spaces of a particular type at all points of the manifold yields a vector bundle, called a **tensor bundle**. The most fundamental tensor bundle is the **cotangent bundle**, defined as

$$T^*M = \coprod_{p\in M} T_p^*M$$

More generally, the **bundle of** (k,l)**-tensors** on M is defined as

$$T^{(k,l)}TM = \coprod_{p\in M} T^{(k,l)}\left(T_pM\right).$$

As special cases, the **bundle of covariant** k**-tensors** is denoted by $T^kT^*M = T^{(0,k)}TM$, and the **bundle of contravariant** k**-tensors** is denoted by $T^kTM = T^{(k,0)}TM$. Similarly, the **bundle of symmetric** k**-tensors** is

$$\omega_kT^*M = \coprod_{p\in M} \omega_k\left(T_p^*M\right)$$

There are the usual identifications among these bundles that follow from [4] Lemma 12.25: for example, $T^1TM = T^{(1,0)}TM = TM$ and $T^1T^*M = T^{(0,1)}TM = \omega_1T^*M = T^*M$

Exercise 1.1.16. *Show that each tensor bundle is a smooth vector bundle over M, with a local trivialization over every open subset that admits a smooth local frame for TM.*

A **tensor field on** M is a section of some tensor bundle over M. A section of $T^1T^*M = T^{(0,1)}TM$ (a covariant 1-tensor field) is also called a **covector field**. As we do with vector fields, we write the value of a tensor field F at $p\in M$ as F_p or $\left.F\right|_p$. Because covariant tensor fields are the most common and important tensor fields we work with, we use the following shorthand notation for the space of all smooth covariant k-tensor fields:

$$\mathcal{T}^k(M) = \Gamma\left(T^kT^*M\right).$$

The space of smooth 0-tensor fields is just $C^\infty(M)$. Let $(E_i) = (E_1,\ldots,E_n)$ be any smooth local frame for TM over an open subset $U\subseteq M$. Associated with such a frame is the **dual coframe**, which we typically denote by $\left(\varepsilon^1,\ldots,\varepsilon^n\right)$; these are smooth covector fields satisfying $\varepsilon^i\left(E_j\right) = \delta_j^i$. For example, given a coordinate frame $\left(\partial/\partial x^1,\ldots,\partial/\partial x^n\right)$ over some open subset $U\subseteq M$, the dual coframe is $\left(dx^1,\ldots,dx^n\right)$, where dx^i is the differential of the coordinate function x^i.

smooth local frame (E_i) and its dual coframe $\left(\varepsilon^i\right)$, the tensor fields $E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_l}$ form a smooth local frame for $T^{(k,l)}\left(T^*M\right)$. In particular, in local coordinates $\left(x^i\right)$, a (k,l)-tensor field F has a coordinate expression of the form

$$F = F_{j_1\ldots j_l}^{i_1\ldots i_k}\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l},$$

where each coefficient $F_{j_1\ldots j_l}^{i_1\ldots i_k}$ is a smooth real-valued function on U.

Exercise 1.1.17. *Suppose $F: M\to T^{(k,l)}TM$ is a rough (k,l)-tensor field. Show that F is smooth on an open set $U\subseteq M$ if and only if whenever $\omega^1,\ldots,\omega^k$ are smooth covector fields and $X_1,\ldots,X_l$ are smooth vector fields defined on U, the real-valued function $F\left(\omega^1,\ldots,\omega^k,X_1,\ldots,X_l\right)$, defined on U by*

$$F\left(\omega^1,\ldots,\omega^k,X_1,\ldots,X_l\right)(p) = F_p\left(\left.\omega^1\right|_p,\ldots,\left.\omega^k\right|_p,\left.X_1\right|_p,\ldots,\left.X_l\right|_p\right),$$

is smooth.

An important property of tensor fields is that they are multilinear over the space of smooth functions. Suppose $F\in\Gamma\left(T^{(k,l)}TM\right)$ is a smooth tensor field. Given smooth covector fields $\omega^1,\ldots,\omega^k\in\mathcal{T}^1(M)$ and smooth vector fields $X_1, \ldots, X_l \in \mathfrak{X}(M)$, above exercise shows that the function $F\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)$ is smooth, and thus F induces a map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M).$$

It is easy to check that this map is **multilinear over** $C^\infty(M)$, that is, for all functions $u, v \in C^\infty(M)$ and smooth vector or covector fields α, β,

$$\mathcal{F}(\ldots, u\alpha + v\beta, \ldots) = u\widetilde{F}(\ldots, \alpha, \ldots) + v\widetilde{F}(\ldots, \beta, \ldots).$$

Even more important is the converse: as the next lemma shows, every such map that is multilinear over $C^\infty(M)$ defines a tensor field. (This lemma is stated and proved in [4] for covariant tensor fields, but the same argument works in the case of mixed tensors.)

Lemma 1.1.18 (Tensor Characterization Lemma). *[4] Lemma 12.24.*
A map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M)$$

is induced by a smooth (k, l)-tensor field as above if and only if it is multilinear over $C^\infty(M)$. Similarly, a map

$$\mathcal{F} : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to \mathfrak{X}(M)$$

is induced by a smooth $(k+1, l)$-tensor field as in Proposition 1.1.5 if and only if is multilinear over $C^\infty(M)$, where $\mathcal{T}^k(M) = \Gamma\left(T^k T^* M\right)$.

1.2 Vector Fields

1.2.1 Lie Bracket

Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. We obtain a smooth map $dF : TM \to TN$, called the **global differential of** F, whose restriction to each tangent space T_pM is the linear map dF_p defined above. In general, the global differential does not take vector fields to vector fields. In the special case that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are vector fields such that $dF\left(X_p\right) = Y_{F(p)}$ for all $p \in M$, we say that the vector fields X and Y are F**-related**.

Lemma 1.2.1. *([4] Prop.8.19 & Cor.8.21) Let $F : M \to N$ be a diffeomorphism between smooth manifolds with or without boundary. For every $X \in \mathfrak{X}(M)$, there is a unique vector field $F_*X \in \mathfrak{X}(N)$, called the **pushforward of** X, that is F-related to X. For every $f \in C^\infty(N)$, it satisfies*

$$\left(\left(F_* X\right) f\right) \circ F = X(f \circ F). \tag{1.4}$$

Suppose $X \in \mathfrak{X}(M)$. Given a real-valued function $f \in C^\infty(M)$, applying X to f yields a new function $Xf \in C^\infty(M)$ by $Xf(p) = X_p f$. The defining equation for tangent vectors translates into the following product rule for vector fields:

$$X(fg) = fXg + gXf. \tag{1.5}$$

A map $X : C^\infty(M) \to C^\infty(M)$ is called a **derivation of** $C^\infty(M)$ (as opposed to a derivation at a point) if it is linear over $\mathbb{R}$ and satisfies (1.5) for all $f, g \in C^\infty(M)$.

Lemma 1.2.2. *([4] Prop.8.15) Let M be a smooth manifold with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some $X \in \mathfrak{X}(M)$.*

Given smooth vector fields $X, Y \in \mathfrak{X}(M)$, define a map $[X, Y] : C^\infty(M) \to C^\infty(M)$ by

$$[X, Y]f = X(Yf) - Y(Xf).$$

The value of the vector field $[X, Y]$ at a point $p \in M$ can be shown to be a deriavtion at p given by the formula $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$. Thus, by Lemma 1.2.2 it defines a smooth vector field, called the **Lie bracket of X and Y.**

Proposition 1.2.3 (Coordinate Formula for the Lie Bracket). *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \partial/\partial x^i$ and $Y = Y^j \partial/\partial x^j$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{1.6}$$

or more concisely,

$$[X, Y] = \left(X(Y^j) - Y(X^j) \right) \frac{\partial}{\partial x^j} \tag{1.7}$$

Proposition 1.2.4 (Properties of Lie Brackets). *([4] Prop.8.28) Let M be a smooth manifold with or without boundary and $X, Y, Z \in \mathfrak{X}(M)$.*

(a) BILINEARITY: $[X, Y]$ is bilinear over $\mathbb{R}$ as a function of X and Y.

(b) ANTISYMMETRY: $[X, Y] = -[Y, X]$.

(c) JACOBI IDENTITY: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

(d) For $f, g \in C^\infty(M), [fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$.

Proposition 1.2.5 (Naturality of Lie Brackets). *([4] Prop.8.30 & Cor.8.31) Let $F : M \to N$ be a smooth map between manifolds with or without boundary, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F-related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$. In particular, if F is a diffeomorphism, then $F_* [X_1, X_2] = [F_* X_1, F_* X_2]$.*

Now suppose $\widetilde{M}$ is a smooth manifold with or without boundary and $M \subseteq \widetilde{M}$ is an immersed or embedded submanifold with or without boundary. The bundle $T\widetilde{M}\big|_M$, obtained by restricting $T\widetilde{M}$ to M, is called the **ambient tangent bundle**. It is a smooth bundle over M whose rank is equal to the dimension of $\widetilde{M}$. The tangent bundle TM is naturally viewed as a smooth subbundle of $T\widetilde{M}\big|_M$, and smooth vector fields on M can also be viewed as smooth sections of $T\widetilde{M}\big|_M$. A vector field $X \in \mathfrak{X}(\widetilde{M})$ always restricts to a smooth section of $T\widetilde{M}\big|_M$, and it restricts to a smooth section of TM if and only if it is **tangent to** M, meaning that $X_p \in T_pM \subseteq T_p\widetilde{M}$ for each $p \in M$.

Corollary 1.2.6 (Brackets of Vector Fields Tangent to Submanifolds). *([4] Cor.8.32) Let $\widetilde{M}$ be a smooth manifold and let M be an immersed submanifold with or without boundary in $\widetilde{M}$. If Y_1 and Y_2 are smooth vector fields on $\widetilde{M}$ that are tangent to M, then $[Y_1, Y_2]$ is also tangent to M.*

Exercise 1.2.7. *Let $\widetilde{M}$ be a smooth manifold with or without boundary and let $M \subseteq \widetilde{M}$ be an embedded submanifold with or without boundary. Show that a vector field $X \in \mathfrak{X}(\widetilde{M})$ is tangent to M if and only if $(Xf)|_M = 0$ whenever $f \in C^\infty(\widetilde{M})$ is a function that vanishes on M.*

1.2.2 Integral Curves and Flows

A **curve** in a smooth manifold M (with or without boundary) is a continuous map $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is some interval. If γ is smooth, then for each $t_0 \in I$ we obtain a vector $\gamma'(t_0) = d\gamma_{t_0}\left(d/\left.dt\right|_{t_0}\right)$, called the **velocity of** γ **at time** t_0. It acts on functions by

$$\gamma'(t_0) f = (f \circ \gamma)'(t_0).$$

In any smooth local coordinates, the coordinate expression for $\gamma'(t_0)$ is exactly the same as it would be in $\mathbb{R}^n$: the components of $\gamma'(t_0)$ are the ordinary t-derivatives of the components of γ.

If $X \in \mathfrak{X}(M)$, then a smooth curve $\gamma : I \to M$ is called an **integral curve of** X if its velocity at each point is equal to the value of X there: $\gamma'(t) = X_{\gamma(t)}$ for each $t \in I$.

The fundamental fact about vector fields (at least in the case of manifolds without boundary) is that there exists a unique maximal integral curve starting at each point, varying smoothly as the point varies. These integral curves are all encoded into a global object called a flow, which we now define.

Given a smooth manifold M (without boundary), a **flow domain for** M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$$

is an open interval containing 0. Given a flow domain $\mathcal{D}$ and a map $\theta : \mathcal{D} \to M$, for each $t \in \mathbb{R}$ we let

$$M_t = \{p \in M : (t, p) \in \mathcal{D}\},$$

and we define maps

$$\theta_t : M_t \to M$$

and

$$\theta^{(p)} : \mathcal{D}^{(p)} \to M$$

by $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$. A **flow on** M is a continuous map $\theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies

$$\theta_0 = \mathrm{Id}_M,$$
$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p) \quad \text{wherever both sides are defined.}$$

If θ is a smooth flow, we obtain a smooth vector field $X \in \mathfrak{X}(M)$ defined by $X_p = \left(\theta^{(p)}\right)'(0)$, called the **infinitesimal generator of** θ.

Theorem 1.2.8 (Fundamental Theorem on Flows)**.** *([4] Thm.9.12) Let X be a smooth vector field on a smooth manifold M (without boundary). There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is X. This flow has the following properties:*

(a) For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of X starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \left\{t - s : t \in \mathcal{D}^{(p)}\right\}$.

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Although the fundamental theorem guarantees only that each point lies on an integral curve that exists for a short time, the next lemma can often be used to prove that a particular integral curve exists for all time.

Lemma 1.2.9 (Escape Lemma)**.** *Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. If $\gamma : I \to M$ is a maximal integral curve of X whose domain I has a finite least upper bound b, then for every $t_0 \in I, \gamma([t_0, b))$ is not contained in any compact subset of M.*

Proposition 1.2.10 (Canonical Form for a Vector Field). *([4] Thm.9.22) Let X be a smooth vector field on a smooth manifold M, and suppose $p \in M$ is a point where $X_p \neq 0$. There exist smooth coordinates $\left(x^i\right)$ on some neighborhood of p in which X has the coordinate representation $\partial / \partial x^1$.*

Recall the pullback

$$(\theta_t^* A)_p(v_1, \cdots, v_k) = d\left(\theta_t\right)_p^*\left(A_{\theta_t(p)}\right)\left(v_1, \ldots, v_k\right)=A_{\theta_t(p)}\left(d\left(\theta_t\right)_p\left(v_1\right), \ldots, d\left(\theta_t\right)_p\left(v_k\right)\right)$$

Given a smooth covariant tensor field A on M, we define the **Lie derivative** of A with respect to V, denoted by $\mathcal{L}_V A$, by

$$\left(\mathcal{L}_V A\right)_p=\left.\frac{d}{d t}\right|_{t=0}\left(\theta_t^* A\right)_p=\lim _{t \rightarrow 0} \frac{d\left(\theta_t\right)_p^*\left(A_{\theta_t(p)}\right)-A_p}{t},$$

provided the derivative exists. Because the expression being differentiated lies in $T^k\left(T_p^* M\right)$ for all t, $\left(\mathcal{L}_V A\right)_p$ makes sense as an element of $T^k\left(T_p^* M\right)$. We say A is invariant under θ if for each t, $\theta_t^* A=A$, i.e., $d(\theta_t)^*_p(A_{\theta_t(p)}) = A_p$. It is a corollary of (g) below that invariance $\iff$ $\mathcal{L}_V A=0$.

Lemma 1.2.11. *With M, V, and A as above, the derivative above exists for every $p \in M$ and defines $\mathcal{L}_V A$ as a smooth tensor field on M.*

Proposition 1.2.12. *([4] Prop.12.32-36) Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose f is a smooth real-valued function (regarded as a 0 -tensor field) on M, and A, B are smooth covariant tensor fields on M.*

(a) $\mathcal{L}_V f=V f$.

(b) $\mathcal{L}_V(f A)=\left(\mathcal{L}_V f\right) A+f \mathcal{L}_V A$.

(c) $\mathcal{L}_V(A \otimes B)=\left(\mathcal{L}_V A\right) \otimes B+A \otimes \mathcal{L}_V B$.

(d) If $X_1, \ldots, X_k$ are smooth vector fields and A is a smooth k-tensor field,

$$\begin{aligned}\mathcal{L}_V\left(A\left(X_1, \ldots, X_k\right)\right)= & \left(\mathcal{L}_V A\right)\left(X_1, \ldots, X_k\right)+A\left(\mathcal{L}_V X_1, \ldots, X_k\right) \\ & +\cdots+A\left(X_1, \ldots, \mathcal{L}_V X_k\right) .\end{aligned}$$

$$\begin{aligned}\left(\mathcal{L}_V A\right)\left(X_1, \ldots, X_k\right)= & V\left(A\left(X_1, \ldots, X_k\right)\right)-A\left(\left[V, X_1\right], X_2, \ldots, X_k\right) \\ & -\cdots-A\left(X_1, \ldots, X_{k-1},\left[V, X_k\right]\right) .\end{aligned}$$

(f) If $f \in C^{\infty}(M)$, then $\mathcal{L}_V(d f)=d\left(\mathcal{L}_V f\right)$.

(g) For any smooth covariant tensor field A and any $\left(t_0, p\right)$ in the domain of θ,

$$\left.\frac{d}{d t}\right|_{t=t_0}\left(\theta_t^* A\right)_p=\left(\theta_{t_0}^*\left(\mathcal{L}_V A\right)\right)_p$$

Proposition 1.2.13. *([4] Thm.9.38) Suppose M is a smooth manifold and $X, Y \in \mathfrak{X}(M)$. The Lie derivative of Y with respect to X is equal to the Lie bracket $[X, Y]$.*

One of the most important applications of the Lie derivative is as an obstruction to invariance under a flow. If θ is a smooth flow, we say that a vector field Y is **invariant under** θ if $\left(\theta_t\right)_* Y=Y$ wherever the left-hand side is defined.

Proposition 1.2.14. *([4] Thm.9.42) Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. A smooth vector field is invariant under the flow of X if and only if its Lie derivative with respect to X is identically zero.*

A k-tuple of vector fields $X_1, \ldots, X_k$ is said to **commute** if $\left[X_i, X_j\right]=0$ for each i and j.

1.3 Smooth Covering Maps

A **covering map** is a surjective continuous map $\pi : \widetilde{M} \to M$ between connected and locally path-connected topological spaces, for which each point of M has connected neighborhood U that is **evenly covered**, meaning that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. It is called a **smooth covering map** if $\widetilde{M}$ and M are smooth manifolds with or without boundary and each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U. For every evenly covered open set $U \subseteq M$, the components of $\pi^{-1}(U)$ are called the **sheets of the covering over** U.

Here are the main properties of covering maps that we need.

Proposition 1.3.1 (Elementary Properties of Smooth Covering Maps)**.**

(a) Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.

(b) An injective smooth covering map is a diffeomorphism.

(c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Proof. See [4] Prop. 4.33. ■

Proposition 1.3.2. *A covering map is a proper map if and only if it is finite-sheeted.*

Exercise 1.3.3. *Prove the preceding proposition.*

If $\pi : \widetilde{M} \to M$ is a covering map and $F : B \to M$ is a continuous map from a topological space B into M, then a lift of $\boldsymbol{F}$ is a continuous map $\widetilde{F} : B \to \widetilde{M}$ such that $\pi \circ \widetilde{F} = F$.

Proposition 1.3.4 (Lifts of Smooth Maps are Smooth)**.** *If $\pi : \widetilde{M} \to M$ is a smooth covering map, B is a smooth manifold with or without boundary, and $F : B \to M$ is a smooth map, then every lift of F is smooth.*

Proof. Since π is a smooth submersion, every lift $\widetilde{F} : B \to \widetilde{M}$ can be written locally as $\widetilde{F} = \sigma \circ F$, where σ is a smooth local section of π (see [4] Thm. 4.26). ■

Proposition 1.3.5 (Lifting Properties of Covering Maps)**.** *Suppose $\pi : \widetilde{M} \to M$ is a covering map.*

(a) UNIQUE LIFTING PROPERTY ([3] Thm. 11.12): If B is a connected topological space and $F : B \to M$ is a continuous map, then any two lifts of F that agree at one point are identical.

(b) PATH LIFTING PROPERTY ([3] Cor. 11.14): Suppose $f : [0,1] \to M$ is a continuous path. For every $\tilde{p} \in \pi^{-1}(f(0))$, there exists a unique lift $\tilde{f} : [0,1] \to \widetilde{M}$ of f such that $\tilde{f}(0) = \tilde{p}$.

(c) MONODROMY THEOREM ([3] Thm. 11.15): Suppose $f, g : [0,1] \to M$ are path-homotopic paths and $\widetilde{f}, \widetilde{g} : [0,1] \to \widetilde{M}$ are their lifts starting at the same point. Then $\tilde{f}$ and $\tilde{g}$ are path-homotopic and $\widetilde{f}(1) = \widetilde{g}(1)$.

Theorem 1.3.6 (Injectivity Theorem)**.** *([3] Thm. 11.16) If $\pi : \widetilde{M} \to M$ is a covering map, then for each point $\tilde{x} \in \widetilde{M}$, the induced fundamental group homomorphism $\pi_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, \pi(\widetilde{x}))$ is injective.*

Theorem 1.3.7 (Lifting Criterion)**.** *([3] Thm. 11.18) Suppose $\pi : \widetilde{M} \to M$ is a covering map, B is a connected and locally path-connected topological space, and $F : B \to M$ is a continuous map. Given $b \in B$ and $\widetilde{x} \in \widetilde{M}$ such that $\pi(\widetilde{x}) = F(b)$, the map F has a lift to $\widetilde{M}$ if and only if $F_*\left(\pi_1(B, b)\right) \subseteq \pi_*\left(\pi_1(\widetilde{M}, \widetilde{x})\right)$.*

Corollary 1.3.8 (Lifting Maps from Simply Connected Spaces). *([3] Cor. 11.19) Suppose $\pi : \widetilde{M} \to M$ and $F : B \to M$ satisfy the hypotheses of Theorem A.56, and in addition B is simply connected. Then every continuous map $F : B \to M$ has a lift to $\widetilde{M}$. Given any $b \in B$, the lift can be chosen to take b to any point in the fiber over $F(b)$.*

Corollary 1.3.9 (Covering Map Homeomorphism Criterion). *A covering map $\pi : \widetilde{M} \to M$ is a homeomorphism if and only if the induced homomorphism $\pi_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, \pi(\widetilde{x}))$ is surjective for some (hence every) $\tilde{x} \in \widetilde{M}$. A smooth covering map is a diffeomorphism if and only if the induced homomorphism is surjective.*

Proof. By Theorem 1.3.7, the hypothesis implies that the identity map Id: $M \to M$ has a lift $\widetilde{\mathrm{Id}} : M \to \widetilde{M}$, which in this case is a continuous inverse for π. If π is a smooth covering map, then the lift is also smooth. ■

Corollary 1.3.10 (Coverings of Simply Connected Spaces). *([3] Cor. 11.33) If M is a simply connected manifold with or without boundary, then every covering of M is a homeomorphism, and if M is smooth, every smooth covering is a diffeomorphism.*

Proposition 1.3.11 (Existence of a Universal Covering Manifold). *([4] Cor. 4.43) If M is a connected smooth manifold, then there exist a simply connected smooth manifold $\widetilde{M}$, called the universal covering manifold of M, and a smooth covering map $\pi : \widetilde{M} \to M$. It is unique in the sense that if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi' : \widetilde{M}' \to M$, then there exists a diffeomorphism $\Phi : \widetilde{M} \to \widetilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

Proposition 1.3.12. *([3] Cor. 11.31)With $\pi : \widetilde{M} \to M$ as in the previous proposition, each fiber of π has the same cardinality as the fundamental group of M.*

Exercise 1.3.13. *Suppose $\pi : \widetilde{M} \to M$ is a covering map. Show that $\widetilde{M}$ is compact if and only if M is compact and π is a finite-sheeted covering.*

1.4 Vector Spaces $T^k(V^*), \Sigma^k(V^*), \Lambda^k(V^*)$

Let V be a f.d. vector space. The vector spaces of all **covariant** k**-tensor**, **contravariant** l**-tensor**, (k,l)**-mixed type tensor** are

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ factors}}, \quad T^l(V) = \underbrace{V \otimes \cdots \otimes V}_{l \text{ factors}}, \quad T^{(k,l)}(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ factors}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ factors}}$$

Suppose (E_i) is any basis for V and (ε^j) be the dual basis for V^*. Then their bases are

$$\begin{aligned} \left\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \le i_1, \ldots, i_k \le n\right\} &\quad \text{for } T^k(V^*) \\ \left\{E_{i_1} \otimes \cdots \otimes E_{i_k} : 1 \le i_1, \ldots, i_k \le n\right\} &\quad \text{for } T^k(V) \\ \left\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \le i_1, \ldots, i_k, j_1, \ldots, j_l \le n\right\} &\quad \text{for } T^{(k,l)}(V) \end{aligned}$$

Therefore, $\dim T^k(V^*) = \dim T^k(V) = n^k$ and $\dim T^{(k,l)}(V) = n^{k+l}$.

Subspace $\Sigma^k(V^*)$

A covariant k-tensor α on V is said to be **symmetric** if its value is unchanged by interchanging any pair of arguments:

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = \alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

whenever $1 \leq i < j \leq k$. These symmetric covariant k-tensors form linear subspace $\Sigma^k(V^*)$ in $T^k(V^*)$. Given a k-tensor α and a permutation $\sigma \in S_k$, we define a new k-tensor ${}^\sigma\alpha$ by

$$ {}^\sigma\alpha(v_1, \ldots, v_k) = \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) $$

Note that ${}^\tau({}^\sigma\alpha) = {}^{\tau\sigma}\alpha$. We define a projection $\mathrm{Sym} : T^k(V^*) \to \Sigma^k(V^*)$ called **symmetrization** by

$$ \mathrm{Sym}\,\alpha = \frac{1}{k!}\sum_{\sigma \in S_k} {}^\sigma\alpha $$

[4] Proposition 12.14 shows that $\mathrm{Sym}\,\alpha$ is indeed symmetric and a form α is symmetric if and only if $\mathrm{Sym}\,\alpha = \alpha$. If $\alpha \in \Sigma^k(V^*)$ and $\beta \in \Sigma^l(V^*)$, we define their **symmetric product** to be the $(k+l)$ tensor $\alpha\beta$ (denoted by juxtaposition) given by

$$ \alpha\beta = \mathrm{Sym}(\alpha \otimes \beta) $$

Example 1.4.1. By [4] p.315 Proposition 12.15, if α and β are covectors, then

$$ \alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha). \tag{1.8} $$

♣

A basis of $\Sigma^k(V^*)$ is given by

$$ \left\{\mathrm{Sym}\left(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}\right), 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n\right\} $$

so that

$$ \dim\left(\Sigma^k(V^*)\right) = \binom{n+k-1}{k} $$

For an attempt to write the basis in form $\{\alpha \otimes \cdots \otimes \alpha\}$, see this post.

Subspace $\Lambda^k(V^*)$

A covariant k-tensor α on V is said to be **alternating** (or **antisymmetric** or **skew-symmetric**) if

$$ \alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) $$

Alternating covariant k-tensors are also variously called **exterior forms**, **multicovectors**, or k**-covectors**. The subspace of all alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*) \subseteq T^k(V^*)$.

Recall that for any permutation $\sigma \in S_k$, the sign of σ, denoted by $\operatorname{sgn}\sigma$, is equal to $+1$ if σ is even (i.e., can be written as a composition of an even number of transpositions), and -1 if σ is odd.

Proposition 1.4.2. *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

(a) α *is alternating.*

(b) For any vectors $v_1, \ldots, v_k$ and any permutation $\sigma \in S_k$,

$$ \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn}\sigma)\alpha(v_1, \ldots, v_k) $$

(c) $\alpha(v_1, \ldots, v_k) = 0$ *whenever the k-tuple $(v_1, \ldots, v_k)$ is linearly dependent.*

(d) α *gives the value zero whenever two of its arguments are equal:*

$$ \alpha(v_1, \ldots, w, \ldots, w, \ldots, v_k) = 0 $$

Proof. See [4] Exercise 12.17 and Lemma 14.1. ■

Example 1.4.3. Every 0-tensor (which is just a real number) is both symmetric and alternating, because there are no arguments to interchange. Similarly, every 1-tensor is both symmetric and alternating. An alternating 2-tensor on V is a skew-symmetric bilinear form. It is interesting to note that every covariant 2-tensor β can be expressed as a sum of an alternating tensor and a symmetric one, because

$$\beta(v,w)=\frac{1}{2}(\beta(v,w)-\beta(w,v))+\frac{1}{2}(\beta(v,w)+\beta(w,v))=\alpha(v,w)+\sigma(v,w)$$

where $\alpha(v,w)=\frac{1}{2}(\beta(v,w)-\beta(w,v))$ is an alternating tensor, and $\sigma(v,w)=\frac{1}{2}(\beta(v,w)+\beta(w,v))$ is symmetric. This is not true for tensors of higher rank, as [4] Problem 12-7 shows. ♣

We define **alteration**, an analogue of symmetrization as the projection $\operatorname{Alt}: T^k\left(V^*\right) \rightarrow \Lambda^k\left(V^*\right)$, as follows:

$$\operatorname{Alt} \alpha=\frac{1}{k!} \sum_{\sigma \in S_k}(\operatorname{sgn} \sigma)\left({ }^\sigma \alpha\right)$$

More explicitly, this means

$$(\operatorname{Alt} \alpha)\left(v_1, \ldots, v_k\right)=\frac{1}{k!} \sum_{\sigma \in S_k}(\operatorname{sgn} \sigma) \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

Example 1.4.4. If α is any 1-tensor, then $\operatorname{Alt} \alpha=\alpha$. If β is a 2 -tensor, then

$$(\operatorname{Alt} \beta)(v,w)=\frac{1}{2}(\beta(v,w)-\beta(w,v))$$

♣

Similar to the properties of symmetrization operator, we have $\operatorname{Alt} \alpha$ is alternating; and that $\operatorname{Alt} \alpha=\alpha \iff \alpha$ is alternating.

To describe the basis of $\Lambda^k(V^*)$, we introduce some notations. For multi-index $I=(i_1,\cdots,i_k)$, we let

$$I_\sigma=\left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right).$$

Note that $I_{\sigma\tau}=\left(I_\sigma\right)_\tau$ for $\sigma,\tau \in S_k$.

For a multi-index $I=\left(i_1, \ldots, i_k\right)$ with increasing components $i_1 \leq \cdots \leq i_k$, define a covariant k-tensor $\varepsilon^I=\varepsilon^{i_1 \ldots i_k}$ by

$$\varepsilon^I\left(v_1, \ldots, v_k\right)=\det\left(\begin{array}{ccc}\varepsilon^{i_1}\left(v_1\right) & \ldots & \varepsilon^{i_1}\left(v_k\right) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}\left(v_1\right) & \ldots & \varepsilon_k^{i_k}\left(v_k\right)\end{array}\right)=\det\left(\begin{array}{ccc}v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \ldots & v_k^{i_k}\end{array}\right).$$

In other words, if $\mathbb{V}$ denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \ldots, v_k$ with respect to the basis $\left(E_i\right)$ dual to $\left(\varepsilon^i\right)$, then $\varepsilon^I\left(v_1, \ldots, v_k\right)$ is the determinant of the $k \times k$ submatrix consisting of rows $i_1, \ldots, i_k$ of $\mathbb{V}$. Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k-tensor. We call ε^I an **elementary alternating tensor** or **elementary k-covector**.

Proposition 1.4.5. *Let V be an n-dimensional vector space. If $\left(\varepsilon^i\right)$ is any basis for V^*, then for each positive integer $k \leq n$, the collection of k-covectors*

$$\mathcal{E}=\left\{\varepsilon^I: I \text{ is an increasing multi-index of length } k\right\}$$

is a basis for $\Lambda^k(V^*)$. *Therefore,*

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If $k > n$, *then* $\dim \Lambda^k(V^*) = 0$.

In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V^*)$ is 1-dimensional and is spanned by $\varepsilon^{1\ldots n}$. By definition, this elementary n covector acts on vectors $(v_1, \ldots, v_n)$ by taking the determinant of their component matrix $\mathbb{V} = \left(v_j^i\right)$. For example, on $\mathbb{R}^n$ with the standard basis, $\varepsilon^{1\ldots n}$ is precisely the determinant function.

Proposition 1.4.6. *Suppose* V *is an* n*-dimensional vector space and* $\omega \in \Lambda^n(V^*)$. *If* $T : V \to V$ *is any linear map and* $v_1, \ldots, v_n$ *are arbitrary vectors in* V, *then*

$$\omega(Tv_1, \ldots, Tv_n) = (\det T)\omega(v_1, \ldots, v_n)$$

Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their wedge product or exterior product to be the following $(k+l)$-covector:

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!}\omega \bar{\wedge} \eta := \frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega \otimes \eta)$$

Proposition 1.4.7 (Properties of Wedge Product; [4] Lemma 14.10 and Proposition 14.11)**.** *Suppose* $\omega, \omega', \eta, \eta'$, *and* ξ *are multicovectors on a finite-dimensional vector space* V.

(a) For any multi-indices I *and* J *of lengths* k *and* l, *we have* $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$ *where* IJ *is the concatenation.*

(a) BILINEARITY: For $a, a' \in \mathbb{R}$,

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega')$$

(b) ASSOCIATIVITY:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$$

(c) ANTICOMMUTATIVITY: For $\omega \in \Lambda^k(V^*)$ *and* $\eta \in \Lambda^l(V^*)$,

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$$

(d) If $\left(\varepsilon^i\right)$ *is any basis for* V^* *and* $I = (i_1, \ldots, i_k)$ *is any multi-index, then*

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I$$

(e) For any covectors $\omega^1, \ldots, \omega^k$ *and vectors* $v_1, \ldots, v_k$,

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = \det\left(\omega^j(v_i)\right)$$

There is an important operation that relates vectors with alternating tensors. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$, called **interior multiplication by** v, as follows:

$$i_v\omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1})$$

In other words, $i_v\omega$ is obtained from ω by inserting v into the first slot. By convention, we interpret $i_v\omega$ to be zero when ω is a 0-covector (i.e., a number). Another common notation is

$$v \lrcorner \omega = i_v\omega$$

This is often read "v into ω."

Lemma 1.4.8 (see [4] Lemma 14.13.). *Let V be a finite-dimensional vector space and $v \in V$.*

(a) $i_v \circ i_v = 0$.

(b) If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta)$$

We make a brief summary.

	spaces	projection	product of k form & l form
symmetric k-tensor	$\Sigma^k(V^*)$	symmetrization Sym	symmetric product $\alpha\beta = \mathrm{Sym}(\alpha \otimes \beta)$
alternating k-tensor	$\Lambda^k(V^*)$	alternation Alt	wedge product $\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$

1.5 Differential Forms

1.6 De Rham Cohomology

Chapter 2

Riemannian Manifolds

Given a vector space V (which we always assume to be real), an **inner product** on V is a map $V \times V \to \mathbb{R}$, typically written $(v, w) \mapsto \langle v, w\rangle$, that satisfies the following properties for all $v, w, x \in V$ and $a, b \in \mathbb{R}$:

(i) SYMMETRY: $\langle v, w\rangle = \langle w, v\rangle$.

(ii) BILINEARITY: $\langle av + bw, x\rangle = a\langle v, x\rangle + b\langle w, x\rangle = \langle x, av + bw\rangle$.

(iii) POSITIVE DEFINITENESS: $\langle v, v\rangle \geq 0$, with equality if and only if $v = 0$.

A vector space endowed with a specific inner product is called an **inner product space**.

An inner product on V allows us to make sense of geometric quantities such as lengths of vectors and angles between vectors. First, we define the **length** or **norm** of a vector $v \in V$ as

$$|v| = \langle v, v\rangle^{1/2}.$$

Polarization identity

$$\langle v, w\rangle = \frac{1}{4}(\langle v + w, v + w\rangle - \langle v - w, v - w\rangle).$$

shows that an inner product is completely determined by knowledge of the lengths of all vectors. The **angle** between two nonzero vectors $v, w \in V$ is defined as the unique $\theta \in [0, \pi]$ satisfying

$$\cos\theta = \frac{\langle v, w\rangle}{|v||w|}$$

Two vectors $v, w \in V$ are said to be **orthogonal** if $\langle v, w\rangle = 0$, which means that either their angle is $\pi/2$ or one of the vectors is zero. If $S \subseteq V$ is a linear subspace, the set $S^{\perp} \subseteq V$, consisting of all vectors in V that are orthogonal to every vector in S, is also a linear subspace, called the **orthogonal complement** of S.

Vectors $v_1, \ldots, v_k$ are called **orthonormal** if they are of length 1 and pairwise orthogonal, or equivalently if $\langle v_i, v_j\rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta symbol). The following well-known proposition shows that every finite-dimensional inner product space has an orthonormal basis.

Proposition 2.0.1 (Gram-Schmidt Algorithm)**.** *Let V be an n-dimensional inner product space, and suppose $(v_1, \ldots, v_n)$ is any ordered basis for V. Then there is an orthonormal ordered basis $(b_1, \ldots, b_n)$ satisfying the following conditions:*

$$\operatorname{span}(b_1, \ldots, b_k) = \operatorname{span}(v_1, \ldots, v_k) \quad \textit{for each } k = 1, \ldots, n$$

Proof. See [5] Proposition 2.3. ■

Let M be a smooth manifold with or without boundary. A **Riemannian metric** on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM; thus g is a symmetric 2-tensor field that is positive definite in the sense that $g_p(v,v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$. A **Riemannian manifold** is a pair (M,g), where M is a smooth manifold and g is a specific choice of Riemannian metric on M. If M is understood to be endowed with a specific Riemannian metric, we sometimes say " M is a Riemannian manifold." The next proposition shows that Riemannian metrics exist in great abundance.

Proposition 2.0.2. *Every smooth manifold admits a Riemannian metric.*

Exercise 2.0.3. *Use a partition of unity to prove the preceding proposition.*

Let g be a Riemannian metric on a smooth manifold M with or without boundary. Because g_p is an inner product on T_pM for each $p \in M$, we often use the following angle-bracket notation for $v, w \in T_pM$:

$$\langle v, w\rangle_g = g_p(v,w).$$

Using this inner product, we can define lengths of tangent vectors, angles between nonzero tangent vectors, and orthogonality of tangent vectors as described above. The length of a vector $v \in T_pM$ is denoted by $|v|_g = \langle v, v\rangle_g^{1/2}$. If the metric is understood, we sometimes omit it from the notation, and write $\langle v, w\rangle$ and $|v|$ in place of $\langle v, w\rangle_g$ and $|v|_g$, respectively.

The starting point for Riemannian geometry is the following fundamental example.

Example 2.0.4 (The Euclidean Metric)**.** The **Euclidean metric** is the Riemannian metric $\bar{g}$ on $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is just the usual dot product on $T_x\mathbb{R}^n$ under the natural identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$. This means that for $v, w \in T_x\mathbb{R}^n$ written in standard coordinates $\left(x^1, \ldots, x^n\right)$ as $v = \sum_i v^i \partial_i\big|_x, w = \sum_j w^j \partial_j\Big|_x$, we have

$$\langle v, w\rangle_{\bar{g}} = \sum_{i=1}^n v^i w^i.$$

When working with $\mathbb{R}^n$ as a Riemannian manifold, we always assume we are using the Euclidean metric unless otherwise specified. ♣

Suppose (M,g) is a Riemannian manifold with or without boundary. If $\left(x^1, \ldots, x^n\right)$ are any smooth local coordinates on an open subset $U \subseteq M$, then g can be written locally in U as

$$g = g_{ij} dx^i \otimes dx^j$$

for some collection of n^2 smooth functions g_{ij} for $i, j = 1, \ldots, n$. The component functions of this tensor field constitute a matrix-valued function (g_{ij}), characterized by $g_{ij}(p) = \left\langle \partial_i|_p, \partial_j|_p \right\rangle$, where $\partial_i = \partial/\partial x^i$ is the i th coordinate vector field; this matrix is symmetric in i and j and depends smoothly on $p \in U$. If $v = v^i \partial_i\big|_p$ is a vector in T_pM such that $g_{ij}(p)v^j = 0$, it follows that $\langle v, v\rangle = g_{ij}(p)v^i v^j = 0$, which implies $v = 0$; thus the matrix $(g_{ij}(p))$ is always nonsingular. The notation for g can be shortened by expressing it in terms of the symmetric product: using the symmetry of g_{ij}, we compute

$$\begin{aligned} g &= g_{ij} dx^i \otimes dx^j \\ &= \frac{1}{2}\left(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j\right) \qquad (g_{ij} = g_{ji}) \\ &= \frac{1}{2}\left(g_{ij} dx^i \otimes dx^j + g_{ij} dx^j \otimes dx^i\right) \qquad (\sum_i \sum_j = \sum_j \sum_i) \\ &= g_{ij} dx^i dx^j \qquad \text{(due to eq. (1.8))} \end{aligned}$$

For example, the Euclidean metric on $\mathbb{R}^n$ (Example 2.0.4) can be expressed in standard coordinates in several ways:

$$\bar{g} = \sum_i dx^i dx^i = \sum_i \left(dx^i\right)^2 = \delta_{ij} dx^i dx^j$$

The matrix of $\bar{g}$ in these coordinates is thus $\bar{g}_{ij} = \delta_{ij}$. More generally, if $(E_1, \ldots, E_n)$ is any smooth local frame for TM on an open subset $U \subseteq M$ and $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ is its dual coframe, we can write g locally in U as

$$g = g_{ij} \varepsilon^i \varepsilon^j, \tag{2.1}$$

where $g_{ij}(p) = \left\langle \left.E_i\right|_p, \left.E_j\right|_p \right\rangle$, and the matrix-valued function (g_{ij}) is symmetric and smooth as before.

A Riemannian metric g acts on smooth vector fields $X, Y \in \mathfrak{X}(M)$ to yield a real-valued function $\langle X, Y\rangle$. In terms of any smooth local frame, this function is expressed locally by $\langle X, Y\rangle = g_{ij} X^i Y^j$ and therefore is smooth. Similarly, we obtain a nonnegative real-valued function $|X| = \langle X, X\rangle^{1/2}$, which is continuous everywhere and smooth on the open subset where $X \neq 0$.

A local frame (E_i) for M on an open set U is said to be an orthonormal frame if the vectors $\left.E_1\right|_p, \ldots, \left.E_n\right|_p$ are an orthonormal basis for T_pM at each $p \in U$. Equivalently, (E_i) is an orthonormal frame if and only if

$$\langle E_i, E_j\rangle = \delta_{ij}$$

in which case g has the local expression

$$g = \left(\varepsilon^1\right)^2 + \cdots + \left(\varepsilon^n\right)^2$$

where $\left(\varepsilon^i\right)^2$ denotes the symmetric product $\varepsilon^i \varepsilon^i = \varepsilon^i \otimes \varepsilon^i$.

Proposition 2.0.5 (Existence of Orthonormal Frames)**.** *Let (M, g) be a Riemannian n-manifold with or without boundary. If (X_j) is any smooth local frame for TM over an open subset $U \subseteq M$, then there is a smooth orthonormal frame (E_j) over U such that* $\operatorname{span}\left(\left.E_1\right|_p, \ldots, \left.E_k\right|_p\right) = \operatorname{span}\left(\left.X_1\right|_p, \ldots, \left.X_k\right|_p\right)$ *for each $k = 1, \ldots, n$ and each $p \in U$. In particular, for every $p \in M$, there is a smooth orthonormal frame (E_j) defined on some neighborhood of p.*

Proof. See [5] Proposition 2.8. ■

Warning: A common mistake made by beginners is to assume that one can find coordinates near p such that the coordinate frame (∂_i) is orthonormal. Above proposition does not show this. In fact, as we will see in Chapter 7, this is possible only when the metric is flat, that is, locally isometric to the Euclidean metric.

For a Riemannian manifold (M, g) with or without boundary, we define the unit tangent bundle to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM = \left\{(p, v) \in TM : |v|_g = 1\right\}.$$

Proposition 2.0.6 (Properties of the Unit Tangent Bundle)**.** *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a smooth, properly embedded codimension-1 submanifold with boundary in TM, with $\partial(UTM) = \pi^{-1}(\partial M)$(where $\pi : UTM \to M$ is the canonical projection). The unit tangent bundle is connected if and only if M is connected (when $\dim M > 1$), and compact if and only if M is compact.*

Exercise 2.0.7. *Use local orthonormal frames to prove the preceding proposition.*

2.1 Pullback Metrics and Isometries

If two vector spaces V and W are both equipped with inner products, denoted by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively, then a map $F: V \rightarrow W$ is called a **linear isometry** if it is a vector space isomorphism that preserves inner products: $\langle F(v), F(v')\rangle_W = \langle v, v'\rangle_V$. If V and W are inner product spaces of dimension n, then given any choices of orthonormal bases $(v_1, \ldots, v_n)$ for V and $(w_1, \ldots, w_n)$ for W, the linear map $F: V \rightarrow W$ determined by $F(v_i) = w_i$ is easily seen to be a linear isometry. Thus all inner product spaces of the same finite dimension are linearly isometric to each other.

Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary. An **isometry from** (M, g) **to** $(\widetilde{M}, \widetilde{g})$ is a diffeomorphism $\varphi: M \rightarrow \widetilde{M}$ such that $\varphi^* \widetilde{g} = g$. We say (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric if there exists an isometry between them.

Proposition 2.1.1. *When $\partial M = \varnothing$, $\varphi: (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ is an isometry if and only if φ is a smooth bijection and each differential $d\varphi_p: T_pM \rightarrow T_{\varphi(p)}\widetilde{M}$ is a linear isometry.*

Proof. "$\Rightarrow$": Notice that

$$\left(\varphi^* \widetilde{g}\right)_p (v, v') = \widetilde{g}_{\varphi(p)} \left(d\varphi_p(v), d\varphi_p(v')\right) = \langle d\varphi_p(v), d\varphi_p(v')\rangle_{\widetilde{g}} \tag{2.2}$$

and

$$g_p(v, v') = \langle v, v'\rangle_g$$

Since φ is an isometry, the RHS of above two equations are equal. So do their LHS. This shows that $d\varphi_p: (T_pM, \langle\cdot,\cdot\rangle_g) \rightarrow \left(T_{\varphi(p)}\widetilde{M}, \langle\cdot,\cdot\rangle_{\widetilde{g}}\right)$ is a linear isometry. φ as a diffeomorphism is smooth and bijective.

"$\Leftarrow$":

Suppose φ is smooth (this condition first ensures $d\varphi_p$ can be defined) and bijective and $d\varphi_p: (T_pM, \langle\cdot,\cdot\rangle_g) \rightarrow \left(T_{\varphi(p)}\widetilde{M}, \langle\cdot,\cdot\rangle_{\widetilde{g}}\right)$ is a linear isometry. We first show that φ is a diffeomorphism: by [4] Theorem 4.14 (c), it suffices to show it has constant rank. But this is resulted from φ being a smooth immersion. That's because isometry implies injectivity by the positivedefiniteness of the norm: for linear map $A: V \rightarrow W$, let $v \in V$ s.t. $Av = \mathbf{0}$; then $0 = \|\mathbf{0}\|_W = \|Av\|_W = \|v\|_V \Rightarrow v = \mathbf{0}$; thus $A^{-1}(\mathbf{0}) = \{\mathbf{0}\} \Rightarrow d\varphi_p$ is injective. The remaining is to pass $\langle v, w\rangle_g = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\widetilde{g}}$ to $\varphi^*\widetilde{g} = g$, but this argument is the same as the " $\Rightarrow$ " direction because the metrics are pointwise defined. ■

A composition of isometries and the inverse of an isometry are again isometries, so being isometric is an equivalence relation on the class of Riemannian manifolds with or without boundary. Our subject, Riemannian geometry, is concerned primarily with properties of Riemannian manifolds that are preserved by isometries.

If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, a map $\varphi: M \rightarrow \widetilde{M}$ is a **local isometry** if each point $p \in M$ has a neighborhood U such that $\varphi|_U$ is an isometry onto an open subset of $\widetilde{M}$. That is, φ is said to be a local isometry if $\forall p \in M$, there is a neighborhood U of p such that $\phi: U \rightarrow \varphi(U)$, defined as the restriction of $\varphi|_U: U \rightarrow \widetilde{M}$ onto codomain $\varphi(U)$, is diffeomorphism from (open) Riemannian submanifold $(U, \iota_U^* g)$ to (open) Riemannian submanifold $\left(\varphi(U), \iota_{\varphi(U)}^* \widetilde{g}\right)$ with $\phi^*\left(\iota_{\varphi(U)}^* \widetilde{g}\right) = \iota_U^* g$. We need to first explain how $\iota^* \widetilde{g}$ gives a Riemannian metric on M (called **pullback metric**). In fact,

Lemma 2.1.2. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold with or without boundary, M is a smooth manifold with or without boundary, and $F: M \rightarrow \widetilde{M}$ is a smooth map. The smooth 2-tensor field $g = F^* \widetilde{g}$ is a Riemannian metric on M if and only if F is an immersion.*

Proof. We have $g_p(v, w) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(w))$. Thus, symmetry and bilinearity of g_p follows from that of $\widetilde{g}_{F(p)}$, and positive definiteness is true iff $dF_p(v) = 0$ implies $v = 0$, i.e., dF_p is injective. ■

A **Riemannian submanifold** (M, g) is then a manifold $M \subseteq \widetilde{M}$ equipped with the metric $g = \iota^* \widetilde{g}$ induced by the pullback of the inclusion $\iota : M \to \widetilde{M}$:

$$g_p(v, w) = \widetilde{g}_p \left(d\iota_p(v), d\iota_p(w) \right).$$

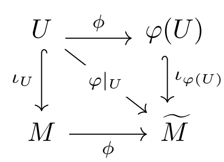
Because we usually identify T_pM with its image in $T_p\widetilde{M}$ under $d\iota_p$, and think of $d\iota_p$ as an inclusion map, what this really amounts to is $g_p(v, w) = \widetilde{g}_p(v, w)$ for $v, w \in T_pM$. In other words, the induced metric g is just the restriction of $\widetilde{g}$ to vectors tangent to M.

We go back to prove an exercise ([5] Exercise 2.7) on local isometry.

Exercise 2.1.3. *Prove that if $(\widetilde{M}, \widetilde{g})$ and (M, g) with $\partial M = \varnothing$ are Riemannian manifolds of the same dimension, a smooth map $\varphi : M \to \widetilde{M}$ is a local isometry if and only if $\varphi^* \widetilde{g} = g$.*

Proof. "$\Leftarrow$":

$\varphi^* \widetilde{g} = g \xRightarrow{\text{proof of (2.2)}}$ each $d\varphi_p$ is a linear isometry $\implies$ $d\varphi_p$ is injective $\implies$ φ is a smooth immersion $\xRightarrow{\text{[4]4.8(b)},\partial M=\varnothing,\dim M=\dim \widetilde{M}=n}$ φ is a local diffeomorphism $\xRightarrow{\text{defn.}}$ $\forall p \in M, \exists$ nbd U of p s.t. $\phi : U \to \varphi(U)$ is a diffeomorphism. The left is to check $\phi^* \left(\iota^*_{\varphi(U)} \widetilde{g} \right) = \iota^*_U g$. The following commutative diagram is helpful. We see that $\iota_{\varphi(U)} \circ \phi = \varphi|_U = \varphi \circ \iota_U$.



Note that $\phi^* \left(\iota^*_{\varphi(U)} \widetilde{g} \right) = \left(\iota_{\varphi(U)} \circ \phi \right)^* \widetilde{g} = \left(\varphi|_U \right)^* \widetilde{g}$. Noticing that $\varphi|_U = \varphi \circ \iota_U$ and that [4] Proposition 3.9 tells us $d\iota_p : T_pU \to T_pM$ is an isomorphism, we for $v, w \in T_pU$ have

$$\begin{aligned}
&\left[\left(\varphi|_U \right)^* \widetilde{g} \right]_p (v, w) \\
&= \widetilde{g}_{\varphi \circ \iota_U(p)} \Big(d\left(\varphi \circ (l_U) \right)_p (v), d\left(\varphi \circ (l_U) \right)_p (w) \Big) \\
&= \widetilde{g}_{\varphi(p)} \Big(d\varphi_{\iota_U(p)} \circ d\left(\iota_U \right)_p (v), d\varphi_{\iota_U(p)} \circ d\left(\iota_U \right)_p (w) \Big) \\
&= \widetilde{g}_{\varphi(p)} \Big(d\varphi_p \Big(d\left(l_U \right)_p (v) \Big), d\varphi_p \Big(d\left(\iota_U \right)_p (w) \Big) \Big) \\
&= \left(\varphi^* \widetilde{g} \right)_p \Big(d\left(\iota_U \right)_p (v), d\left(\iota_U \right)_p (w) \Big) \\
&\overset{\text{given}}{=} g_p \Big(d\left(l_U \right)_p (v), d\left(\iota_U \right)_p (w) \Big) \\
&= \left(\iota^*_U g \right)_p (v, w)
\end{aligned} \tag{2.3}$$

Thus $\left(\varphi|_U \right)^* \widetilde{g} = \iota^*_U g$ on T_pU.

"$\Rightarrow$":

Now φ is a local isometry. $\forall p \in M$, there exists a neighborhood U of p s.t. $\phi : U \to \varphi(U)$ is a diffeomorphism from (open) Riemannian submanifold $(U, \iota^*_U g)$ to (open) Riemannian submanifold $\left(\varphi(U), \iota^*_{\varphi(U)} \widetilde{g} \right)$ with $\left(\varphi|_U \right)^* \left(l^*_{\varphi(U)} \widetilde{g} \right) = \left(\varphi|_U \right)^* \widetilde{g} = \iota^*_U g$. Since $d\left(\iota_U \right)_p : T_pU \to T_pM$ is an isomorphism (see [4] Proposition

3.9), we for $\widehat{v}, \widehat{w} \in T_pM$ have $v = \left[d\left(\iota_U\right)_p\right]^{-1}(\widehat{v}), w = \left[d\left(\iota_U\right)_p\right]^{-1}(\widehat{w}) \in T_pU$ and

$$\begin{aligned} g_p(\widehat{v}, \widehat{w}) &= g_p\left(d\left(\iota_U\right)_p(v), d\left(\iota_U\right)_p(w)\right) \\ &= \left(\iota_U^* g\right)_p(v, w) \\ &\overset{\text{given}}{=} \left[\left(\varphi|_U\right)^* \widetilde{g}\right]_p(v, w) \\ &\overset{(2.2)}{=\!=} \left(\varphi^* \widetilde{g}\right)_p\left(d\left(\iota_U\right)_p(v), d\left(\iota_U\right)_p(w)\right) \\ &= \left(\varphi^* \widetilde{g}\right)_p(\widehat{v}, \widehat{w}) \end{aligned}$$

This shows $g = \varphi^* \widetilde{g}$. ■

Remark 2.1.4. We enforced $\partial M = \varnothing$ to use [4] Proposition 4.8 (b), which is used in the proof of [4] Theorem 4.14 (c). Also note that we don't need $\partial \widetilde{M} = \varnothing$ due to [4] 4.9. ♠

2.2 Methods for Constructing Riemannian Metrics

2.2.1 Riemannian submanifold

As we have seen, every submanifold M of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ inherits a Riemannian metric $g = \iota^* \widetilde{g}$.

Example 2.2.1 (Spheres)**.** For each positive integer n, the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold. The Riemannian metric induced on $\mathbb{S}^n$ by the Euclidean metric is denoted by $\mathring{g}$ and known as the **round metric** or **standard metric on** $\mathbb{S}^n$. ♣

The next proposition describes one of the most important tools for studying Riemannian submanifolds. If $(\widetilde{M}, \widetilde{g})$ is an m-dimensional smooth Riemannian manifold and $M \subseteq \widetilde{M}$ is an n-dimensional submanifold (both with or without boundary), a local frame $(E_1, \ldots, E_m)$ for $\widetilde{M}$ on an open subset $\widetilde{U} \subseteq \widetilde{M}$ is said to be **adapted to** M if the first n vector fields $(E_1, \ldots, E_n)$ are tangent to M. (see remark below.)

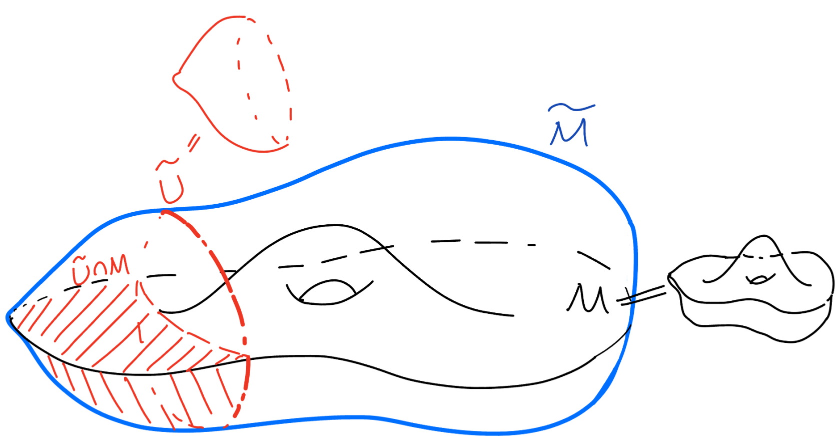


Figure 2.1: Adapted local frame

Remark 2.2.2. From [4] p.116, we can see that T_pM can be seen as a subspace of $T_p\widetilde{M}$. Thus, $n = \dim T_pM \leq \dim T_p\widetilde{M} = m$. When we say $(E_1, \ldots, E_n)$ are tangent to M we mean for each $p \in \widetilde{U} \cap M$, we have $(E_i)_p \in T_pM$ (notice that $(E_i)_p$ is defined in $T_p\widetilde{M}$ but not necessarily in T_pM, which, as we just said in last senetence, is a subspace of $T_p\widetilde{M}$.) ♠

Proposition 2.2.3 (Existence of Adapted Orthonormal Frames). *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold (without boundary), and let $M \subseteq \widetilde{M}$ be an embedded smooth submanifold with or without boundary. Given $p \in M$, there exist a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame for $\widetilde{M}$ on $\widetilde{U}$ that is adapted to M.*

Exercise 2.2.4. *Prove the preceding proposition. [Hint: Apply the Gram-Schmidt algorithm to a coordinate frame in slice coordinates (see [5] Prop. A.22).]*

Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold (without boundary) and $M \subseteq \widetilde{M}$ is a smooth submanifold with or without boundary in $\widetilde{M}$. Given $p \in M$, a vector $\nu \in T_p\widetilde{M}$ is said to be **normal to** M if $\langle \nu, w \rangle = 0$ for every $w \in T_pM$. The space of all vectors normal to M at p is a subspace of $T_p\widetilde{M}$, called the **normal space at** p and denoted by $N_pM = (T_pM)^\perp$. At each $p \in M$, the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \oplus N_pM$. A section N of the ambient tangent bundle $T\widetilde{M}\big|_M$ is called a **normal vector field along** M if $N_p \in N_pM$ for each $p \in M$. The set

$$NM = \coprod_{p \in M} N_pM$$

is called the normal bundle of M. Fig. 2.2 gives an example where vector $v \in T_p\widetilde{M}$ is normal to T_pM for $\widetilde{M} \subseteq \mathbb{R}^3$.

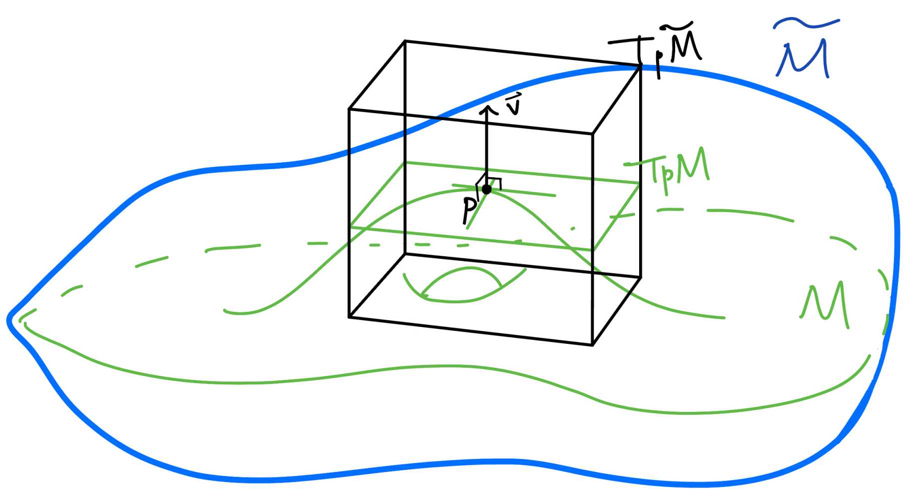


Figure 2.2: Tangent space of Riemannian submanifold

Proposition 2.2.5 (The Normal Bundle). *If $\widetilde{M}$ is a Riemannian m-manifold (without boundary) and $M \subseteq \widetilde{M}$ is an immersed or embedded n-dimensional submanifold with or without boundary, then NM is a smooth rank-$(m-n)$ vector subbundle of the ambient tangent bundle $T\widetilde{M}\big|_M$. There are smooth bundle homomorphisms*

$$\pi^\top : T\widetilde{M}\big|_M \to TM, \quad \pi^\perp : T\widetilde{M}\big|_M \to NM$$

called the **tangential** *and* **normal projections***, that for each $p \in M$ restrict to orthogonal projections from $T_p\widetilde{M}$ to T_pM and N_pM, respectively.*

Proof. See [5] Proposition 2.16. ■

In case $\widetilde{M}$ is a manifold with boundary, the preceding constructions do not always work, because there is not a fully general construction of slice coordinates in that case. However, there is a satisfactory result in case the submanifold is the boundary itself, using boundary coordinates in place of slice coordinates.

Suppose (M, g) is a Riemannian manifold with boundary. We will always consider ∂M to be a Riemannian submanifold with the induced metric.

Proposition 2.2.6 (Existence of Outward-Pointing Normal)**.** *If (M, g) is a smooth Riemannian manifold with boundary, the normal bundle to ∂M is a smooth rank-1 vector bundle over ∂M, and there is a unique smooth outward-pointing unit normal vector field along all of ∂M.*

Exercise 2.2.7. *Prove this proposition. [Hint: Use the paragraph preceding [5] Prop. B.17 as a starting point.]*

Computations on a submanifold $M \subseteq \widetilde{M}$ are usually carried out most conveniently in terms of a **smooth local parametrization**: this is a smooth map $X : U \to \widetilde{M}$, where U is an open subset of $\mathbb{R}^n$ (or $\mathbb{R}^n_+$ in case M has a boundary), such that $X(U)$ is an open subset of M, and such that X, regarded as a map from U into M, is a diffeomorphism onto its image. Note that we can think of X either as a map into M or as a map into $\widetilde{M}$; both maps are typically denoted by the same symbol X. If we put $V = X(U) \subseteq M$ and $\varphi = X^{-1} : V \to U$, then (V, φ) is a smooth coordinate chart on M.

Suppose (M, g) is a Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ and $X : U \to \widetilde{M}$ is a smooth local parametrization of M. The coordinate representation of g in these coordinates is given by the following 2-tensor field on U:

$$\left(\varphi^{-1}\right)^* g = X^* g = X^* \iota^* \widetilde{g} = (\iota \circ X)^* \widetilde{g}.$$

Since $\iota \circ X$ is just the map X itself, regarded as a map into $\widetilde{M}$, this is really just $X^*\widetilde{g}$. The simplicity of the formula for the pullback of a tensor field makes this expression exceedingly easy to compute, once a coordinate expression for $\widetilde{g}$ is known. For example, if M is an immersed n-dimensional Riemannian submanifold of $\mathbb{R}^m$ and $X : U \to \mathbb{R}^m$ is a smooth local parametrization of M, the induced metric on U is just

$$g = X^* \bar{g} \overset{\text{[4] Cor.12.28}}{=\!=\!=} \sum_{i=1}^{m} \left(dX^i\right)^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \frac{\partial X^i}{\partial u^j} du^j \right)^2 = \sum_{i=1}^{m} \sum_{j,k=1}^{n} \frac{\partial X^i}{\partial u^j} \frac{\partial X^i}{\partial u^k} du^j du^k. \tag{2.4}$$

where (u^i) stands for the coordinates of $\mathbb{R}^n \supseteq U$.

Example 2.2.8 (Metrics in Graph Coordinates)**.** If $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ is a smooth function, then the **graph of** f is the subset $\Gamma(f) = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^{n+1}$, which is an embedded submanifold of dimension n. It has a global parametrization $X : U \to \mathbb{R}^{n+1}$ called a **graph parametrization**, given by $X(u) = (u, f(u))$; the corresponding coordinates $\left(u^1, \ldots, u^n\right)$ on M are called **graph coordinates**. In graph coordinates, by (2.4), the induced metric of $\Gamma(f)$ is

$$X^* \bar{g} = \sum_{i=1}^{n} \left(\overbrace{\sum_{j=1}^{n} \underbrace{\frac{\partial u^i}{\partial u^j}}_{=\delta_{ij}} du^j}^{=du^i} \right)^2 + \left(\overbrace{\sum_{j=1}^{n} \frac{\partial f(u)}{\partial u^j} du^j}^{=df} \right)^2 = \left(du^1\right)^2 + \cdots + \left(du^n\right)^2 + df^2.$$

Applying this to the upper hemisphere of $\mathbb{S}^2$ with the parametrization $X : \mathbb{B}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

we see that the round metric on $\mathbb{S}^2$ can be written locally as

$$\begin{aligned}\mathring{g} = X^*\bar{g} &= du^2 + dv^2 + \left(\frac{udu + vdv}{\sqrt{1-u^2-v^2}}\right)^2 \\ &= \frac{\left(1-v^2\right)du^2 + \left(1-u^2\right)dv^2 + 2uvdudv}{1-u^2-v^2}.\end{aligned}$$

♣

Example 2.2.9 (Surfaces of Revolution)**.** Let H be the half-plane $\{(r,z): r>0\}$, and suppose $C \subseteq H$ is an embedded 1-dimensional submanifold. The **surface of revolution** determined by C is the subset $S_C \subseteq \mathbb{R}^3$ given by

$$S_C = \left\{(x,y,z) : \left(\sqrt{x^2+y^2}, z\right) \in C\right\}.$$

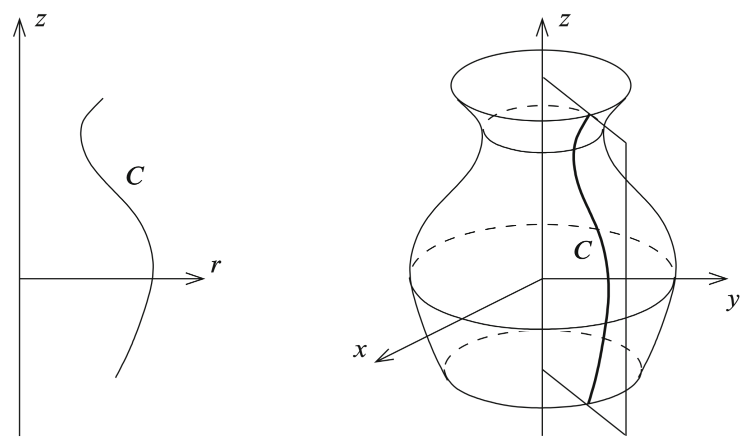


Figure 2.3: A surface of revolution

The set C is called its **generating curve** (see Fig. 2.3). Every smooth local parametrization $\gamma(t) = (a(t), b(t))$ for C yields a smooth local parametrization for S_C of the form

$$X(t,\theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t)),$$

provided that (t,θ) is restricted to a sufficiently small open set in the plane. The t-coordinate curves $t \mapsto X(t,\theta_0)$ are called **meridians**, and the θ-coordinate curves $\theta \mapsto X(t_0,\theta)$ are called **latitude circles**. The induced metric on S_C is

$$\begin{aligned}X^*\bar{g} =& d(a(t)\cos\theta)^2 + d(a(t)\sin\theta)^2 + d(b(t))^2 \\ =& \left(a'(t)\cos\theta dt - a(t)\sin\theta d\theta\right)^2 \\ &+ \left(a'(t)\sin\theta dt + a(t)\cos\theta d\theta\right)^2 + \left(b'(t)dt\right)^2 \\ =& \left(a'(t)^2 + b'(t)^2\right)dt^2 + a(t)^2 d\theta^2.\end{aligned}$$

In particular, if γ is a **unit-speed curve** (meaning that $|\gamma'(t)|^2 = a'(t)^2 + b'(t)^2 \equiv 1$), this reduces to $dt^2 + a(t)^2 d\theta^2$. Here are some examples of surfaces of revolution and their induced metrics.

- If C is the semicircle $r^2 + z^2 = 1$, parametrized by $\gamma(\varphi) = (\sin\varphi, \cos\varphi)$ for $0 < \varphi < \pi$, then S_C is the unit sphere (minus the north and south poles). The map $X(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ constructed above is called the **spherical coordinate parametrization**, and the induced metric is $d\varphi^2 + \sin^2\varphi d\theta^2$. (This example is the source of the terminology for meridians and latitude circles.)

- If C is the circle $(r-2)^2+z^2=1$, parametrized by $\gamma(t)=(2+\cos t, \sin t)$, we obtain a torus of revolution, whose induced metric is $dt^2+(2+\cos t)^2 d\theta^2$.
- If C is a vertical line parametrized by $\gamma(t)=(1,t)$, then S_C is the unit cylinder $x^2+y^2=1$, and the induced metric is $dt^2+d\theta^2$. Note that this means that the parametrization $X:\mathbb{R}^2\to\mathbb{R}^3$ is an isometric immersion.

♣

Example 2.2.10 (The $\boldsymbol{n}$-Torus as a Riemannian Submanifold)**.** The n**-torus** is the manifold $\mathbb{T}^n=\mathbb{S}^1\times\cdots\times\mathbb{S}^1$, regarded as the subset of $\mathbb{R}^{2n}$ defined by $(x^1)^2+(x^2)^2+\cdots+(x^{2n-1})^2+(x^{2n})^2=1$. The smooth covering map $X:\mathbb{R}^n\to\mathbb{T}^n$, defined by $X(u^1,\cdots,u^n)=(\cos u^1,\sin u^1,\cdots,\cos u^n,\sin u^n)$, restricts to a smooth local parametrization on any sufficiently small open subset of $\mathbb{R}^n$, and the induced metric is equal to the Euclidean metric in $\left(u^i\right)$ coordinates, and therefore the induced metric on $\mathbb{T}^n$ is flat. ♣

Exercise 2.2.11. *Verify the claims in above three examples.*

2.2.2 Riemannian Products

Next we consider products. If (M_1,g_1) and (M_2,g_2) are Riemannian manifolds, the product manifold $M_1\times M_2$ has a natural Riemannian metric $g=g_1\oplus g_2$, called the product metric, defined by

$$g_{(p_1,p_2)}\left((v_1,v_2),(w_1,w_2)\right)=\left.g_1\right|_{p_1}(v_1,w_1)+\left.g_2\right|_{p_2}(v_2,w_2),$$

where (v_1,v_2) and (w_1,w_2) are elements of $T_{p_1}M_1\oplus T_{p_2}M_2$, which is naturally identified with $T_{(p_1,p_2)}(M_1\times M_2)$. Smooth local coordinates $\left(x^1,\ldots,x^n\right)$ for M_1 and $\left(x^{n+1},\ldots,x^{n+m}\right)$ for M_2 give coordinates $\left(x^1,\ldots,x^{n+m}\right)$ for $M_1\times M_2$. In terms of these coordinates, the product metric has the local expression $g=g_{ij}dx^idx^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij})=\begin{pmatrix}(g_1)_{ab} & 0\\ 0 & (g_2)_{cd}\end{pmatrix}$$

here the indices a,b run from 1 to n, and c,d run from $n+1$ to $n+m$. Product metrics on products of three or more Riemannian manifolds are defined similarly.

Exercise 2.2.12. *Show that the induced metric on $\mathbb{T}^n$ described in Example 2.2.10 is equal to the product metric obtained from the usual induced metric on $\mathbb{S}^1\subseteq\mathbb{R}^2$.*

Here is an important generalization of product metrics. Suppose (M_1,g_1) and (M_2,g_2) are two Riemannian manifolds, and $f:M_1\to\mathbb{R}^+$is a strictly positive smooth function. The **warped product** $M_1\times_f M_2$ is the product manifold $M_1\times M_2$ endowed with the Riemannian metric $g=g_1\oplus f^2g_2$, defined by

$$g_{(p_1,p_2)}\left((v_1,v_2),(w_1,w_2)\right)=\left.g_1\right|_{p_1}(v_1,w_1)+f(p_1)^2\left.g_2\right|_{p_2}(v_2,w_2),$$

where $(v_1,v_2),(w_1,w_2)\in T_{p_1}M_1\oplus T_{p_2}M_2$ as before. (Despite the similarity with the notation for product metrics, $g_1\oplus f^2g_2$ is generally not a product metric unless f is constant.) A wide variety of metrics can be constructed in this way; here are just a few examples.

Example 2.2.13 (Warped Products)**.**

(a) With $f\equiv 1$, the warped product $M_1\times fM_2$ is just the space $M_1\times M_2$ with the product metric.

(b) Every surface of revolution can be expressed as a warped product, as follows. Let H be the half-plane $\{(r,z):r>0\}$, let $C\subseteq H$ be an embedded smooth 1-dimensional submanifold, and let $S_C\subseteq\mathbb{R}^3$ denote the corresponding surface of revolution as in Example 2.2.9. Endow C with the Riemannian metric induced from the Euclidean metric on H, and let $\mathbb{S}^1$ be endowed with its standard metric. Let $f:C\to\mathbb{R}$ be the distance to the z-axis: $f(r,z)=r$. Then [5] Problem 2-3 shows that S_C is isometric to the warped product $C\times{}_f\mathbb{S}^1$.

(c) If we let ρ denote the standard coordinate function on $\mathbb{R}^+ \subseteq \mathbb{R}$, then the map $\Phi(\rho, \omega) = \rho\omega$ gives an isometry from the warped product $\mathbb{R}^+ \times_\rho \mathbb{S}^{n-1}$ to $\mathbb{R}^n \backslash \{0\}$ with its Euclidean metric (see [5] Problem 2-4).

♣

2.2.3 Riemannian Submersions

Unlike submanifolds and products, the quotient of Riemannian manifolds only inherit Riemannian metrics under very special circumstances. Suppose $\widetilde{M}$ and M are smooth manifolds, $\pi : \widetilde{M} \to M$ is a smooth submersion, and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$. By the submersion level set theorem (see [4] Cor.5.13), each level set $\widetilde{M_y} = \pi^{-1}(y)$ is regular (as π is a smooth submersion) and a properly embedded smooth submanifold of $\widetilde{M}$, and π is a defining map for $\widetilde{M_y}$ (see [4] p.107). Then by [4] Prop.5.38, $T_x\widetilde{M_y} = \operatorname{Ker}\left(d\pi_x : T_x\widetilde{M} \to T_{\pi(x)}M\right)$ for any $x \in \widetilde{M_y}$. Therefore, at each point $x \in \widetilde{M}$, we define two subspaces of the tangent space $T_x\widetilde{M}$ as follows: the **vertical tangent space at** x is

$$V_x = \operatorname{Ker} d\pi_x = T_x\left(\widetilde{M}_{\pi(x)}\right)$$

(that is, the tangent space to the fiber containing x), and the **horizontal tangent space at** x is its orthogonal complement:

$$H_x = (V_x)^\perp := \left\{v \in T_x\widetilde{M} \mid \forall w \in T_x\left(\widetilde{M}_{\pi(x)}\right), \langle v, w\rangle_g = 0\right\}$$

Then the tangent space $T_x\widetilde{M}$ decomposes as an orthogonal direct sum $T_x\widetilde{M} = H_x \oplus V_x$. Note that the vertical space is well defined for every submersion, because it does not refer to the metric; but the horizontal space depends on the metric.

A vector field on $\widetilde{M}$ is said to be a **horizontal vector field** if its value at each point lies in the horizontal space at that point; a **vertical vector field** is defined similarly. Given a vector field X on M, a vector field $\widetilde{X}$ on $\widetilde{M}$ is called a **horizontal lift of** X if $\widetilde{X}$ is horizontal and π-related to X. (The latter property means that $d\pi_x\left(\widetilde{X}_x\right) = X_{\pi(x)}$ for each $x \in \widetilde{M}$.) In other words, the following diagram is commutative ($\widetilde{X}$ is so-called "lift"):

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widetilde{X}} & T\widetilde{M} = \coprod_{x \in \widetilde{M}} T_x\widetilde{M} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle d\pi} \\
M & \xrightarrow[X]{} & TM = \coprod_{x \in M} T_xM
\end{array}$$

The next proposition is the principal tool for doing computations on Riemannian submersions.

Proposition 2.2.14 (Properties of Horizontal Vector Fields). *Let $\widetilde{M}$ and M be smooth manifolds, let $\pi : \widetilde{M} \to M$ be a smooth submersion, and let $\widetilde{g}$ be a Riemannian metric on $\widetilde{M}$.*

(a) Every smooth vector field W on $\widetilde{M}$ can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.

(b) Every smooth vector field on M has a unique smooth horizontal lift to $\widetilde{M}$.

(c) For every $x \in \widetilde{M}$ and $v \in H_x$, there is a vector field $X \in \mathfrak{X}(M)$ whose horizontal lift $\widetilde{X}$ satisfies $\widetilde{X}_x = v$.

Proof. [5] Proposition 2.25. ■

The fact that every horizontal vector at a point of $\widetilde{M}$ can be extended to a horizontal lift on all of $\widetilde{M}$ (part (c) of the preceding proposition) is highly useful for computations. It is important to be aware, though, that not every horizontal vector field on $\widetilde{M}$ is a horizontal lift, as the next exercise shows.

Exercise 2.2.15. *Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection map $\pi(x, y) = x$, and let W be the smooth vector field $y\partial_x$ on $\mathbb{R}^2$. Show that W is horizontal, but there is no vector field on $\mathbb{R}$ whose horizontal lift is equal to W.*

Now we can identify some quotients of Riemannian manifolds that inherit metrics of their own. Let us begin by describing what such a metric should look like.

Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds, and $\pi : \widetilde{M} \to M$ is a smooth submersion. Then π is said to be a **Riemannian submersion** if for each $x \in \widetilde{M}$, the differential $d\pi_x$ restricts to a linear isometry from H_x onto $T_{\pi(x)}M$. In other words, $\widetilde{g}_x(v, w) = g_{\pi(x)}\left(d\pi_x(v), d\pi_x(w)\right)$ whenever $v, w \in H_x$.

Remark 2.2.16. Note that $d\pi_x : T_x\widetilde{M} = V_x \oplus H_x = \operatorname{Ker}(d\pi_x) \oplus (V_x)^{\perp} \to T_{\pi(x)}M$ is a C^{∞} submersion and is thus onto. Thus, $\forall v' \in T_{\pi(x)}M, \exists v = v_{V_x} + v_{H_x}$ s.t $v' = d\pi_x(v) = d\pi_x(v_{V_x}) + d\pi_x(v_{H_x}) = 0 + d\pi_x(v_{H_x}) = d\pi_x(v_{H_x})$. This shows that $d\pi_x|_{H_x} : H_x \to T_{\pi(x)}M$ is also onto. Therefore, in the above definition, the only requirement is linear isometry. ♠

Example 2.2.17 (Riemannian Submersions)**.**

(a) The projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a Riemannian submersion if $\mathbb{R}^{n+k}$ and $\mathbb{R}^n$ are both endowed with their Euclidean metrics.

(b) If M and N are Riemannian manifolds and $M \times N$ is endowed with the product metric, then both projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Riemannian submersions.

(c) If $M \times_f N$ is a warped product manifold, then the projection $\pi_M : M \times_f N \to M$ is a Riemannian submersion, but π_N typically is not.

♣

Exercise 2.2.18. *Verify above example.*

Solution. [Incomplete soln] We do (a).

For

$$\begin{gathered}\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n \\ (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}) \mapsto (x_1, \cdots, x_n)\end{gathered}$$

we have components $\pi_i(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}) = x_i$ for $1 \le i \le n$ and Jacobian

$$J_\pi(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times (n+k)}$$

Then

$$d\pi_x\left(\sum_i^{n+k} v^i \frac{\partial}{\partial x^i}\right) = \left[J_\pi(x)\begin{pmatrix} v^1 \\ \vdots \\ v^{n+k}\end{pmatrix}\right] \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n}\end{pmatrix} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n}\end{pmatrix} = \sum_i^n v^i \frac{\partial}{\partial x^i}$$

Since

$$V_x = \operatorname{Ker}(d\pi_x) = \left\{ \sum_i^{n+k} v^i \frac{\partial}{\partial x^i} \,\middle|\, v^1 = \cdots = v^n = 0 \right\}$$

and

$$H_x = (V_x)^{\perp} = \left\{ \sum_i^{n+k} v^i \frac{\partial}{\partial x^i} \middle| v^{n+1} = \cdots = v^{n+k} = 0 \right\},$$

we see $\forall v, w \in H_x$,

$$\begin{aligned} \widetilde{g}(v,w) = \bar{g}(v,w) = \sum\nolimits^{n+k} |v_i - w_i|^2 = \sum\nolimits^{n} |v_i - w_i|^2 + \sum\nolimits^{k} |v_{n+i} - w_{n+i}|^2 \\ = \sum\nolimits^{n} |v_i - w_i|^2 = \bar{g}\left(d\pi_x(v), d\pi_x(w)\right) = g\left(d\pi_x(v), d\pi_x(w)\right) \end{aligned}$$

The map π is thus a Riemannian submersion. ♦

Given a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and a surjective submersion $\pi : \widetilde{M} \to M$, it is almost never the case that there is a metric on M that makes π into a Riemannian submersion. It is not hard to see why: for this to be the case, whenever $p_1, p_2 \in \widetilde{M}$ are two points in the same fiber $\pi^{-1}(y)$, the linear maps $\left(d\pi_{p_i}|_{H_{p_i}}\right)^{-1} : T_yM \to H_{p_i}$ both have to pull $\widetilde{g}$ back to the same inner product on T_yM.

There is, however, an important special case in which there is such a metric. Suppose $\pi : \widetilde{M} \to M$ is a smooth surjective submersion, and G is a group acting on $\widetilde{M}$. (See [5] Appendix C for a review of the basic definitions and terminology regarding group actions on manifolds.) We say that the action is **vertical** if every element $\varphi \in G$ takes each fiber to itself, meaning that $\pi(\varphi \cdot p) = \pi(p)$ for all $p \in \widetilde{M}$. The action is **transitive on fibers** if for each $p, q \in \widetilde{M}$ such that $\pi(p) = \pi(q)$, there exists $\varphi \in G$ such that $\varphi \cdot p = q$.

If in addition $\widetilde{M}$ is endowed with a Riemannian metric, the action is said to be an **isometric action** or an **action by isometries**, and the metric is said to be **invariant under** G, if the map $x \mapsto \varphi \cdot x$ is an isometry for each $\varphi \in G$. In that case, provided the action is effective (so that different elements of G define different isometries of $\widetilde{M}$), we can identify G with a subgroup of $\mathrm{Iso}(\widetilde{M}, g)$. Since an isometry is, in particular, a diffeomorphism, every isometric action is an action by diffeomorphisms.

Theorem 2.2.19. *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, let $\pi : \widetilde{M} \to M$ be a surjective smooth submersion, and let G be a group acting on $\widetilde{M}$. If the action is isometric, vertical, and transitive on fibers, then there is a unique Riemannian metric on M such that π is a Riemannian submersion.*

Proof. Problem 2.5.4. ■

The next corollary describes one important situation to which the preceding theorem applies.

Corollary 2.2.20. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and G is a Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then the orbit space $M = \widetilde{M}/G$ has a unique smooth manifold structure and Riemannian metric such that π is a Riemannian submersion.*

Proof. Under the given hypotheses, the quotient manifold theorem (see [4] Theorem 21.10) shows that M has a unique smooth manifold structure such that the quotient map $\pi : \widetilde{M} \to M$ is a smooth submersion. It follows easily from the definitions in that case that the given action of G on $\widetilde{M}$ is vertical and transitive on fibers. Since the action is also isometric, Theorem 2.2.19 shows that M inherits a unique Riemannian metric making π into a Riemannian submersion. ■

Here is an important example of a Riemannian metric defined in this way. A larger class of such metrics is described in Problem 2.5.5.

Example 2.2.21 (The Fubini-Study Metric)**.** Let n be a positive integer, and consider the complex projective space $\mathbb{CP}^n$ defined in [5] Example C.19. That example shows that the map $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ sending each point in $\mathbb{C}^{n+1} \backslash \{0\}$ to its span is a surjective smooth submersion. Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ endowed with its Euclidean metric, we can view the unit sphere $\mathbb{S}^{2n+1}$ with its round metric $\mathring{g}$ as an embedded Riemannian submanifold of $\mathbb{C}^{n+1} \backslash \{0\}$. Let $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ denote the restriction of the map π. Then p is smooth, and it is surjective, because every 1-dimensional complex subspace contains elements of unit norm. We need to show that it is a submersion. Let $z_0 \in \mathbb{S}^{2n+1}$ and set $\zeta_0 = p(z_0) \in \mathbb{CP}^n$. Since π is a smooth submersion, it has a smooth local section $\sigma : U \to \mathbb{C}^{n+1}$ defined on a neighborhood U of ζ_0 and satisfying $\sigma(\zeta_0) = z_0$ (Thm. A.17). Let $v : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{S}^{2n+1}$ be the radial projection onto the sphere:

$$\nu(z) = \frac{z}{|z|}.$$

Since dividing an element of $\mathbb{C}^{n+1}$ by a nonzero scalar does not change its span, it follows that $p \circ \nu = \pi$. Therefore, if we set $\widetilde{\sigma} = \nu \circ \sigma$, we have $p \circ \widetilde{\sigma} = p \circ \nu \circ \sigma = \pi \circ \sigma = \mathrm{Id}_U$, so $\widetilde{\sigma}$ is a local section of p. By the local secrtion theorem (see [4] Theorem 4.26), this shows that p is a submersion. Define an action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ by complex multiplication:

$$\lambda \cdot \left(z^1, \ldots, z^{n+1}\right) = \left(\lambda z^1, \ldots, \lambda z^{n+1}\right)$$

for $\lambda \in \mathbb{S}^1$ (viewed as a complex number of norm 1) and $z = \left(z^1, \ldots, z^{n+1}\right) \in \mathbb{S}^{2n+1}$. This is easily seen to be isometric, vertical, and transitive on fibers of p. By Theorem 2.2.19, therefore, there is a unique metric on $\mathbb{CP}^n$ such that the map $p : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion. This metric is called the Fubini-Study metric. ♣

2.2.4 Riemannian Coverings

Another important special case of Riemannian submersions occurs in the context of covering maps. Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds. A smooth covering map $\pi : M \to M$ is called a **Riemannian covering** if it is a local isometry.

Proposition 2.2.22. *Suppose $\pi : \widetilde{M} \to M$ is a smooth normal covering map, and $\widetilde{g}$ is any metric on $\widetilde{M}$ that is invariant under all covering automorphisms. Then there is a unique metric g on M such that π is a Riemannian covering.*

Proof. "invariant under $\Gamma = \mathrm{Aut}_\pi(\widetilde{M})$," is defined in last subsection; for normal covering map and beyond, see [3] p.293, 309-314 and [4] Chapter 21; for proof of the proposition, see [5] Proposition 2.31. ■

Proposition 2.2.23. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and Γ is a discrete Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then $\widetilde{M}/\Gamma$ has a unique Riemannian metric such that the quotient map $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$ is a normal Riemannian covering.*

Proof. This is an immediate consequence of [4] Thm.21.13 and above proposition. ■

Corollary 2.2.24. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds, $\pi : \widetilde{M} \to M$ is a normal Riemannian covering map, and $\Gamma = \mathrm{Aut}_\pi(\widetilde{M})$. Then M is isometric to $\widetilde{M}/\Gamma$.*

Proof. Proof by [4] Prop.21.12 & [4] Thm.21.13 & [4] Thm.4.31. ■

Example 2.2.25. The two-element group $\Gamma = \{\pm 1\}$ acts smoothly, freely, properly, and isometrically on $\mathbb{S}^n$ by multiplication. [5] Example C.24 shows that the quotient space is diffeomorphic to the real projective space $\mathbb{RP}^n$ and the quotient map $q : \mathbb{S}^n \to \mathbb{RP}^n$ is a smooth normal covering map. Because the action is isometric, Proposition 2.2.23 shows that there is a unique metric on $\mathbb{RP}^n$ such that q is a Riemannian covering. ♣

Example 2.2.26 (The Open Möbius Band)**.** The **open Möbius band** is the quotient space $M = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}^2$ by $n \cdot (x, y) = (x + n, (-1)^n y)$. This action is smooth, free, proper, and isometric, and therefore M inherits a flat Riemannian metric such that the quotient map is a Riemannian covering. (See Problem 2.5.6) ♣

Exercise 2.2.27. *Let $\mathbb{T}^n \subseteq \mathbb{R}^{2n}$ be the n-torus with its induced metric. Show that the map $X : \mathbb{R}^n \to \mathbb{T}^n$ of Example 2.2.10 is a Riemannian covering.*

2.3 Basic Constructions on Riemannian Manifolds

2.3.1 Raising and Lowering Indices

Given a Riemannian metric g on a smooth manifold M with or without boundary, we define a bundle homomorphism $\widehat{g} : TM \to T^*M$ as follows. For each $p \in M$ and each $v \in T_pM$, we let $\widehat{g}(v) \in T_p^*M$ be the covector defined by

$$\widehat{g}(v)(w) = g_p(v, w) \quad \text{for all } w \in T_pM.$$

To see that this is a smooth bundle homomorphism, it is easiest to consider its action on smooth vector fields:

$$\widehat{g}(X)(Y) = g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Because $\widehat{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y, it follows from the tensor characterization lemma 1.1.18 that $\widehat{g}(X)$ is a smooth covector field; and because $\widehat{g}(X)$ is linear over $C^\infty(M)$ as a function of X, this defines $\widehat{g}$ as a smooth bundle homomorphism by the bundle homomorphism characterization lemma ([4] Lemma 10.29). As usual, we use the same symbol for both the pointwise bundle homomorphism $\widehat{g} : TM \to T^*M$ and the linear map on sections $\widehat{g} : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.

Note that $\widehat{g}$ is injective at each point, because $\widehat{g}(v) = 0$ for some $v \in T_pM$ implies

$$0 = \widehat{g}(v)(v) = \langle v, v \rangle_g,$$

which in turn implies $v = 0$. For dimensional reasons, therefore, $\widehat{g}$ is bijective, so it is a bundle isomorphism (see [4] Proposition 10.26).

Given a smooth local frame (E_i) and its dual coframe $\left(\varepsilon^i\right)$, let $g = g_{ij}\varepsilon^i\varepsilon^j$ be the local expression for g (see (2.1)). If $X = X^iE_i$ is a smooth vector field, the covector field $\widehat{g}(X)$ has the coordinate expression $\widehat{g}(X) = \left(g_{ij}X^i\right)\varepsilon^j$, as

$$\widehat{g}(X)(E_k) = g_{ij}\varepsilon^i\varepsilon^j(X, E_k) = g_{ij}X^i\varepsilon^j(E_k) \implies \widehat{g}(X) = \left(g_{ij}X^i\right)\varepsilon^j.$$

Exercise 2.3.1. *Write down the matrix of $\widehat{g}$ and conclude that the matrix of $\widehat{g}$ in any local frame is the same as the matrix of g itself.*

Solution. For each p, $\widehat{g}$ as a linear mapping from vector space T_pM to vector space T_p^*M sends X_p to the mappping defined by

$$\begin{aligned} T_pM &\to \mathbb{R} \\ Y_p &\mapsto g(X_p, Y_p), \end{aligned}$$

Since

$$\widehat{g}\left(E_k\right)=(g_{ij}(E_k)^i)\varepsilon^j=\sum_j g_{kj}\varepsilon^j.$$

We see by definition 1.1.8 that the matrix of $\widehat{g}$ is

$$A=\begin{pmatrix} g_{11} & \cdots & g_{n1} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{nn} \end{pmatrix}$$

We note that this is the transpose of the matrix of g. However, since the matrix of g is symmetric, we proved the statement. ♦

Given a vector field X, it is standard practice to denote the components of the covector field $\widehat{g}(X)$ by

$$X_j=g_{ij}X^i,$$

so that

$$\widehat{g}(X)=X_j\varepsilon^j,$$

and we say that $\widehat{g}(X)$ is obtained from X by **lowering an index**. With this in mind, the covector field $\widehat{g}(X)$ is denoted by $X^\flat$ and called X **flat**, borrowing from the musical notation for lowering a tone. That is, we also use $\flat$ to denote $\widehat{g}$, which is a smooth bundle isomorphism, as we remarked above.

The matrix of the inverse map $\widehat{g}^{-1}: T_p^*M \to T_pM$ is the inverse of (g_{ij}). (Because (g_{ij}) is the matrix of the isomorphism $\widehat{g}$, it is invertible at each point.) We let $\left(g^{ij}\right)$ denote the matrix-valued function whose value at $p\in M$ is the inverse of the matrix $(g_{ij}(p))$, so that

$$g^{ij}g_{jk}=g_{kj}g^{ji}=\delta^i_k. \tag{2.5}$$

Because g_{ij} is a symmetric matrix, so is g^{ij}, as you can easily check. Thus for a covector field $\omega\in\mathfrak{X}^*(M)$, the vector field $\widehat{g}^{-1}(\omega)$ has the coordinate representation

$$\widehat{g}^{-1}(\omega)=\omega^iE_i$$

where

$$\omega^i=g^{ij}\omega_j. \tag{2.6}$$

If ω is a covector field, the vector field $\widehat{g}^{-1}(\omega)$ is called (what else?) ω **sharp** and denoted by $\omega^\sharp$, and we say that it is obtained from ω by **raising an index**. Likewise, as the inverse of $\widehat{g}$, the map $\widehat{g}^{-1}=\sharp:\Gamma(T^*M)\to\Gamma(TM); T^*M\to TM$ is a smooth bundle isomorphism.

The two inverse isomorphisms $\flat$ and $\sharp$ are known as the **musical isomorphisms**.

Probably the most important application of the sharp operator is to extend the classical gradient operator to Riemannian manifolds. If g is a Riemannian metric on M and $f: M\to\mathbb{R}$ is a smooth function, the **gradient of** f is the vector field $\operatorname{grad} f=(df)^\sharp$ obtained from df by raising an index. Unwinding the definitions, we see that $\operatorname{grad} f$ is characterized by the fact that

$$df_p(w)=\left\langle \operatorname{grad} f|_p, w\right\rangle \quad \text{for all } p\in M, w\in T_pM,$$

since RHS$=g(\operatorname{grad} f|_p, w)=\widehat{g}\left(\operatorname{grad} f|_p\right)(w)=\widehat{g}\left(\widehat{g}^{-1}(df_p)\right)(w)=df_p(w)$. $\operatorname{grad} f$ and has the local basis expression

$$\operatorname{grad} f=\left(g^{ij}(df)_i\right)E_j \xlongequal{(df)_i=(df)(E_i)\overset{[4]14.24}{=}E_i f}\left(g^{ij}E_if\right)E_j.$$

Thus if (E_i) is an orthonormal frame (then $(g_{ij}) = I \implies (g^{ij}) = I^{-1} = I$), then $\operatorname{grad} f$ is the vector field whose components are the same as the components of df; but in other frames, this will not be the case.

The next proposition shows that the gradient has the same geometric interpretation on a Riemannian manifold as it does in Euclidean space. If f is a smooth real-valued function on a smooth manifold M, recall that a point $p \in M$ is called a **regular point of** f if $df_p \neq 0$, and a **critical point of** f otherwise; and a level set $f^{-1}(c)$ is called a **regular level set** if every point of $f^{-1}(c)$ is a regular point of f. [5] Corollary A.26 shows that each regular level set is an embedded smooth hypersurface in M.

Proposition 2.3.2. *Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $\mathcal{R} \subseteq M$ is the set of regular points of f. For each $c \in \mathbb{R}$, the set $M_c = f^{-1}(c) \cap \mathcal{R}$, if nonempty, is an embedded smooth hypersurface in M, and grad f is everywhere normal to M_c.*

Proof. Problem 2.5.7. ∎

Definition 2.3.3. *The flat and sharp operators can be applied to tensors of any rank, in any index position, to convert tensors from covariant to contravariant or vice versa. Formally, this operation is defined as follows: if F is any (k, l)-tensor and $i \in \{1, \ldots, k+l\}$ is any covariant index position for F (meaning that the i th argument is a vector, not a covector), we can form a new tensor $F^\sharp$ of type $(k+1, l-1)$ by setting*

$$F^\sharp(\alpha_1, \ldots, \alpha_{k+l}) = F\left(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i^\sharp, \alpha_{i+1}, \ldots, \alpha_{k+l}\right)$$

whenever $\alpha_1, \ldots, \alpha_{k+l}$ are vectors or covectors as appropriate. In any local frame, the components of $F^\sharp$ are obtained by multiplying the components of F by g^{pq} and contracting one of the indices of g^{pq} with the i th index of F. Similarly, if i is a contravariant index position, we can define a $(k-1, l+1)$-tensor $F^\flat$ by

$$F^\flat(\alpha_1, \ldots, \alpha_{k+l}) = F\left(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i^\flat, \alpha_{i+1}, \ldots, \alpha_{k+l}\right).$$

In components, it is computed by multiplying by g_{pq} and contracting.

Example 2.3.4. For example, if A is a mixed 3-tensor given in terms of a local frame by

$$A = A_i{}^j{}_k \varepsilon^i \otimes E_j \otimes \varepsilon^k,$$

we can lower its middle index to obtain a covariant 3-tensor $A^\flat$ with components

$$A_{ijk} = g_{jl} A_i{}^l{}_k.$$

♣

To avoid overly cumbersome notation, we use the symbols $F^\sharp$ and $F^\flat$ without explicitly specifying which index position the sharp or flat operator is to be applied to; when there is more than one choice, we will always stipulate in words what is meant.

Another important application of the flat and sharp operators is to extend the trace operator introduced to covariant tensors. If h is any covariant k-tensor field on a Riemannian manifold with $k \geq 2$, we can raise one of its indices (say the last one for definiteness) and obtain a $(1, k-1)$-tensor $h^\sharp$. The trace of $h^\sharp$ is thus a well-defined covariant $(k-2)$-tensor field. We define the **trace of** h **with respect to** g as

$$\operatorname{tr}_g h = \operatorname{tr}\left(h^\sharp\right).$$

Sometimes we may wish to raise an index other than the last, or to take the trace on a pair of indices other than the last covariant and contravariant ones. In each such case, we will say in words what is meant.

The most important case is that of a covariant 2-tensor field. In this case, $h^\sharp$ is a $(1, 1)$-tensor field, which can equivalently be regarded as an endomorphism field, and $\operatorname{tr}_g h$ is just the ordinary trace of this endomorphism field.

Proposition 2.3.5. *In terms of a basis, this is*

$$\operatorname{tr}_g h = h_i{}^i = g^{ij} h_{ij}.$$

Proof. By proposition 1.1.12, we see

$$\operatorname{tr}(h^\sharp) = (h^\sharp)_i^i = (h^\sharp)_i{}^i$$

where we note that ${}_i{}^i$ is merely used to emphasize the order of indices. Mimicing example 2.3.4, we may denote $(h^\sharp)_i{}^i$ just as $h_i{}^i$. Besides,

$$(h^\sharp)_i{}^j = h_i{}^j = g^{jk} h_{ik}.$$

Thus

$$\operatorname{tr}(h^\sharp) = (h^\sharp)_i^i = (h^\sharp)_i{}^i = g^{ik} h_{ik} = g^{ij} h_{ij}$$

∎

In particular, in an orthonormal frame this is the ordinary trace of the matrix (h_{ij}) (the sum of its diagonal entries); but if the frame is not orthonormal, then this trace is different from the ordinary trace.

Exercise 2.3.6. *If g is a Riemannian metric on M and (E_i) is a local frame on M, there is a potential ambiguity about what the expression $\left(g^{ij}\right)$ represents: we have defined it to mean the inverse matrix of (g_{ij}), but one could also interpret it as the components of the contravariant 2-tensor field $g^{\sharp\sharp}$ obtained by raising both of the indices of g. Show that these two interpretations lead to the same result.*

2.3.2 Inner Products of Tensors

A Riemannian metric yields, by definition, an inner product on tangent vectors at each point. Because of the musical isomorphisms between vectors and covectors, it is easy to carry the inner product over to covectors as well.

Suppose g is a Riemannian metric on M, and $x \in M$. We can define an inner product on the cotangent space $T_x^* M$ by

$$\langle \omega, \eta \rangle_g = \left\langle \omega^\sharp, \eta^\sharp \right\rangle_g.$$

(Just as with inner products of vectors, we might sometimes omit g from the notation when the metric is understood.) To see how to compute this, we just use the basis formula (2.6) for the sharp operator, together with the relation $g_{kl} g^{ki} = g_{lk} g^{ki} = \delta_l^i$, to obtain

$$\begin{aligned} \langle \omega, \eta \rangle &= \left\langle g^{ki} \omega_i E_k, g^{lj} \eta_j E_l \right\rangle \\ &= \langle E_k, E_l \rangle \left(g^{ki} \omega_i\right) \left(g^{lj} \eta_j\right) \\ &= g_{kl} \left(g^{ki} \omega_i\right) \left(g^{lj} \eta_j\right) \\ &= \delta_l^i g^{lj} \omega_i \eta_j \\ &= g^{ij} \omega_i \eta_j. \end{aligned}$$

In other words, the inner product on covectors is represented by the inverse matrix $\left(g^{ij}\right)$. Using our convention (2.6), this can also be written

$$\langle \omega, \eta \rangle = \omega_i \eta^i = \omega^j \eta_j.$$

Exercise 2.3.7. *Let (M, g) be a Riemannian manifold with or without boundary, let (E_i) be a local frame for M, and let $\left(\varepsilon^i\right)$ be its dual coframe. Show that the following are equivalent:*

(a) (E_i) is orthonormal.

(b) $\left(\varepsilon^i\right)$ is orthonormal.

(c) $\left(\varepsilon^i\right)^\sharp = E_i$ *for each* i.

This construction can be extended to tensor bundles of any rank, as the following proposition shows. First a bit of terminology: if $E \to M$ is a smooth vector bundle, a smooth fiber metric on $\boldsymbol{E}$ is an inner product on each fiber E_p that varies smoothly, in the sense that for any (local) smooth sections σ, τ of E, the inner product $\langle\sigma, \tau\rangle$ is a smooth function.

Proposition 2.3.8 (Inner Products of Tensors)**.** *Let* (M, g) *be an n-dimensional Riemannian manifold with or without boundary. There is a unique smooth fiber metric on each tensor bundle* $T^{(k,l)}TM$ *with the property that if* $\alpha_1, \ldots, \alpha_{k+l}$*,* $\beta_1, \ldots, \beta_{k+l}$ *are vector or covector fields as appropriate, then*

$$\langle\alpha_1 \otimes \cdots \otimes \alpha_{k+l}, \beta_1 \otimes \cdots \otimes \beta_{k+l}\rangle = \langle\alpha_1, \beta_1\rangle \cdot \cdots \cdot \langle\alpha_{k+l}, \beta_{k+l}\rangle . \tag{2.7}$$

With this inner product, if $(E_1, \ldots, E_n)$ *is a local orthonormal frame for* TM *and* $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ *is the corresponding dual coframe, then the collection of tensor fields* $E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}$ *as all the indices range from 1 to* n *forms a local orthonormal frame for* $T^{(k,l)}TM$*. In terms of any (not necessarily orthonormal) frame, this fiber metric satisfies*

$$\langle F, G\rangle = g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F_{j_1 \ldots j_l}^{i_1 \ldots i_k} G_{s_1 \ldots s_l}^{r_1 \ldots r_k} . \tag{2.8}$$

If F *and* G *are both covariant, this can be written*

$$\langle F, G\rangle = F_{j_1 \ldots j_l} G^{j_1 \ldots j_l},$$

where the last factor on the right represents the components of G *with all of its indices raised:*

$$G^{j_1 \ldots j_l} = g^{j_1 s_1} \ldots g^{j_l s_l} G_{s_1 \ldots s_l} .$$

Proof. Problem 2.5.9. ∎

2.3.3 Volume Form and Integration

2.4 Generalizations of Riemannian Metrics

There are other common ways of measuring "lengths" of tangent vectors on smooth manifolds. Let's digress briefly to mention three that play important roles in other branches of mathematics: pseudo-Riemannian metrics, sub-Riemannian metrics, and Finsler metrics. Each is defined by relaxing one of the requirements in the definition of Riemannian metric: a pseudoRiemannian metric is obtained by relaxing the requirement that the metric be positive; a sub-Riemannian metric by relaxing the requirement that it be defined on the whole tangent space; and a Finsler metric by relaxing the requirement that it be quadratic on each tangent space.

Pseudo-Riemannian Metrics

A **pseudo-Riemannian metric** (occasionally also called a **semi-Riemannian metric**) on a smooth manifold M is a symmetric 2-tensor field g that is nondegenerate at each point $p \in M$. This means that the only vector orthogonal to everything is the zero vector. More formally, $g(X, Y) = 0$ for all $Y \in T_p M$ if and only if $X = 0$. If $g = g_{ij} \varphi^i \varphi^j$ in terms of a local coframe, nondegeneracy just means that the matrix g_{ij} is invertible. If g is Riemannian, nondegeneracy follows immediately from positive-definiteness, so every Riemannian metric is also a pseudo-Riemannian metric; but in general pseudo-Riemannian metrics need not be positive.

Given a pseudo-Riemannian metric g and a point $p \in M$, by a simple extension of the Gram-Schmidt algorithm one can construct a basis $(E_1, \ldots, E_n)$ for $T_p M$ in which g has the expression

$$g = -\left(\varphi^1\right)^2 - \cdots - \left(\varphi^r\right)^2 + \left(\varphi^{r+1}\right)^2 + \cdots + \left(\varphi^n\right)^2 \tag{2.9}$$

for some integer $0 \leq r \leq n$. This integer r, called the index of g, is equal to the maximum dimension of any subspace of T_pM on which g is negative definite. Therefore the index is independent of the choice of basis, a fact known classically as *Sylvester's law of inertia*.

By far the most important pseudo-Riemannian metrics (other than the Riemannian ones) are the **Lorentz metrics**, which are pseudo-Riemannian metrics of index 1. The most important example of a Lorentz metric is the **Minkowski metric**; this is the Lorentz metric m on $\mathbb{R}^{n+1}$ that is written in terms of coordinates $\left(\xi^1, \ldots, \xi^n, \tau\right)$ as

$$m = \left(d\xi^1\right)^2 + \cdots + \left(d\xi^n\right)^2 - (d\tau)^2. \tag{2.10}$$

In the special case of $\mathbb{R}^4$, the Minkowski metric is the fundamental invariant of Einstein's special theory of relativity, which can be expressed succinctly by saying that in the absence of gravity, the laws of physics have the same form in any coordinate system in which the Minkowski metric has the expression (2.10). The differing physical characteristics of "space" (the ξ directions) and "time" (the τ direction) arise from the fact that they are subspaces on which g is positive definite and negative definite, respectively. The general theory of relativity includes gravitational effects by allowing the Lorentz metric to vary from point to point.

Many aspects of the theory of Riemannian metrics apply equally well to pseudo-Riemannian metrics. Although we do not treat pseudo-Riemannian geometry directly in this book, we will attempt to point out as we go along which aspects of the theory apply to pseudo-Riemannian metrics. As a rule of thumb, proofs that depend only on the invertibility of the metric tensor, such as existence and uniqueness of the Riemannian connection and geodesics, work fine in the pseudo-Riemannian setting, while proofs that use positivity in an essential way, such as those involving distance-minimizing properties of geodesics, do not.

For an introduction to the mathematical aspects of pseudo-Riemannian metrics, see the excellent book [O'N83] (Barrett O'Neill, *Semi-Riemannian Geometry with Applications to General Relativity*); a more physical treatment can be found in [HE73] (Stephen W. Hawking and George F. R. Ellis, *The Large-Scale Structure of Space-Time*.)

Sub-Riemannian Metrics

A **sub-Riemannian metric** (aka. **singular Riemannian metric** or **Carnot-Carathéodory metric**) on a manifold M is a fiber metric on a smooth distribution $S \subset TM$ (i.e., a k-plane field or sub-bundle of TM). Since lengths make sense only for vectors in S, the only curves whose lengths can be measured are those whose tangent vectors lie everywhere in S. Therefore one usually imposes some condition on S that guarantees that any two nearby points can be connected by such a curve. This is, in a sense, the opposite of the Frobenius integrability condition, which would restrict every such curve to lie in a single leaf of a foliation.

Sub-Riemannian metrics arise naturally in the study of the abstract models of real submanifolds of complex space $\mathbb{C}^n$, called *CR manifolds*. (Here CR stands for "Cauchy-Riemann.") CR manifolds are real manifolds endowed with a distribution $S \subset TM$ whose fibers carry the structure of complex vector spaces (with an additional integrability condition that need not concern us here). In the model case of a submanifold $M \subset \mathbb{C}^n$, S is the set of vectors tangent to M that remain tangent after multiplication by $i = \sqrt{-1}$ in the ambient complex coordinates. If S is sufficiently far from being integrable, choosing a fiber metric on S results in a sub-Riemannian metric whose geometric properties closely reflect the complex-analytic properties of M as a subset of $\mathbb{C}^n$.

Another motivation for studying sub-Riemannian metrics arises from *control theory*. In this subject, one is given a manifold with a vector field depending on parameters called controls, with the goal being to vary the controls so as to obtain a solution curve with desired properties, often one that minimizes some function such as arc length. If the vector field is everywhere tangent to a distribution S on the manifold (for example, in the case of a robot arm whose motion is restricted by the orientations of its hinges), then the function can often be modeled as a sub-Riemannian metric and optimal solutions modeled as sub-Riemannian geodesics.

A useful introduction to the geometry of sub-Riemannian metrics is provided in the article [Str86] (Robert s. Strichartz, *Sub-Riemannian Geometry*.)

Finsler Metrics

A **Finsler metric** on a manifold M is a continuous function $F : TM \to \mathbb{R}$, smooth on the complement of the zero section, that defines a norm on each tangent space T_pM. This means that $F(X) > 0$ for $X \neq 0, F(cX) = |c|F(X)$ for $c \in \mathbb{R}$, and $F(X+Y) \leq F(X)+F(Y)$. Again, the norm function associated with any Riemannian metric is a special case.

The inventor of Riemannian geometry himself, G. F. B. Riemann, clearly envisaged an important role in n-dimensional geometry for what we now call Finsler metrics; he restricted his investigations to the "Riemannian" case purely for simplicity (see Spivak, volume 2). However, only very recently have Finsler metrics begun to be studied seriously from a geometric point of view.

The recent upsurge of interest in Finsler metrics has been motivated largely by the fact that two different Finsler metrics appear very naturally in the theory of several complex variables: at least for bounded strictly convex domains in $\mathbb{C}^n$, the Kobayashi metric and the Carathéodory metric are intrinsically defined, biholomorphically invariant Finsler metrics. Combining differential-geometric and complex-analytic methods has led to striking new insights into both the function theory and the geometry of such domains.

2.5 Problems

Exercise 2.5.1. *([5] 2-1) Show that every Riemannian* 1*-manifold is flat.*

Exercise 2.5.2. *([5] 2-2) Suppose V and W are finite-dimensional real inner product spaces of the same dimension, and $F : V \to W$ is any map (not assumed to be linear or even continuous) that preserves the origin and all distances: $F(0) = 0$ and $|F(x) - F(y)| = |x - y|$ for all $x, y \in V$. Prove that F is a linear isometry. [Hint:First show that F preserves inner products, and then show that it is linear.]*

Exercise 2.5.3. *([5] 2-5) Prove parts (b) and (c) of Proposition 2.2.14 (properties of horizontal vector fields).*

Exercise 2.5.4. *([5] 2-6) Prove Theorem 2.2.19 (if $\pi : \widetilde{M} \to M$ is a surjective smooth submersion, and a group acts on $\widetilde{M}$ isometrically, vertically, and transitively on fibers, then M inherits a unique Riemannian metric such that π is a Riemannian submersion).*

Exercise 2.5.5. *([5] 2-7) For $0 < k < n$, the set $\mathrm{G}_k(\mathbb{R}^n)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ is called a **Grassmann manifold** or **Grassmannian**. The group $\mathrm{GL}(n, \mathbb{R})$ acts transitively on $\mathrm{G}_k(\mathbb{R}^n)$ in an obvious way, and $\mathrm{G}_k(\mathbb{R}^n)$ has a unique smooth manifold structure making this action smooth (see [4] Example 21.21).*

*(a) Let $\mathrm{V}_k(\mathbb{R}^n)$ denote the set of orthonormal ordered k-tuples of vectors in $\mathbb{R}^n$. By arranging the vectors in k columns, we can view $\mathrm{V}_k(\mathbb{R}^n)$ as a subset of the vector space $\mathrm{M}(n \times k, \mathbb{R})$ of all $n \times k$ real matrices. Prove that $\mathrm{V}_k(\mathbb{R}^n)$ is a smooth submanifold of $\mathrm{M}(n \times k, \mathbb{R})$ of dimension $k(2n - k - 1)/2$, called a **Stiefel manifold**. [Hint: Consider the map $\Phi : \mathrm{M}(n \times k, \mathbb{R}) \to \mathrm{M}(k \times k, \mathbb{R})$ given by $\Phi(A) = A^TA$.]*

(b) Show that the map $\pi : \mathrm{V}_k(\mathbb{R}^n) \to \mathrm{G}_k(\mathbb{R}^n)$ that sends a k-tuple to its span is a surjective smooth submersion.

(c) Give $\mathrm{V}_k(\mathbb{R}^n)$ the Riemannian metric induced from the Euclidean metric on $\mathrm{M}(n \times k, \mathbb{R})$. Show that the right action of $\mathrm{O}(k)$ on $\mathrm{V}_k(\mathbb{R}^n)$ by matrix multiplication on the right is isometric, vertical, and transitive on fibers of π, and thus there is a unique metric on $\mathrm{G}_k(\mathbb{R}^n)$ such that π is a Riemannian submersion. [Hint: It might help to note that the Euclidean inner product on $\mathrm{M}(n \times k, \mathbb{R})$ can be written in the form $\langle A, B\rangle = \mathrm{tr}\left(A^TB\right)$.]

Exercise 2.5.6. *([5] 2-8) Prove that the action of $\mathbb{Z}$ on $\mathbb{R}^2$ defined in Example 2.2.26 is smooth, free, proper, and isometric, and therefore the open Möbius band inherits a flat Riemannian metric such that the quotient map is a Riemannian covering.*

Exercise 2.5.7. *([5] 2-9) Prove Proposition 2.3.2 (the gradient is orthogonal to regular level sets).*

Exercise 2.5.8. *([5] 2-10) Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$ is a nowhere-vanishing vector field. Prove that $X = \operatorname{grad} f$ if and only if $Xf \equiv |X|_g^2$ and X is orthogonal to the level sets of f at all regular points of f.*

Exercise 2.5.9. *([5] 2-11) Prove Proposition 2.7 (inner products on tensor bundles).*

Chapter 3

Model Riemannian Manifolds and Their Geodesics

Chapter 4

Connections

4.1 The Problem of Differentiating Vector Fields

See [5] for more details. In essense, we cannot define the acceleration of a curve $\gamma : I \to M$ for an abstract manifold as in the case $M \subseteq \mathbb{R}^n$ (the definition of velocity though is still valid: $\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\big|_{t_0}\right)$) because to define $\gamma''(t)$ by differentiating $\gamma'(t)$ with respect to t, we have to take a limit of a difference quotient involving the vectors $\gamma'(t+h)$ and $\gamma'(t)$ who, however, live in different vector spaces $T_{\gamma(t+h)}M$ and $T_{\gamma(t)}M$.

4.2 Connections

Definition 4.2.1. *Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary, and let $\Gamma(E)$ denote the space of smooth sections of E. A **connection in** E is a map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

(i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X : for $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$$

(ii) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y : for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$$

(iii) ∇ satisfies the following product rule: for $f \in C^\infty(M)$,

$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y.$$

*The symbols ∇ reads as "del" or "nabla," and $\nabla_X Y$ is called the **covariant derivative of** Y **in the direction** X.*

There is a variety of types of connections that are useful in different circumstances. The type of connection we have defined here is sometimes called a **Koszul connection** to distinguish it from other types. Since we have no need to consider other types of connections in this book, we refer to Koszul connections simply as connections.

Although a connection is defined by its action on global sections, it follows from the definitions that it is actually a local operator, as the next lemma shows.

Lemma 4.2.2 (Locality). *Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every $X \in \mathfrak{X}(M), Y \in \Gamma(E)$, and $p \in M$, the covariant derivative $\nabla_X Y|_p$ depends only on the values of X and Y in an arbitrarily small neighborhood of p. More precisely, if $X = \widetilde{X}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} Y\big|_p$; if $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_X \widetilde{Y}|_p$. (The proof is similar to that of [4] Proposition 3.8)*

Proof. First consider Y. Replacing Y by $Y - \widetilde{Y}$ shows that it suffices to prove $\nabla_X Y|_p = 0$ if Y vanishes on a neighborhood of p.

Thus suppose Y is a smooth section of E that is identically zero on a neighborhood U of p. Choose a bump function $\varphi \in C^\infty(M)$ (see [4] p.42) with support in U such that $\varphi(p) = 1$. The hypothesis that Y vanishes on U implies that $\varphi Y \equiv 0$ on all of M, so for every $X \in \mathfrak{X}(M)$, we have $\nabla_X(\varphi Y) = \nabla_X(0 \cdot \varphi Y) = 0 \nabla_X(\varphi Y) = 0$. Thus the product rule gives

$$0 = \nabla_X(\varphi Y) = \overbrace{(X\varphi)Y}^{=0} + \varphi(\nabla_X Y) \Rightarrow 0 = \varphi(\nabla_X Y)$$

Now $Y \equiv 0$ on the support of φ, so the first term on the right is identically zero. Evaluating above equation at p shows that $\nabla_X Y|_p = 0$. The argument for X is similar: use property (i) of connection to get

$$0 \xlongequal{\varphi \text{ bump spt in } U \text{ w/ } \varphi(p)=1} \nabla_{\varphi X} Y = \varphi \nabla_X Y.$$

Then evaluate both sides at p. ■

Proposition 4.2.3 (Restriction of a Connection). *Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every open subset $U \subseteq M$, there is a unique connection ∇^U on the restricted bundle $E|_U$ that satisfies the following relation for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$:*

$$\nabla^U_{(X|_U)}(Y|_U) = (\nabla_X Y)|_U . \tag{4.1}$$

Remark 4.2.4. We recall from [4] p.255 Example 10.8 to see that $E|_U$ is a smooth vector bundle. Also recall $X|_U$ is a smooth vector field on U. See [4] p.185 proposition 8.23. Lastly, notice the comments given after local and global section on [4]p.255. $Y, \nabla_X(Y) \in \Gamma(E)$ naturally restricts to a global smooh section on U. From the first two notions, we see that the LHS of (4.1) is well-defined. By the last notion, the RHS of (4.1) is also clear. ♠

Proof. [5] p.90 proposition 4.3. ■

In the situation of this proposition, we typically just refer to the restricted connection as ∇ instead of ∇^U; the proposition guarantees that there is no ambiguity in doing so.

Lemma 4.2.2 tells us that we can compute the value of $\nabla_X Y$ at p knowing only the values of X and Y in a neighborhood of p. In fact, as the next proposition shows, we need only know the value of X at p itself.

Proposition 4.2.5. *Under the hypotheses of Lemma 4.2.2, $\nabla_X Y|_p$ depends only on the values of Y in a neighborhood of p and the value of X at p. (Since the claim about Y was proved in Lemma 4.2.2, this is to prove $X_p = \widetilde{X}_p \Rightarrow \nabla_X Y|_p = \nabla_{\widetilde{X}} Y\big|_p$. Equivalently, $(X - \widetilde{X})_p = 0_p \in T_pM \Rightarrow \nabla_{X-\widetilde{X}} Y\Big|_p =$ zero section ζ.)*

Proof. The claim about Y was proved in Lemma 4.2.2. To prove the claim about X, it suffices by linearity to assume that $X_p = 0$ and show that $\nabla_X Y|_p = 0$. Choose a coordinate neighborhood U of p, and write

$X = X^i\partial_i$ in coordinates on U, with $X^i(p) = 0$. Thanks to Proposition 4.2.3, it suffices to work with the restricted connection on U, which we also denote by ∇. For every $Y \in \Gamma\left(E|_U\right)$, we have

$$\nabla_X Y\big|_p = \nabla_{X^i\partial_i} Y\big|_p = X^i(p)\nabla_{\partial_i} Y\big|_p = 0.$$

■

Remark 4.2.6. Thanks to Propositions 4.2.3 and 4.2.5, we can make sense of the expression $\nabla_v Y$ when v is some element of T_pM and Y is a smooth local section of E defined only on some neighborhood of p. To evaluate it, let X be a vector field on a neighborhood of p whose value at p is v, and set $\nabla_v Y = \nabla_X Y\big|_p$. Proposition 4.2.5 shows that the result does not depend on the extension chosen. Henceforth, we will interpret covariant derivatives of local sections of bundles in this way without further comment. ♠

4.2.1 Connections in the Tangent Bundle

For Riemannian or pseudo-Riemannian geometry, our primary concern is with connections in the tangent bundle, so for the rest of the chapter we focus primarily on that case. A connection in the tangent bundle is often called simply a **connection on** M. (The terms **affine connection** and **linear connection** are also sometimes used in this context, but there is little agreement on the precise definitions of these terms, so we avoid them.)

Suppose M is a smooth manifold with or without boundary. By the definition we just gave, a connection in TM is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying properties (i)-(iii) above. Although the definition of a connection resembles the characterization of $(1, 2)$-tensor fields given by the tensor characterization lemma (Lemma B.6), a connection in TM is not a tensor field because it is not linear over $C^\infty(M)$ in its second argument, but instead satisfies the product rule.

For computations, we need to examine how a connection appears in terms of a local frame. Let (E_i) be a smooth local frame for TM on an open subset $U \subseteq M$. For every choice of the indices i and j, we can express the vector field $\nabla_{E_i}\bar{E}_j$ in terms of this same frame:

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma_{ij}^k E_k \tag{4.2}$$

As i, j, and k range from 1 to $n = \dim M$, this defines n^3 smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, called the **connection coefficients** of ∇ with respect to the given frame. The following proposition shows that the connection is completely determined in U by its connection coefficients.

Proposition 4.2.7. *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a smooth local frame over an open subset $U \subseteq M$, and let $\left\{\Gamma_{ij}^k\right\}$ be the connection coefficients of ∇ with respect to this frame. For smooth vector fields $X, Y \in \mathfrak{X}(U)$, written in terms of the frame as $X = X^iE_i, Y = Y^jE_j$, one has*

$$\nabla_X Y = \left(X\left(Y^k\right) + X^iY^j\Gamma_{ij}^k\right) E_k. \tag{4.3}$$

Proof. Just use the defining properties of a connection and compute:

$$\begin{aligned}
\nabla_X Y &= \nabla_X\left(Y^jE_j\right)\\
&\overset{\text{(iii)}}{=\!=} Y^j\nabla_X E_j + X\left(Y^j\right)E_j \qquad (Y^j : U \to \mathbb{R} \text{ are component functions})\\
&\overset{\text{(i)}}{=\!=} X^iY^j\nabla_{E_i}E_j + X\left(Y^j\right)E_j \qquad (X = X^iE_i)\\
&= X\left(Y^j\right)E_j + X^iY^j\Gamma_{ij}^kE_k
\end{aligned}$$

Renaming the dummy index in the first term yields (4.3). ■

Once the connection coefficients (and thus the connection) have been determined in some local frame, they can be determined in any other local frame on the same open set by the result of the following proposition.

Proposition 4.2.8 (Transformation Law for Connection Coefficients)**.** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose we are given two smooth local frames (E_i) and $\left(\widetilde{E}_j\right)$ for TM on an open subset $U \subseteq M$, related by $\widetilde{E}_i = A_i^j E_j$ for some matrix of functions $\left(A_i^j\right)$. Let Γ_{ij}^k and $\widetilde{\Gamma}_{ij}^k$ denote the connection coefficients of ∇ with respect to these two frames. Then*

$$\widetilde{\Gamma}_{ij}^k = \left(A^{-1}\right)_p^k A_i^q A_j^r \Gamma_{qr}^p + \left(A^{-1}\right)_p^k A_i^q E_q\left(A_j^p\right).$$

Proof. We note that

$$\begin{pmatrix} \widetilde{E}_1 \\ \vdots \\ \widetilde{E}_n \end{pmatrix} = \begin{pmatrix} A_1^1 & \cdots & A_1^n \\ \vdots & \ddots & \vdots \\ A_n^1 & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix}$$

Hence, $E_p = (A^{-1})_p^k \widetilde{E}_k$. By (4.3), we see that

$$\begin{aligned} \nabla_{\widetilde{E}_i} \widetilde{E}_j &= \left[\widetilde{E}_i(\widetilde{E}_j^p) + \widetilde{E}_i^q \widetilde{E}_j^r \Gamma_{qr}^p\right] E_p \\ &= \left[\left(A_i^q E_q\right)\left(A_j^p\right) + A_i^q A_j^r \Gamma_{qr}^p\right]\left((A^{-1})_p^k \widetilde{E}_k\right) \\ &= \left(A^{-1}\right)_p^k A_i^q A_j^r \Gamma_{qr}^p + \left(A^{-1}\right)_p^k A_i^q E_q\left(A_j^p\right) \end{aligned}$$

■

4.2.2 Existence of Connections

So far, we have studied properties of connections but have not produced any, so you might be wondering whether they are plentiful or rare. In fact, they are quite plentiful, as we will show shortly. Let us begin with the simplest example.

Example 4.2.9 (The Euclidean Connection)**.** In $T\mathbb{R}^n$, define the Euclidean connection $\bar{\nabla}$ by the following formula ([5] (4.3)).

$$\bar{\nabla}_X Y = X\left(Y^1\right) \frac{\partial}{\partial x^1} + \cdots + X\left(Y^n\right) \frac{\partial}{\partial x^n}$$

It is easy to check that this satisfies the required properties for a connection, and that its connection coefficients in the standard coordinate frame are all zero: It is easy to verify (i)-(iii). Computation of connection coefficients is also straightforward:

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \frac{\partial}{\partial x^i}\left(\delta_{jk}\right) \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i}(1) \frac{\partial}{\partial x^j} = 0 \frac{\partial}{\partial x^j} = 0$$

♣

Here is a way to construct a large class of examples.

Example 4.2.10 (The Tangential Connection on a Submanifold of $\mathbb{R}^n$)**.** Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Define a connection $\nabla^{\top}$ on TM, called the tangential connection, by setting

$$\nabla_X^{\top} Y = \pi^{\top}\left(\left.\bar{\nabla}_{\widetilde{X}} \widetilde{Y}\right|_M\right),$$

where $\pi^\top$ is the orthogonal projection onto TM, $\bar{\nabla}$ is the Euclidean connection on $\mathbb{R}^n$ (Example 4.2.9), and $\widetilde{X}$ and $\widetilde{Y}$ are smooth extensions of X and Y to an open set in $\mathbb{R}^n$. (Such extensions exist by the result of [5] Exercise A.23.) Since the value of $\bar{\nabla}_{\widetilde{X}}\widetilde{Y}$ at a point $p \in M$ depends only on $\widetilde{X}_p = X_p$, this just boils down to defining $\left(\nabla_X^\top Y\right)_p$ to be equal to the tangential directional derivative $\nabla_{X_p}^\top Y$ we intuitively defined in [5] (4.4). To show it is indeed a connection, see [5] Example 4.9. ♣

In fact, there are many connections on $\mathbb{R}^n$, or indeed on every smooth manifold that admits a global frame (for example, every manifold covered by a single smooth coordinate chart). The following lemma shows how to construct all of them explicitly.

Lemma 4.2.11. *Suppose M is a smooth n-manifold with or without boundary, and M admits a global frame (E_i). Formula (4.3) gives a one-to-one correspondence between connections in TM and choices of n^3 smooth real-valued functions $\left\{\Gamma_{ij}^k\right\}$ on M.*

Proof. Every connection determines functions $\left\{\Gamma_{ij}^k\right\}$ by (4.2), and we have shown that those functions satisfy (4.3). On the other hand, given $\left\{\Gamma_{ij}^k\right\}$, we can define $\nabla_X Y$ by (4.3); it is easy to see that the resulting expression is smooth if X and Y are smooth, linear over $\mathbb{R}$ in Y, and linear over $C^\infty(M)$ in X. To prove that it is a connection, only the product rule requires checking; this is a straightforward computation: we check that

$$\nabla_X Y := \left(X\left(Y^k\right) + X^i Y^j \Gamma_{ij}^k\right) E_k$$

satisfies the product rule (iii). For $f \in C^\infty(M)$,

$$\begin{aligned}
\nabla_X(fY) &= \left(X\left(fY^k\right) + X^i f Y^j \Gamma_{ij}^k\right) E_k \\
&\overset{[4]\ (8.5)}{=\!=\!=} \left(fX\left(Y^k\right) + Y^k X f + f X^i Y^j \Gamma_{ij}^k\right) E_k \\
&= f\left(X\left(Y^k\right) + X^i Y^j \Gamma_{ij}^k\right) E_k + \left(Y^k X f\right) E_k \\
&= f\nabla_X Y + (Xf) Y^k E_k \\
&= f\nabla_X Y + (Xf) Y.
\end{aligned}$$

■

Proposition 4.2.12. *The tangent bundle of every smooth manifold with or without boundary admits a connection.*

Proof. Let M be a smooth manifold with or without boundary, and cover M with coordinate charts $\{U_\alpha\}$; the preceding lemma guarantees the existence of a connection ∇^α on each U_α. Choose a partition of unity $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$. We would like to patch the various ∇^α 's together by the formula

$$\nabla_X Y = \sum_\alpha \varphi_\alpha \nabla_X^\alpha Y.$$

Because the set of supports of the φ_α 's is locally finite, the sum on the right-hand side has only finitely many nonzero terms in a neighborhood of each point, so it defines a smooth vector field on M. It is immediate from this definition that $\nabla_X Y$ is linear over $\mathbb{R}$ in Y and linear over $C^\infty(M)$ in X. We have to be a bit careful with the product rule, though, since a linear combination of connections is not necessarily a connection. (You can check, for example, that if ∇^0 and ∇^1 are connections, then neither $2\nabla^0$ nor $\nabla^0 + \nabla^1$ satisfies the product rule.) By direct computation,

$$\begin{aligned}\nabla_X(fY) &= \sum_\alpha \varphi_\alpha \nabla^\alpha_X(fY)\\ &= \sum_\alpha \varphi_\alpha\left((Xf)Y + f\nabla^\alpha_X Y\right)\\ &= (Xf)Y\sum_\alpha \varphi_\alpha + f\sum_\alpha \varphi_\alpha \nabla^\alpha_X Y\\ &= (Xf)Y + f\nabla_X Y.\end{aligned}$$

∎

Although a connection is not a tensor field, the next proposition shows that the difference between two connections is.

Proposition 4.2.13 (The Difference Tensor)**.** *Let M be a smooth manifold with or without boundary. For any two connections ∇^0 and ∇^1 in TM, define a map $D : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by*

$$D(X,Y) = \nabla^1_X Y - \nabla^0_X Y.$$

Then D is bilinear over $C^\infty(M)$, and thus defines a (1,2)-tensor field called the difference tensor between ∇^0 and ∇^1.

Proof. It is immediate from the definition that D is linear over $C^\infty(M)$ in its first argument, because both ∇^0 and ∇^1 are. To show that it is linear over $C^\infty(M)$ in the second argument, expand $D(X, fY)$ using the product rule, and note that the two terms in which f is differentiated cancel each other. The last sentence of the proposition is a consequence of Lemma 1.1.18:

$$D : \underbrace{}_{\text{0 factor}} \times \underbrace{\mathfrak{X}(M)\times\mathfrak{X}(M)}_{\text{2 factors}} \to \mathfrak{X}(M)$$

is bilinear and then defines a (1,2)-tensor field. ∎

Now that we know there is always one connection in TM, we can use the result of the preceding proposition to say exactly how many there are.

Theorem 4.2.14. *Let M be a smooth manifold with or without boundary, and let ∇^0 be any connection in TM. Then the set $\mathcal{A}(TM)$ of all connections in TM is equal to the following affine space:*

$$\mathcal{A}(TM) = \left\{\nabla^0 + D : D \in \Gamma\left(T^{(1,2)}TM\right)\right\},$$

where $D \in \Gamma\left(T^{(1,2)}TM\right)$ is interpreted as a map from $\mathfrak{X}(M)\times\mathfrak{X}(M)$ to $\mathfrak{X}(M)$ as in Proposition 1.1.5, and $\nabla^0 + D : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ is defined by

$$\left(\nabla^0 + D\right)_X Y = \nabla^0_X Y + D(X,Y).$$

Proof. [5] Problem 4-4. ∎

4.3 Covariant Derivatives of Tensor Fields

We first defined a connection in E, the total space of a vector bundle $\pi : E \to M$:

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

and then in particular a connection in $E = TM$, where $\Gamma(TM) = \mathfrak{X}(M)$:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

We show in this section that every connection in TM automatically induces connections in all tensor bundles over M,

$$\nabla : \mathfrak{X}(M) \times \Gamma\left(T^{(k,l)}(TM)\right) \to \Gamma\left(T^{(k,l)}(TM)\right)$$

and thus gives a way to compute covariant derivatives of tensor fields of any type.

Proposition 4.3.1. *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Then ∇ uniquely determines a connection in each tensor bundle $T^{(k,l)}TM$, also denoted by ∇, such that the following four conditions are satisfied.*

(i) *In $T^{(1,0)}TM = TM, \nabla$ agrees with the given connection.*

(ii) *In $T^{(0,0)}TM = M \times \mathbb{R}, \nabla$ is given by ordinary differentiation of functions:*

$$\nabla_X f = Xf$$

(For the identification $T^{(0,0)}TM = M \times \mathbb{R}$, see [4] p.317: for any vector space V, [4] p. 312 notes that $T^0 V = \mathbb{R}$ by convention. Now

$$T^0 T^* M = \coprod_{p \in M} T^0\left(T_p^* M\right) = \coprod_{p \in M} \mathbb{R} = M \times \mathbb{R}$$

Similarly, $T^0 TM = M \times \mathbb{R}$. Thus, $T^{(0,0)}TM$, either interpreted as $T^0 T^* M$ or $T^0 TM$, equals to $M \times \mathbb{R}$. And the space of smooth sections $\Gamma\left(T^{(0,0)}TM\right) = \Gamma(M \times \mathbb{R}) = C^\infty(M)$ is just the space of smooth functions.)

(iii) *∇ obeys the following product rule with respect to tensor products:*

$$\nabla_X(F \otimes G) = \left(\nabla_X F\right) \otimes G + F \otimes \left(\nabla_X G\right).$$

(iv) *∇ commutes with all contractions: if "tr" denotes a trace on any pair of indices, one covariant and one contravariant, then*

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}\left(\nabla_X F\right)$$

This connection also satisfies the following additional properties:

(a) *∇ obeys the following product rule with respect to the natural pairing between a covector field ω and a vector field Y :*

$$\nabla_X\langle\omega, Y\rangle = \left\langle\nabla_X \omega, Y\right\rangle + \left\langle\omega, \nabla_X Y\right\rangle.$$

(Note: $\langle\omega, Y\rangle_p := \left\langle\omega_p, Y_p\right\rangle = \omega_p\left(Y_p\right)$. So $\langle\omega, Y\rangle \in C^\infty(M)$. See [4] p.274)

(b) (b) For all $F \in \Gamma\left(T^{(k,l)}TM\right)$, smooth 1-forms $\omega^1, \ldots, \omega^k$, and smooth vector fields $Y_1, \ldots, Y_l$,

$$\begin{aligned}\left(\nabla_X F\right)\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right) &= X\left(F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right)\right) \\ &- \sum_{i=1}^{k} F\left(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l\right) \\ &- \sum_{j=1}^{l} F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l\right).\end{aligned} \tag{4.4}$$

Proof. First we show that every family of connections on all tensor bundles satisfying (i)-(iv) also satisfies (a) and (b). Suppose we are given such a family of connections, all denoted by ∇. Recall for $\omega \in \mathfrak{X}^*(M), Y \in \mathfrak{X}(M), \omega \otimes Y$ denotes the tensor fields defined by $(\omega \otimes Y)_p := \omega_p \otimes Y_p$ (see [4] p.317), and $\langle \omega, Y\rangle$ is also pointwise defined: $\langle \omega, Y\rangle_p := \langle \omega_p, Y_p\rangle = \omega_p\left(Y_p\right)$. Also note that

$$\omega_p \otimes Y_p \in T_p^* M \otimes T_p M = T^{(1,1)} T_p^* M \cong \operatorname{End}\left(T_p^* M\right)$$

so that $\omega \otimes Y \in \Gamma\left(T^{(1,1)} T^* M\right)$. Then the trace of $\omega_p \otimes Y_p \in T^{(1,1)} T_p^* M$ is the sum of the diagonal elements of the matrix representation of $\omega_p \otimes Y_p$ identified as a linear endomorphism. Plugging $k = l = 0$ into formula (1.3) gives

$$\operatorname{tr}\left(\omega_p \otimes Y_p\right) = \sum_{1 \le m \le n} \left(\omega_p \otimes Y_p\right)_m^m \overset{[4]\ 12.22}{=\!=\!=} \left(\omega_p\right)_m \left(Y_p\right)^m.$$

On the other hand, if $Y_p = \left(Y_p\right)^i E_i, \omega_p = \left(\omega_p\right)_j \varepsilon^j$ then $\varepsilon^j\left(E_i\right) = \delta_i^j$ gives that

$$\omega_p\left(Y_p\right) = \left(\omega_p\right)_j \varepsilon^j\left[\left(Y_p\right)^i E_i\right] = \left(\omega_p\right)_j \left(Y_p\right)^j = \left(\omega_p\right)_m \left(Y_p\right)^m$$

Thus $\operatorname{tr}\left(\omega_p \otimes Y_p\right) = \omega_p\left(Y_p\right)$ and $\langle \omega, Y\rangle = \operatorname{tr}(\omega \otimes Y)$. Therefore, (i)-(iv) imply

$$\begin{aligned}\nabla_X \omega(Y) = \nabla_X\langle \omega, Y\rangle = \nabla_X(\operatorname{tr}(\omega \otimes Y)) &= \operatorname{tr}\left(\nabla_X(\omega \otimes Y)\right) \\ &= \operatorname{tr}\left(\left(\nabla_X \omega\right) \otimes Y + \omega \otimes \left(\nabla_X Y\right)\right) \quad \text{(by (iv))} \\ &= \operatorname{tr}\left(\left(\nabla_X \omega\right) \otimes Y\right) + \operatorname{tr}\left(\omega \otimes \left(\nabla_X Y\right)\right) \quad \text{(linearity of } \operatorname{tr}) \\ &= \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle \quad \left(\nabla_X \omega \text{ is a 1-form, } \in \Gamma(T^{(0,1)}TM) \text{ while } \nabla_X Y \text{ is a vector field, } \in \Gamma(T^{(1,0)}TM)\right)\end{aligned} \tag{4.5}$$

Then (b) is proved by induction using a similar computation applied to

$$F\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right) = \underbrace{\operatorname{tr} \circ \cdots \circ \operatorname{tr}}_{k+l}\left(F \otimes \omega^1 \otimes \cdots \otimes \omega^k \otimes Y_1 \otimes \cdots \otimes Y_l\right),$$

where each trace operator acts on an upper index of F and the lower index of the corresponding 1-form, or a lower index of F and the upper index of the corresponding vector field. In fact, (4.4) can be easily generalized from the case $k = l = 1$:

$$\begin{aligned}\nabla_X F(\omega, Y) &= \nabla_X(\operatorname{tr} \circ \operatorname{tr}(F \otimes \omega \otimes Y)) \\ &= \operatorname{tr} \circ \operatorname{tr}\left(\nabla_X(F \otimes (\omega \otimes Y))\right) \\ &= \operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes (\omega \otimes Y) + F \otimes (\nabla_X(\omega \otimes Y))) \\ &= \operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes (\omega \otimes Y) + F \otimes ((\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y))) \\ &= \operatorname{tr} \circ \operatorname{tr}((\nabla_X F) \otimes \omega \otimes Y + F \otimes \nabla_X \omega \otimes Y + F \otimes \omega \otimes \nabla_X Y) \\ &= (\nabla_X F)(\omega, Y) + F(\nabla_X \omega, Y) + F(\omega, \nabla_X Y) \\ \Longrightarrow (\nabla_X F)(\omega, Y) &= \nabla_X F(\omega, Y) - F(\nabla_X \omega, Y) - F(\omega, \nabla_X Y)\end{aligned}$$

Next we address uniqueness. Assume again that ∇ represents a family of connections satisfying (i)-(iv), and hence also (a) and (b). Observe that (ii) and (a) imply that the covariant derivative of every 1-form ω can be computed by

$$\left(\nabla_X \omega\right)(Y)=X(\omega(Y))-\omega\left(\nabla_X Y\right). \tag{4.6}$$

(this is just the same as (4.5) since a one-form is also a covector field.)

It follows that the connection on 1-forms is uniquely determined by the original connection in TM, which is $\nabla_X Y$. Similarly, (b) gives a formula determining the covariant derivative of every tensor field F in terms of covariant derivatives of vector fields and 1-forms, so the connection in every tensor bundle is uniquely determined.

Now to prove existence, we first define covariant derivatives of 1-forms by (4.6), and then we use (4.4) to define ∇ on all other tensor bundles. The first thing that needs to be checked is that the resulting expression is multilinear over $C^{\infty}(M)$ in each ω^i and Y_j, and therefore defines a smooth tensor field. This is done by inserting $f\omega^i$ in place of ω^i, or fY_j in place of Y_j, and expanding the right-hand side, noting that the two terms in which f is differentiated cancel each other out. Once we know that $\nabla_X F$ is a smooth tensor field, we need to check that it satisfies the defining properties of a connection. Linearity over $C^{\infty}(M)$ in X and linearity over $\mathbb{R}$ in F are both evident from (4.4) and (4.6), and the product rule in F follows easily from the fact that differentiation of functions by X satisfies the product rule. It is then a straightforward computational exercise to show that the resulting connection satisfies conditions (i)-(iii). To prove (iv), first observe that every (k,l)-tensor field can be written locally as a sum of tensor fields of the form $Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l$, and for such a tensor field the trace on the i th contravariant index and the j th covariant one satisfies

$$\operatorname{tr}\left(Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l\right)=\zeta^j\left(Z_i\right) Z_1 \otimes \cdots \otimes \widehat{Z_i} \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \widehat{\zeta^j} \otimes \cdots \otimes \zeta^l .$$

Then (iv) follows by applying (4.4) and (4.6) to this formula. ■

While (4.4) and (4.6) are useful for proving the existence and uniqueness of the connections in tensor bundles, they are not very practical for computation, because computing the value of $\nabla_X F$ at a point requires extending all of its arguments to vector fields and covector fields in an open set, and computing a great number of derivatives. For computing the components of a covariant derivative in terms of a local frame, the formulas in the following proposition are far more useful.

Proposition 4.3.2. *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a local frame for M, $\left(\varepsilon^j\right)$ is its dual coframe, and $\left\{\Gamma_{ij}^k\right\}$ are the connection coefficients of ∇ with respect to this frame. Let X be a smooth vector field, and let $X^i E_i$ be its local expression in terms of this frame.*

(a) The covariant derivative of a 1-form $\omega=\omega_i \varepsilon^i$ is given locally by

$$\nabla_X(\omega)=\left(X\left(\omega_k\right)-X^j \omega_i \Gamma_{jk}^i\right) \varepsilon^k .$$

(b) If $F \in \Gamma\left(T^{(k,l)} TM\right)$ is a smooth mixed tensor field of any rank, expressed locally as

$$F=F_{j_1 \ldots j_l}^{i_1 \ldots i_k} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l},$$

then the covariant derivative of F is given locally by

$$\nabla_X F=\left(X\left(F_{j_1 \ldots j_l}^{i_1 \ldots i_k}\right)+\sum_{s=1}^k X^m F_{j_1 \ldots j_l}^{i_1 \ldots p \ldots i_k} \Gamma_{mp}^{i_s}-\sum_{s=1}^l X^m F_{j_1 \ldots p \ldots j_l}^{i_1 \ldots i_k} \Gamma_{mj_s}^p\right) \times$$
$$E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} .$$

Proof. To show (a), we only need to show

$$(\nabla_X \omega)(E_k) = X(\omega_k) - X^j \omega_i \Gamma^i_{jk}$$

By (4.6), we see

$$\begin{aligned}
\nabla_X \omega(E_k) &= X(\omega(E_k)) - \omega(\nabla_X E_k) \\
&= X(\omega_k) - \omega \left[(X(\underbrace{\delta^i_k}_{\text{constant}}) + X^j \delta^r_k \Gamma^i_{jr}) E_i \right] \\
&= X(\omega_k) - \omega \left[(0 + X^j \Gamma^i_{jk}) E_i \right] \\
&= X(\omega_k) - \omega_i X^j \Gamma^i_{jk}
\end{aligned}$$

To show (b), we only need to show

$$(\nabla_X F)(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}) = X\left(F^{i_1 \dots i_k}_{j_1 \dots j_l}\right) + \sum_{s=1}^{k} X^m F^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m F^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} \Gamma^p_{m j_s}$$

By (4.4), we see

$$\begin{aligned}
&(\nabla_X F)\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}\right) \\
=& X\left(F(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l})\right) - \sum_{s=1}^{k} F\left(\varepsilon^{i_1}, \dots, \nabla_X \varepsilon^{i_s}, \dots, \varepsilon^{i_k}, E_{j_1}, \dots, E_{j_l}\right) \\
&- \sum_{s=1}^{l} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \dots, \nabla_X E_{j_s}, \dots, E_{j_l}\right) \\
=& F^{i_1 \dots i_k}_{j_1 \dots j_l} - \sum_{s=1}^{k} F\left(\varepsilon^{i_1}, \dots, -X^m \Gamma^{i_s}_{mp} \varepsilon^p, \dots, \varepsilon^{i_k}, E_{j_1}, \dots, E_{j_l}\right) \\
&- \sum_{s=1}^{l} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \dots, X^m \Gamma^p_{m j_s} E_p, \dots, E_{j_l}\right) \quad \text{(by (a) and (4.3))} \\
=& F^{i_1 \dots i_k}_{j_1 \dots j_l} + \sum_{s=1}^{k} X^m F^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m F^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} \Gamma^p_{m j_s}
\end{aligned}$$

■

Because the covariant derivative $\nabla_X F$ of a tensor field (or, as a special case, a vector field) is linear over $C^\infty(M)$ in X, the covariant derivatives of F in all directions can be handily encoded in a single tensor field whose rank is one more than the rank of F, as follows.

Proposition 4.3.3 (The Total Covariant Derivative). *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every $F \in \Gamma\left(T^{(k,l)}TM\right)$, the map*

$$\nabla F : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ copies}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1 \text{ copies}} \to C^\infty(M)$$

given by

$$(\nabla F)\left(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l, X\right) = (\nabla_X F)\left(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l\right) \tag{4.7}$$

defines a smooth $(k, l+1)$-tensor field on M called the ***total covariant derivative of*** *F.*

Proof. This follows immediately from the tensor characterization lemma (Lemma 1.1.18): $\nabla_X F$ is a tensor field, so it is multilinear over $C^\infty(M)$ in its $k+l$ arguments; and it is linear over $C^\infty(M)$ in X by definition of a connection. ∎

Remark 4.3.4. Note that the smooth $(k, l+1)$-tensor field induced by ∇F is called the total covariant derivative of F and is denoted by ∇F as well. One can think of the covariant derivative of a tensor field as directional derivatives while the total covariant derivative is the total derivative of the tensor field. ♠

When we write the components of a total covariant derivative in terms of a local frame, it is standard practice to use a semicolon to separate indices resulting from differentiation from the indices resulting from the "+1" insertion. For example, let Y be a vector field. That is, $Y \in \mathfrak{X}(M) = \Gamma\left(T^{(1.0)}TM\right)$ where $k=1, l=0$ in the above proposition. We write it in coordinates as $Y = Y^i E_i$. Then the components of the $(1,1)$-tensor field ∇Y are written as $Y^i{}_{;j}$, i.e.,

$$\nabla Y = Y^i{}_{;j} E_i \otimes \varepsilon^j$$

where $Y^i{}_{;j}$ is obtained by the following:

$$\begin{aligned} Y^i{}_{;j} &= \nabla Y\left(\varepsilon^i, E_j\right) = \left(\nabla_{E_j} Y\right)\left(\varepsilon^i\right) \\ &\overset{(4.3)}{=\!=} \left(E_j\left(Y^l\right) + \left(E^j\right)^m Y^k \Gamma^l_{mk}\right) E_l\left(\varepsilon^i\right) \\ &= E_j\left(Y^i\right) + \left(E^j\right)^m Y^k \Gamma^i_{mk} \\ &= E_j\left(Y^i\right) + Y^k \Gamma^i_{jk} \end{aligned}$$

For a one-form $\omega \in \Gamma(T^{(0,1)}TM)$ where $k=0, l=1$ in above proposition, we have a $(0,2)$-tensor field $\nabla\omega$. If we write $\omega = \omega_m \varepsilon^m$, then the components of the $\nabla\omega$ are written as $\omega_{i;j}$, i.e.,

$$\nabla\omega = \omega_{i;j} \varepsilon^i \otimes \varepsilon^j$$

where $\omega_{i;j}$ is obtained by the following:

$$\begin{aligned} \omega_{i;j} &= \nabla\omega\left(E_i, E_j\right) = \left(\nabla_{E_j}\omega\right)\left(E_i\right) \\ &\overset{(4.6)}{=\!=} E_j\left(\omega\left(E_i\right)\right) - \omega\left(\nabla_{E_j} E_i\right) \\ &\overset{(4.2)}{=\!=} E_j\left(\omega_m \varepsilon^m\left(E_i\right)\right) - \omega_m \varepsilon^m\left(\Gamma^k_{ji} E_k\right) \\ &= E_j \omega_i - \omega_k \Gamma^k_{ji} \end{aligned}$$

More generally, replacing (4.3) and (4.6) with (4.4) and using the definition of coefficient Γ^k_{ij} we get a formula for the components of total covariant derivatives of arbitrary tensor fields as shown in the next lemma.

Proposition 4.3.5. *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM; and let (E_i) be a smooth local frame for TM and $\left\{\Gamma^k_{ij}\right\}$ the corresponding connection coefficients. The components of the total covariant derivative of a (k,l)-tensor field F with respect to this frame are given by*

$$F^{i_1\ldots i_k}_{j_1\ldots j_l;m} = E_m\left(F^{i_1\ldots i_k}_{j_1\ldots j_l}\right) + \sum_{s=1}^{k} F^{i_1\ldots p\ldots i_k}_{j_1\ldots j_l}\Gamma^{i_s}_{mp} - \sum_{s=1}^{l} F^{i_1\ldots i_k}_{j_1\ldots p\ldots j_l}\Gamma^p_{mj_s}.$$

Proof.

$$\begin{aligned}
\Gamma\left(T^{(k,l+1)}TM\right) \ni \nabla F = F^{i_1,\cdots,i_k}_{j_1,\cdots,j_i;m} E_{i_1} \otimes \cdots \otimes F_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} \otimes \varepsilon^m \\
F^{i_1\cdots i_j}_{j\cdots\cdots j;m} &= \nabla F\left(\varepsilon^i, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}, E_m\right) \\
&\overset{(4.7)}{=\!=} \nabla_{E_m} F\left(\varepsilon^{i_1}, \cdots, \varepsilon^{i_k}, E_{j_1}, \cdots, E_{j_l}\right) \\
&\overset{\text{prop. 4.3.2(b)}}{=\!=\!=} E_m\left(F^{i_1\cdots i_k}_{j_1\cdots j_l}\right) + \sum_{s=1}^{k} (E_m)^q F^{i_1\cdots p\cdots i_k}_{j,\cdots j_l} \Gamma^{i_s}_{qp} - \sum_{s=1}^{l} (E_m)^q F^{i_1\cdots i_k}_{j_1\cdots q\cdots j_l} \Gamma^p_{qj_s} \\
&= E_m\left(F^{i_1\cdots i_k}_{j_1\cdots j_l}\right) + \sum_{s=1}^{k} F^{i_1\cdots p\cdots i_k}_{j_1\cdots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} F^{i_1\cdots i_k}_{j,\cdots,j_l} \Gamma^p_{mj_s}
\end{aligned}$$

■

Exercise 4.3.6. *Suppose F is a smooth (k,l)-tensor field and G is a smooth (r,s) tensor field. Show that the components of the total covariant derivative of $F \otimes G$ are given by*

$$(\nabla(F \otimes G))^{i_1\ldots i_k p_1\ldots p_r}_{j_1\ldots j_l q_1\ldots q_s;m} = F^{i_1\ldots i_k}_{j_1\ldots j_l;m} G^{p_1\ldots p_r}_{q_1\ldots q_s} + F^{i_1\ldots i_k}_{j_1\ldots j_l} G^{p_1\ldots p_r}_{q_1\ldots q_s;m}.$$

[Remark: This formula is often written in the following way, more suggestive of the product rule for ordinary derivatives:

$$\left(F^{i_1\ldots i_k}_{j_1\ldots j_l} G^{p_1\ldots p_r}_{q_1\ldots q_s}\right)_{;m} = F^{i_1\ldots i_k}_{j_1\ldots j_l;m} G^{p_1\ldots p_r}_{q_1\ldots q_s} + F^{i_1\ldots i_k}_{j_1\ldots j_l} G^{p_1\ldots p_r}_{q_1\ldots q_s;m}.$$

Notice that this does not say that $\nabla(F \otimes G) = (\nabla F) \otimes G + F \otimes (\nabla G)$, because in the first term on the right-hand side of this latter formula, the index resulting from differentiation is not the last lower index.]

4.3.1 Second Covariant Derivative

Having defined the tensor field ∇F for a (k,l)-tensor field F, we can in turn take its total covariant derivative and obtain a $(k, l+2)$-tensor field $\nabla^2 F = \nabla(\nabla F)$. Given vector fields $X, Y \in \mathfrak{X}(M)$, let us introduce the notation $\nabla^2_{X,Y} F$ for the (k,l)-tensor field obtained by inserting X, Y in the last two slots of $\nabla^2 F$:

$$\nabla^2_{X,Y} F(\ldots) = \nabla^2 F(\ldots, Y, X).$$

Note the reversal of order of X and Y : this is necessitated by our convention that the last index position in ∇F is the one resulting from differentiation, while it is conventional to let $\nabla^2_{X,Y}$ stand for differentiating first in the Y direction, then in the X direction. (For this reason, some authors adopt the convention that the new index position introduced by differentiation is the first instead of the last. As usual, be sure to check each author's conventions when you read.)

It is important to be aware that $\nabla^2_{X,Y} F$ is not the same as $\nabla_X(\nabla_Y F)$. The main reason is that the former is linear over $C^\infty(M)$ in Y, while the latter is not. The relationship between the two expressions is given in the following proposition.

Proposition 4.3.7. *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth vector field or tensor field F,*

$$\nabla^2_{X,Y} F = \nabla_X(\nabla_Y F) - \nabla_{(\nabla_X Y)} F.$$

Proof. For $Y \in \mathfrak{X}(M) = \Gamma\left(T^{(1,0)}TM\right), \nabla F \in \Gamma\left(T^{(k,l+1)}TM\right)$, we have $\nabla F \otimes Y \in \Gamma\left(T^{(k+1,l+1)}TM\right)$. The covariant derivative $(\nabla_Y F)(\cdots) \overset{(4.7)}{=\!=} \nabla F(\cdots, Y)$ can be expressed as the trace of $\nabla F \otimes Y$ on its last two indices. We have

$$\nabla_Y F = \operatorname{tr}(\nabla F \otimes Y) = C^{k+1}_{l+1}(\nabla F \otimes Y) \tag{4.8}$$

as we can verify by computing their components: proposition 4.3.2 shows that

$$(\nabla_Y F)^{i_1\cdots i_k}_{j_1\cdots j_l} = Y\left(F^{i_1\cdots i_k}_{j_1\cdots j_l}\right) + \sum_{s=1}^{k} Y^m F^{i_1\cdots p\cdots i_k}_{j_1\cdots j_l}\Gamma^{i_s}_{mp} - \sum_{s=1}^{l} Y^m F^{i_1\cdots i_k}_{j_1\cdots p\cdots j_l}\Gamma^{p}_{mj_s} \xlongequal{\text{prop. 4.3.5}} F^{i_1\cdots\cdots j_k}_{j_1\cdots j_l;m} Y^m \tag{4.9}$$

On the other hand,

$$\begin{aligned}
[\operatorname{tr}(\nabla F\otimes Y)]^{i_1\cdots i_k}_{j_1\cdots j_l} &\xlongequal{(1.3)} (\nabla F\otimes Y)^{i_1\cdots i_k m}_{j_1\cdots j_l m} \\
&= (\nabla F\otimes Y)\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},\varepsilon^{m},E_{j_1},\cdots,E_{j_l},E_m\right) \\
&= \nabla F\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},E_{j_1},\cdots,E_{j_l},E_m\right) Y\left(\varepsilon^{m}\right) \\
&= F^{i_1\cdots i_k}_{j_1\cdots j_l;m} Y^m
\end{aligned} \tag{4.10}$$

Similarly, $\nabla^2_{X,Y}F$ can be expressed as an iterated trace:

$$\nabla^2_{X,Y}F = \operatorname{tr}\left(\operatorname{tr}\left(\nabla^2 F\otimes X\right)\otimes Y\right).$$

(First trace the last index of $\nabla^2 F$ with that of X, and then trace the last remaining free index-originally the second-to-last in $\nabla^2 F$-with that of Y.)

We notice that for $X\in\mathfrak{X}(M)=\Gamma\left(T^{(1,0)}TM\right), \nabla F\in\Gamma\left(T^{(k,l+1)}TM\right)$, we have $\nabla_X(\nabla F)\in\Gamma\left(T^{(k,l+1)}TM\right)$, $\nabla(\nabla F)\in\Gamma\left(T^{(k,l+2)}TM\right)$, $\nabla(\nabla F)\otimes X\in\Gamma\left(T^{(k+1,l+2)}TM\right)$, and $\nabla_X(\nabla F)\otimes Y\in\Gamma\left(T^{(k+1,l+1)}TM\right)$. We write the iterated expression as

$$C^{k+1}_{l+1}\left(C^{k+1}_{l+2}(\nabla(\nabla F)\otimes X)\otimes Y\right) \xlongequal{(4.8)} C^{k+1}_{l+1}\left(\nabla_X(\nabla F)\otimes Y\right) = \nabla(\nabla F)(\cdots,Y,X) := \nabla^2_{X,Y}F,$$

where the second equality comes from the following reasoning:

$$[C^{k+1}_{l+1}(\nabla_X(\nabla F)\otimes Y)]^{i_1\cdots i_k}_{j_1\cdots j_l} \xlongequal{(4.10)} [\nabla_X(\nabla F)]^{i_1\cdots i_k}_{j_1\cdots j_l\underbrace{j_{l+1}}_{=q}} Y^q \xlongequal{(4.9)} (\nabla F)^{i_1\cdots i_k}_{j_1\cdots j_l\underbrace{j_{l+1}}_{=q}\underbrace{j_{l+2}}_{=m}} X^m Y^q,$$

where $j_{l+1}=q$ and $j_{l+2}=m$ are just renaming of indices. On the other hand, $F\in T^{(k,l)}(V)\Rightarrow\nabla F\in T^{(k,l+1)}(V)\Rightarrow[\nabla(\nabla F)]\in T^{(k,l+2)}(V)\Rightarrow[\nabla(\nabla F)(\cdots,Y,X)]\in T^{(k,l)}(V)$ where

$$\nabla(\nabla F)(\cdots,Y,X):\left(\omega^1,\cdots,\omega^k,Y_1,\cdots,Y_l\right)\mapsto\nabla(\nabla F)\left(\omega^1,\cdots,\omega^k,Y_1,\cdots,Y_l,Y,X\right)$$

Now,

$$\begin{aligned}
&[\nabla(\nabla F)(\cdots,Y,X)]^{i_1\cdots i_k}_{j_1\cdots j_l} = \nabla(\nabla F)\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},E_{j_1},\cdots,E_{j_l},Y,X\right) \\
&= \nabla_X(\nabla F)\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},E_{j_1},\cdots,E_{j_l},Y\right) \\
&\xlongequal{\text{prop. 4.3.2}} \left[X\left((\nabla F)^{i_1\cdots j_k}_{j_1\cdots j_{l+1}}\right) + \sum_{s=1}^{k} X^m(\nabla F)^{i_1\cdots p\cdots i_k}_{j_1\cdots j_{l+1}}\Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m(\nabla F)^{i_1\cdots i_k}_{j_1\cdots p\cdots j_{l+1}}\Gamma^{p}_{mj_s}\right] \\
&\times \underbrace{E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes\varepsilon^{j_1}\otimes\cdots\otimes\varepsilon^{j_{l+1}}\left(\varepsilon^{i_1},\cdots,\varepsilon^{i_k},E_{j_1},\cdots,E_{j_l},Y\right)}_{=Y^{j_{l+1}}} \\
&\xlongequal{\text{prop. 4.3.5}} (\nabla F)^{i_1\cdots i_k}_{j_1\cdots j_l\underbrace{j_{l+1}}_{=q}\underbrace{j_{l+2}}_{=m}} X^m Y^q
\end{aligned}$$

This shows $\left[C^{k+1}_{l+1}\left(\nabla_X(\nabla F)\otimes Y\right)\right]^{i_1\cdots,i_k}_{j_1\cdots j_l} = [\nabla(\nabla F)(\cdots,Y,X)]^{i_1\cdots,i_k}_{j_1\cdots j_l}$. So

$$\nabla^2_{X,Y}F = C^{k+1}_{l+1}\left(C^{k+1}_{l+2}(\nabla(\nabla F)\otimes X)\otimes Y\right)$$

Therefore, since ∇_X commutes with contraction (see prop. 4.3.1 (iv)) and satisfies the product rule with respect to tensor products (see prop. 4.3.1 (iii)), we have

$$\begin{aligned}\nabla_X\left(\nabla_Y F\right) &= \nabla_X(\operatorname{tr}(\nabla F \otimes Y)) \\ &= \operatorname{tr}\left(\nabla_X(\nabla F \otimes Y)\right) \\ &= \operatorname{tr}\left(\nabla_X(\nabla F) \otimes Y + \nabla F \otimes \nabla_X Y\right) \\ &= \operatorname{tr}\left(\operatorname{tr}\left(\nabla^2 F \otimes X\right) \otimes Y\right) + \operatorname{tr}\left(\nabla F \otimes \nabla_X Y\right) \\ &= \nabla^2_{X,Y} F + \nabla_{(\nabla_X Y)} F\end{aligned}$$

■

Example 4.3.8 (The Covariant Hessian)**.** Let u be a smooth function on M. Then $\nabla u \in \Gamma\left(T^{(0,1)}TM\right) = \Omega^1(M)$ is just the 1-form du, because both tensors have the same action on vectors: $\nabla u(X) = \nabla_X u = Xu = du(X)$. The 2-tensor $\nabla^2 u = \nabla(du)$ is called the **covariant Hessian of** u. Above proposition shows that its action on smooth vector fields X, Y can be computed by the following formula:

$$\nabla^2 u(Y, X) = \nabla^2_{X,Y} u = \nabla_X\left(\nabla_Y u\right) - \nabla_{(\nabla_X Y)} u = X(Yu) - \left(\nabla_X Y\right) u.$$

In any local coordinates, it is

$$\nabla^2 u = u_{;ij} dx^i \otimes dx^j, \quad \text{with } u_{;ij} = \partial_j \partial_i u - \Gamma^k_{ji} \partial_k u.$$

♣

4.4 Vector and Tensor Fields Along Curves

Let M be a smooth manifold with or without boundary. Given a smooth curve $\gamma : I \to M$, a **vector field along** γ is a continuous map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$; it is a **smooth vector field along** γ if it is smooth as a map from I to TM. We let $\mathfrak{X}(\gamma)$ denote the set of all smooth vector fields along γ. It is a real vector space under pointwise vector addition and multiplication by constants, and it is a module over $C^\infty(I)$ with multiplication defined pointwise:

$$(fX)(t) = f(t)X(t).$$

The most obvious example of a vector field along a smooth curve γ is the curve's velocity: $\gamma'(t) \in T_{\gamma(t)}M$ for each t, and its coordinate expression

$$\gamma'(t) = \dot\gamma^1(t) \left.\frac{\partial}{\partial x^1}\right|_{\gamma(t)} + \cdots \dot\gamma^n(t) \left.\frac{\partial}{\partial x^n}\right|_{\gamma(t)}$$

shows that it is smooth. Here is another example: if γ is a curve in $\mathbb{R}^2$, let $N(t) = R\gamma'(t)$, where R is counterclockwise rotation by $\pi/2$, so $N(t)$ is normal to $\gamma'(t)$. In standard coordinates, $N(t) = \left(-\dot\gamma^2(t), \dot\gamma^1(t)\right)$, so N is a smooth vector field along γ.

A large supply of examples is provided by the following construction: suppose $\gamma : I \to M$ is a smooth curve and $\widetilde V$ is a smooth vector field on an open subset of M containing the image of γ. Define $V : I \to TM$ by setting $V(t) = \widetilde V_{\gamma(t)}$ for each $t \in I$. Since V is equal to the composition $\widetilde V \circ \gamma$, it is smooth. A smooth vector field along γ is said to be **extendible** if there exists a smooth vector field $\widetilde V$ on a neighborhood of the image of γ that is related to V in this way (Fig.4.1).

Not every vector field along a curve need be extendible; for example, if $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$ (Fig.4.2), then γ' is not extendible. Even if γ is injective, its velocity need not be extendible, as the next example shows.

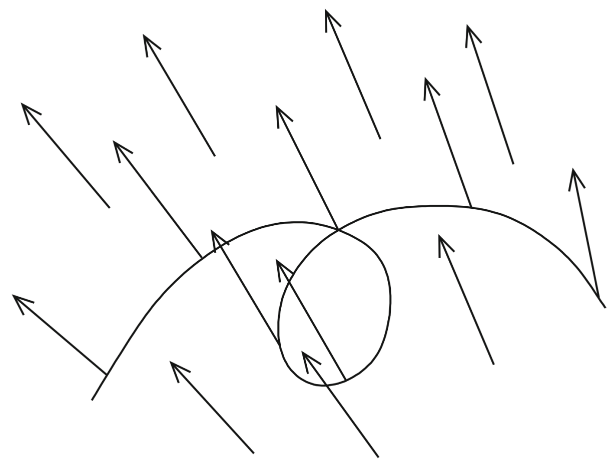


Figure 4.1: Extendible vector field

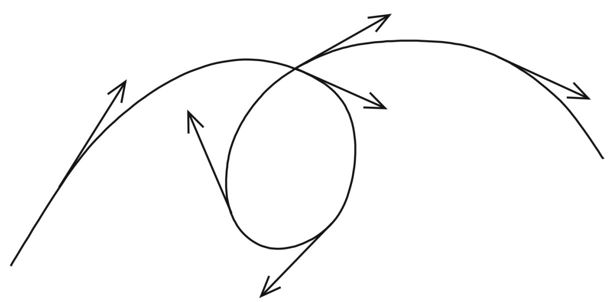


Figure 4.2: Nonextendible vector field

Example 4.4.1. Consider the figure eight curve $\gamma : (-\pi, \pi) \to \mathbb{R}^2$ defined by

$$\gamma(t) = (\sin 2t, \sin t).$$

Its image is a set that looks like a figure eight in the plane (Fig.4.3). [5] Problem 4-7 asks to show that γ is an injective smooth immersion, but its velocity vector field is not extendible. For problem 4-7, we can verify a more general claim given by [4] Example 4.2 (b): if $\gamma : J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$. For (b), suppose that the smooth curve γ is a smooth immersion. Then $d\gamma_{t_0}$ is injective for every $t_0 \in J$. Then $\gamma'(t_0) = d\gamma_{t_0}\left(d/\,dt|_{t_0}\right) \neq 0$. Conversely, suppose $\gamma'(t_0) \neq 0$ for every $t_0 \in J$. Suppose that $d\gamma_{t_0}(v) = 0$ for some $v \in T_{t_0}J$. Since $T_{t_0}J$ is spanned by d/dtt_{t_0}, we have that $v = \alpha d/\,dt|_{t_0}$ for some $\alpha \in \mathbb{R}$. Then $0 = d\gamma_{t_0}\left(\alpha d/\,dt|_{t_0}\right) = \alpha d\gamma_{t_0}\left(d/\,dt|_{t_0}\right) = \alpha\gamma'(t_0)$, implying $\alpha = 0$. Hence, $d\gamma_{t_0}$ is injective, and therefore γ is a smooth immersion.

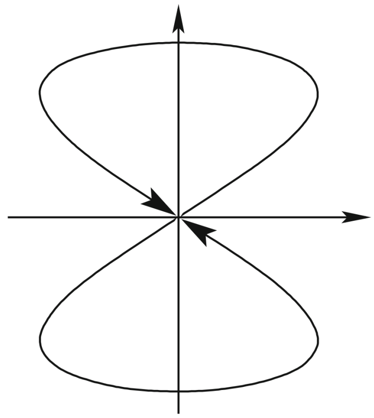


Figure 4.3: The image of the figure eight curve

We compute
$$\gamma'(t) = \frac{d\gamma^1}{dt}(t)\frac{\partial}{\partial x} + \frac{d\gamma^2}{dt}(t)\frac{\partial}{\partial y} = 2\cos 2t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y} = \begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix}$$

$2\cos 2t$'s zeros are $\pm\frac{\pi}{4} \cdot \pm\frac{3\pi}{4}$ and $\cos t$'s zeros are $\pm\frac{\pi}{2}$, so the velocity is nonvanishing and γ is a smooth immersion. The injectivity of γ is clear. The claim that $V = \gamma'$ is non-extendible is equivalent of saying that there exists no smooth vector field $\widetilde{V}(p) = \widetilde{V}_x(p)\frac{\partial}{\partial x} + \widetilde{V}_y(p)\frac{\partial}{\partial y}$ of which γ is an integral curve (see [4] p.206), or that there are no smooth functions $\widetilde{V}_x, \widetilde{V}_y : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix} = \gamma'(t) = \widetilde{V}(\gamma(t)) = \begin{pmatrix} \widetilde{V}_x(\gamma(t)) \\ \widetilde{V}_y(\gamma(t)) \end{pmatrix} = \begin{pmatrix} \widetilde{V}_x(\sin 2t, \sin t) \\ \widetilde{V}_y(\sin 2t, \sin t) \end{pmatrix}$$

For example, we assume there is $g : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$ such that $2\cos 2t = g(\sin 2t, \sin t)$ and use the implicit function theorem ... ♣

More generally, a **tensor field along** γ is a continuous map σ from I to some tensor bundle $T^{(k,l)}TM$ such that $\sigma(t) \in T^{(k,l)}\left(T_{\gamma(t)}M\right)$ for each $t \in I$. It is a **smooth tensor field along** γ if it is smooth as a map from I to $T^{(k,l)}TM$, and it is extendible if there is a smooth tensor field $\widetilde{\sigma}$ on a neighborhood of $\gamma(I)$ such that $\sigma = \widetilde{\sigma} \circ \gamma$.

4.4.1 Continuous Derivatives Along Curves

Here is the promised interpretation of a connection as a way to take derivatives of vector fields along curves.

Theorem 4.4.2 (Covariant Derivative Along a Curve)**.** *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For each smooth curve $\gamma : I \to M$, the connection determines a unique operator*
$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$
called the ***covariant derivative along*** γ*, satisfying the following properties:*

(i) LINEARITY OVER $\mathbb{R}$:
$$D_t(aV + bW) = aD_tV + bD_tW \quad \textit{for } a, b \in \mathbb{R}.$$

(ii) PRODUCT RULE:

$$D_t(fV) = f'V + fD_tV \quad \text{for } f \in C^\infty(I).$$

(iii) If $V \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\widetilde{V}$ of V,

$$\left(D_tV\right)(t) = \nabla_{\gamma'(t)}\widetilde{V}$$

where $D_tV \in \mathfrak{X}(\gamma)$ is a vector field along γ, i.e. $D_tV : I \to TM$ where $\left(D_tV\right)(t) \in T_{\gamma(t)}M$. For the RHS, $\nabla_{\gamma'(t)}\widetilde{V}$ is understood in terms of remark 4.2.6: let X be a vector field on a nieghborhood U of the point $p = \gamma(t)$ such that $X_p = \gamma'(t) = v \in T_pM$, and $\nabla_{\gamma'(t)}\widetilde{V} = \nabla_v\widetilde{V} = \left(\nabla_X\widetilde{V}\right)_p$.

Remark 4.4.3. There are analogous operators on the space of C^∞ tensor fields of any type along γ. For example, the above explains $T^{(1,0)}TM = TM$ case. Another of peculiarity is $T^{(0,0)}TM = M \times \mathbb{R}$ which as explained in prop. 4.3.1 gives rise to smooth functions along the curve γ, i.e., $f : \operatorname{Im}(\gamma) \to \mathbb{R}$. Then analogusly, we have $D_t(af + bg) = aD_tf + bD_tg$; $D_t(fg) = f'g + fD_tg$; and for extension $\widetilde{f} \in C^\infty(U)$ where $\operatorname{Im}(\gamma) \subseteq U$, we by similar notations above, $\left(D_tf\right)(t) = \nabla_{\gamma'(t)}\widetilde{f} = \nabla_v\widetilde{f} = \left(\nabla_X\widetilde{f}\right)_p \xlongequal{\text{prop. 4.3.1}} (X\widetilde{f})_p = X_p\widetilde{f} = v(\widetilde{f}) = \gamma'(t)(\widetilde{f}) \xlongequal{\text{[4]p.69}} (\widetilde{f} \circ \gamma)'(t) = (f \circ \gamma)'(t) = \frac{d}{dt}(f \circ \gamma)(t)$. ♠

Proof. For simplicity, we prove the theorem for the case of vector fields along γ; the proof for arbitrary tensor fields is essentially identical except for notation.

First we show uniqueness. Suppose D_t is such an operator, and let $t_0 \in I$ be arbitrary. An argument similar to that of Lemma 4.2.2 shows that the value of D_tV at t_0 depends only on the values of V in any interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ containing t_0. (If t_0 is an endpoint of I, extend γ to a slightly bigger open interval, prove the lemma there, and then restrict back to I. If M has nonempty boundary, we can do this after first embedding M into a smooth manifold $\widetilde{M}$ without boundary and extending ∇ arbitrarily to a connection on $\widetilde{M}$.) Choose smooth coordinates $\left(x^i\right)$ for M in a neighborhood of $\gamma\left(t_0\right)$, and write

$$V(t) = V^j(t)\partial_j\big|_{\gamma(t)}$$

for t near t_0, where $V^1, \ldots, V^n$ are smooth real-valued functions defined on some neighborhood of t_0 in I. By the properties of D_t, since each ∂_j is extendible,

$$\begin{aligned} D_tV(t) &= \dot{V}^j(t)\partial_j\Big|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\partial_j\big|_{\gamma(t)} \\ &= \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t))\right)\partial_k\Big|_{\gamma(t)}. \end{aligned} \tag{4.11}$$

We spare some sapce to explain above equation: We interpret ∂_j as a vector field along γ. Namely,

$$\begin{aligned} \partial_j : I &\to TM \\ t &\mapsto \partial_j|_{\gamma(t)} \end{aligned}$$

where $\left(\partial_j|_{\gamma(t)}\right)$ spans $T_{\gamma(t)}M$.

$$\begin{aligned} \left(D_tV\right)(t) = D_t\left(\sum_j V^j\partial_j\right)(t) &\overset{(i)}{=} \left[\sum_j D_t\left(V^j\partial_j\right)\right] \\ &\overset{(ii)}{=} \left[\sum_j \dot{V}^j\partial_j + V^jD_t\left(\partial_j\right)\right](t) \\ &\xlongequal{\text{Einstein summation}} \dot{V}^j(t)\partial_j(t) + V^j(t)D_t\left(\partial_j\right)(t) \\ &\overset{\text{(iii)}}{=} \dot{V}^j(t)\partial_j\Big|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\left(\widetilde{\partial_j}\right) \end{aligned}$$

$\partial_j \in \mathfrak{X}(\gamma)$ is natrurally extended to the coordinate vector field in $\mathfrak{X}(U)$ (see [4] p.176 Example 8.2), still denoted as ∂_j (i.e., $\widetilde{\partial}_J = \partial_j$). Let X be a vector field on a nieghborhood of the point $p = \gamma(t)$ such that $X_p = \gamma'(t) = v \in T_pM$, and $\nabla_{\gamma'(t)}\left(\widetilde{\partial}_J\right) = \nabla_{\gamma'(t)}\left(\partial_j\right) = \left(\nabla_X\left(\partial_j\right)\right)_p$. Now,

$$\nabla_X\left(\partial_j\right) = \left(X\left(\left(\partial_j\right)^k\right) + X^i\left(\partial_j\right)^m \Gamma_{im}^k\right)\partial_k = X^i\left(\partial_j\right)^m \Gamma_{im}^k \partial_k = X^i \Gamma_{ij}^k \partial_k$$

where the k-th component funciton $\left(\partial_j\right)^k$ is a constant funciton δ_j^k and thus is evaluated by the vector field X to be zero (see [4] p.180 and [4] 3.4(a)). Then,

$$\begin{aligned}
\nabla_{\gamma'(t)}\left(\widetilde{\partial}_j\right) &= \left(\nabla_X\left(\partial_j\right)\right)_{\gamma(t)} = \left(X^i \Gamma_{ij}^k \partial_k\right)_{\gamma(t)} = X^i(\gamma(t))\Gamma_{ij}^k(\gamma(t))\partial_k\big|_{\gamma(t)} \\
&= \left(X_{\gamma(t)}\right)^i \Gamma_{ij}^k(\gamma(t))\partial_k\Big|_{\gamma(t)} = \dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t))\partial_k\big|_{\gamma(t)}
\end{aligned}$$

where we notice that X^i, Γ_{ij}^k are all functions. Therefore,

$$\begin{aligned}
\left(D_t V\right)(t) &= \dot{V}^j(t)\partial_j\Big|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\left(\widetilde{\partial_J}\right) \\
&= \dot{V}^j(t)\partial_j\Big|_{\gamma(t)} + V^j(t)\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t))\partial_k\big|_{\gamma(t)} \\
&= \left(\dot{V}^k(t) + V^j(t)\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t))\right)\partial_k\Big|_{\gamma(t)}
\end{aligned}$$

This shows that such an operator is unique if it exists. For existence, if $\gamma(I)$ is contained in a single chart, we can define $D_t V$ by (4.11); the easy verification that it satisfies the requisite properties is left as an exercise. In the general case, we can cover $\gamma(I)$ with coordinate charts and define $D_t V$ by this formula in each chart, and uniqueness implies that the various definitions agree whenever two or more charts overlap. ■

(It is worth noting that in the physics literature, the covariant derivative along a curve is sometimes called the **absolute derivative**.)

Exercise 4.4.4. *Complete the proof of theorem by showing that the operator D_t defined in coordinates by (4.11) satisfies properties (i)-(iii).*

Apart from its use in proving existence of the covariant derivative along a curve, (4.11) also gives a practical formula for computing such covariant derivatives in coordinates.

Now we can further improve proposition 4.2.5 by showing that $\nabla_v Y$ actually depends only on the values of Y along any curve through p whose velocity is v.

Proposition 4.4.5. *Let M be a smooth manifold with or without boundary, let ∇ be a connection in TM, and let $p \in M$ and $v \in T_pM$. Suppose Y and $\widetilde{Y}$ are two smooth vector fields that agree at points in the image of some smooth curve $\gamma : I \to M$ such that $\gamma\left(t_0\right) = p$ and $\gamma'\left(t_0\right) = v$. Then $\nabla_v Y = \nabla_v \widetilde{Y}$.*

Proof. We can define a smooth vector field Z along γ by $Z(t) = Y_{\gamma(t)} = \widetilde{Y}_{\gamma(t)}$. Since both Y and $\widetilde{Y}$ are extensions of Z, it follows from condition (iii) in above theorem that both $\nabla_v Y$ and $\nabla_v \widetilde{Y}$ are equal to $D_t Z\left(t_0\right)$. ■

4.5 Geodesics

Armed with the notion of covariant differentiation along curves, we can now define acceleration and geodesics.

Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth curve $\gamma : I \to M$, we define the **acceleration of** γ to be the vector field $D_t \gamma'$ along γ. A smooth curve γ is called a **geodesic** (with respect to ∇) if its acceleration is zero: $D_t\gamma' \equiv 0$. In terms of smooth coordinates $\left(x^i\right)$, if we write the component functions of γ as $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$, then it follows from (4.11) that γ is a geodesic if and only if its component functions satisfy the following **geodesic equation**:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(x(t)) = 0, \tag{4.12}$$

where we use $x(t)$ as an abbreviation for the n-tuple of component functions $\left(x^1(t), \ldots, x^n(t)\right)$. This is a system of second-order ordinary differential equations (ODEs) for the real-valued functions $x^1, \ldots, x^n$. The next theorem uses ODE theory to prove existence and uniqueness of geodesics with suitable initial conditions. (Because difficulties can arise when a geodesic starts on the boundary or later hits the boundary, we state and prove this theorem only for manifolds without boundary.)

Theorem 4.5.1 (Existence and Uniqueness of Geodesics)**.** *Let M be a smooth manifold and ∇ a connection in TM. For every $p \in M, w \in T_pM$, and $t_0 \in \mathbb{R}$, there exist an open interval $I \subseteq \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma\left(t_0\right) = p$ and $\gamma'\left(t_0\right) = w$. Any two such geodesics agree on their common domain.*

Proof. Let $\left(x^i\right)$ be smooth coordinates on some neighborhood U of p. A smooth curve in U, written as $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$, is a geodesic if and only if its component functions satisfy (4.12). The standard trick for proving existence and uniqueness for such a second-order system is to introduce auxiliary variables $v^i = \dot{x}^i$ to convert it to the following equivalent first-order system in twice the number of variables:

$$\begin{aligned} \dot{x}^k(t) &= v^k(t), \\ \dot{v}^k(t) &= -v^i(t)v^j(t)\Gamma_{ij}^k(x(t)). \end{aligned} \tag{4.13}$$

Treating $\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right)$ as coordinates on $U \times \mathbb{R}^n$, we can recognize (4.13) as the equations for the flow of the vector field $G \in \mathfrak{X}\left(U \times \mathbb{R}^n\right)$ given by

$$G_{(x,v)} = v^k \left.\frac{\partial}{\partial x^k}\right|_{(x,v)} - v^i v^j \Gamma_{ij}^k(x) \left.\frac{\partial}{\partial v^k}\right|_{(x,v)}. \tag{4.14}$$

By the fundamental theorem on flows 1.2.8, for each $(p, w) \in U \times \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, there exist an open interval I_0 containing t_0 and a unique smooth solution $\zeta : I_0 \to U \times \mathbb{R}^n$ to this system satisfying the initial condition $\zeta\left(t_0\right) = (p, w)$. If we write the component functions of ζ as $\zeta(t) = \left(x^i(t), v^i(t)\right)$, then we can easily check that the curve $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$ in U satisfies the existence claim of the theorem.

To prove the uniqueness claim, suppose $\gamma, \tilde{\gamma} : I \to M$ are both geodesics defined on some open interval with $\gamma\left(t_0\right) = \widetilde{\gamma}\left(t_0\right)$ and $\gamma'\left(t_0\right) = \widetilde{\gamma}'\left(t_0\right)$. In any local coordinates around $\gamma\left(t_0\right)$, we can define smooth curves $\zeta, \widetilde{\zeta} : \left(t_0 - \varepsilon, t_0 + \varepsilon\right) \to U \times \mathbb{R}^n$ as above. These curves both satisfy the same initial value problem for the system (4.13), so by the uniqueness of ODE solutions, they agree on $\left(t_0 - \varepsilon, t_0 + \varepsilon\right)$ for some $\varepsilon > 0$. Suppose for the sake of contradiction that $\gamma(b) \neq \widetilde{\gamma}(b)$ for some $b \in I$. First suppose $b > t_0$, and let β be the infimum of numbers $b \in I$ such that $b > t_0$ and $\gamma(b) \neq \tilde{\gamma}(b)$ (Fig.4.4).

Then $\beta \in I$, and by continuity, $\gamma(\beta) = \tilde{\gamma}(\beta)$ and $\gamma'(\beta) = \widetilde{\gamma}'(\beta)$. Applying local uniqueness in a neighborhood of β, we conclude that γ and $\tilde{\gamma}$ agree on a neighborhood of β, which contradicts our choice of β. Arguing similarly to the left of t_0, we conclude that $\gamma \equiv \tilde{\gamma}$ on all of I. ■

A geodesic $\gamma : I \to M$ is said to be **maximal** if it cannot be extended to a geodesic on a larger interval, that is, if there does not exist a geodesic $\tilde{\gamma} : \widetilde{I} \to M$ defined on an interval $\widetilde{I}$ properly containing I and satisfying $\left.\widetilde{\gamma}\right|_I = \gamma$. A **geodesic segment** is a geodesic whose domain is a compact interval.

Corollary 4.5.2. *Let M be a smooth manifold and let ∇ be a connection in TM. For each $p \in M$ and $v \in T_pM$, there is a unique maximal geodesic $\gamma : I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing 0 .*

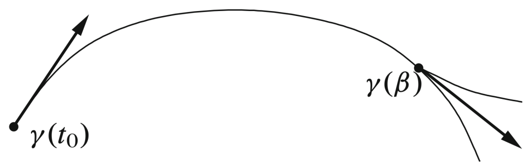


Figure 4.4: Uniqueness of geodesics

Proof. Given $p \in M$ and $v \in T_pM$, let I be the union of all open intervals containing 0 on which there is a geodesic with the given initial conditions. By Theorem 4.5.1 , all such geodesics agree where they overlap, so they define a geodesic $\gamma : I \to M$, which is obviously the unique maximal geodesic with the given initial conditions. ∎

Exercise 4.5.3. *Show that the maximal geodesics on $\mathbb{R}^n$ with respect to the Euclidean connection given by formula ([5] (4.3))*

$$\bar{\nabla}_X Y = X\left(Y^1\right)\frac{\partial}{\partial x^1} + \cdots + X\left(Y^n\right)\frac{\partial}{\partial x^n}$$

are exactly the constant curves and the straight lines with constant-speed parametrizations.

The unique maximal geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is often called simply the **geodesic with initial point** p **and initial velocity** v, and is denoted by γ_v. (For simplicity, we do not specify the initial point p in the notation; it can implicitly be recovered from v by $p = \pi(v)$, where $\pi : TM \to M$ is the natural projection.)

4.6 Parallel Transport

Another construction involving covariant differentiation along curves that will be useful later is called parallel transport. Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. A smooth vector or tensor field V along a smooth curve γ is said to be parallel along γ (with respect to ∇) if $D_t V \equiv 0$ (Fig.4.5). Thus a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve.

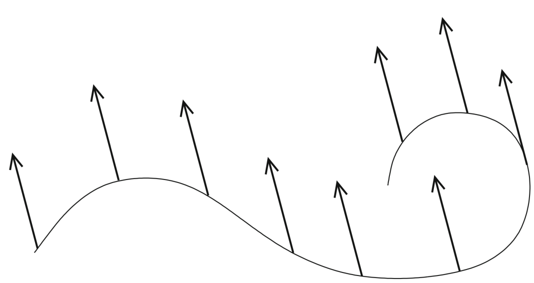


Figure 4.5: A parallel vector field along a curve

Exercise 4.6.1. *Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve, and let V be a smooth vector field along γ. Show that V is parallel along γ with respect to the Euclidean connection if and only if its component functions (with respect to* *the standard basis) are constants.*

The fundamental fact about parallel vector and tensor fields along curves is that every tangent vector or tensor at any point on a curve can be uniquely extended to a parallel field along the entire curve. Before we prove this claim, let us examine what the equation of parallelism looks like in coordinates. Given a smooth curve γ with a local coordinate representation $\gamma(t) = \left(\gamma^1(t), \ldots, \gamma^n(t)\right)$, formula (4.11) shows that a vector field V is parallel along γ if and only if

$$\dot{V}^k(t) = -V^j(t)\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t)), \quad k = 1, \ldots, n, \tag{4.15}$$

with analogous expressions based on Proposition 4.3.5 for tensor fields of other types. In each case, this is a system of first-order linear ordinary differential equations for the unknown coefficients of the vector or tensor field-in the vector case, the functions $\left(V^1(t), \ldots, V^n(t)\right)$. The usual ODE theorem guarantees the existence and uniqueness of a solution for a short time, given any initial values at $t = t_0$; but since the equation is linear, we can actually show much more: there exists a unique solution on the entire parameter interval.

Theorem 4.6.2 (Existence, Uniqueness, and Smoothness for Linear ODEs)**.** *Let $I \subseteq \mathbb{R}$ be an open interval, and for $1 \leq j, k \leq n$, let $A_j^k : I \to \mathbb{R}$ be smooth functions. For all $t_0 \in I$ and every initial vector $\left(c^1, \ldots, c^n\right) \in \mathbb{R}^n$, the linear initial value problem*

$$\begin{aligned} \dot{V}^k(t) &= A_j^k(t)V^j(t) \\ V^k\left(t_0\right) &= c^k \end{aligned}$$

has a unique smooth solution on all of I, and the solution depends smoothly on $(t, c) \in I \times \mathbb{R}^n$.

Theorem 4.6.3 (Existence and Uniqueness of Parallel Transport)**.** *Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given a smooth curve $\gamma : I \to M, t_0 \in I$, and a vector $v \in T_{\gamma(t_0)}M$ or tensor $v \in T^{(k,l)}\left(T_{\gamma(t)}M\right)$, there exists a unique parallel vector or tensor field V along γ such that $V\left(t_0\right) = v$.*

Proof. As in the proof of Theorem 4.4.2, we carry out the proof for vector fields. The case of tensor fields differs only in notation.

First suppose $\gamma(I)$ is contained in a single coordinate chart. Then V is parallel along γ if and only if its components satisfy the linear system of ODEs (4.15). Theorem 4.6.2 guarantees the existence and uniqueness of a solution on all of I with any initial condition $V\left(t_0\right) = v$.

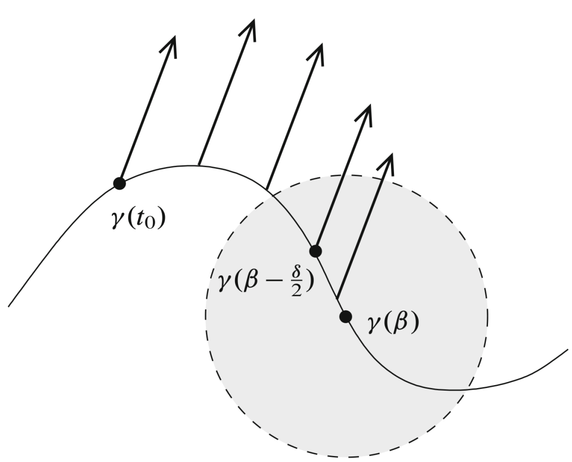


Figure 4.6: Existence and uniqueness of parallel transports

Now suppose $\gamma(I)$ is not covered by a single chart. Let β denote the supremum of all $b > t_0$ for which a unique parallel transport exists on $[t_0, b]$. (The argument for $t < t_0$ is similar.) We know that $\beta > t_0$, since for b close enough to $t_0, \gamma([t_0, b])$ is contained in a single chart and the above argument applies. Then a unique parallel transport V exists on $[t_0, \beta)$ (Fig.4.6). If β is equal to sup I, we are done. If not, choose smooth coordinates on an open set containing $\gamma(\beta - \delta, \beta + \delta)$ for some positive δ. Then there exists a unique parallel vector field $\widetilde{V}$ on $(\beta - \delta, \beta + \delta)$ satisfying the initial condition $\widetilde{V}(\beta - \delta/2) = V(\beta - \delta/2)$. By uniqueness, $V = \widetilde{V}$ on their common domain, and therefore $\widetilde{V}$ is a parallel extension of V past β, which is a contradiction. ■

The vector or tensor field whose existence and uniqueness are proved in Theorem 4.6.3 is called the **parallel transport of** v **along** γ. For each $t_0, t_1 \in I$, we define a map

$$P_{t_0 t_1}^{\gamma} : T_{\gamma(t_0)} M \rightarrow T_{\gamma(t_1)} M,$$

called the **parallel transport map**, by setting $P_{t_0 t_1}^{\gamma}(v) = V(t_1)$ for each $v \in T_{\gamma(t_0)} M$, where V is the parallel transport of v along γ. This map is linear, because the equation of parallelism is linear. It is in fact an isomorphism, because $P_{t_1 t_0}^{\gamma}$ is an inverse for it.

It is also useful to extend the parallel transport operation to curves that are merely piecewise smooth. Given an admissible curve $\gamma : [a, b] \rightarrow M$, a map $V : [a, b] \rightarrow TM$ such that $V(t) \in T_{\gamma(t)} M$ for each t is called a **piecewise smooth vector field along** γ if V is continuous and there is an admissible partition $(a_0, \ldots, a_k)$ for γ such that V is smooth on each subinterval $[a_{i-1}, a_i]$. We will call any such partition an **admissible partition for** V. A piecewise smooth vector field V along γ is said to be **parallel along** γ if $D_t V = 0$ wherever V is smooth.

Corollary 4.6.4 (Parallel Transport Along Piecewise Smooth Curves). *Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given an admissible curve $\gamma : [a, b] \rightarrow M$ and a vector $v \in T_{\gamma(t_0)} M$ or tensor $v \in T^{(k,l)}(T_{\gamma(t)} M)$, there exists a unique piecewise smooth parallel vector or tensor field V along γ such that $V(a) = v$, and V is smooth wherever γ is.*

Proof. Let $(a_0, \ldots, a_k)$ be an admissible partition for γ. First define $\left.V\right|_{[a_0, a_1]}$ to be the parallel transport of v along the first smooth segment $\left.\gamma\right|_{[a_0, a_1]}$; then define $\left.V\right|_{[a_1, a_2]}$ to be the parallel transport of $V(a_1)$ along the next smooth segment $\left.\gamma\right|_{[a_1, a_2]}$; and continue by induction. ■

Here is an extremely useful tool for working with parallel transport. Given any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)} M$, we can parallel transport the vectors b_i along γ, thus obtaining an n-tuple of parallel vector fields $(E_1, \ldots, E_n)$ along γ. Because each parallel transport map is an isomorphism, the vectors $(E_i(t))$ form a basis for $T_{\gamma(t)} M$ at each point $\gamma(t)$. Such an n-tuple of vector fields along γ is called a **parallel frame along** γ. Every smooth (or piecewise smooth) vector field along γ can be expressed in terms of such a frame as $V(t) = V^i(t) E_i(t)$, and then the properties of covariant derivatives along curves, together with the fact that the E_i 's are parallel, imply

$$D_t V(t) = \dot{V}^i(t) E_i(t) \tag{4.16}$$

wherever V and γ are smooth. This means that a vector field is parallel along γ if and only if its component functions with respect to the frame (E_i) are constants.

The parallel transport map is the means by which a connection "connects" nearby tangent spaces. The next theorem and its corollary show that parallel transport determines covariant differentiation along curves, and thereby the connection itself.

Theorem 4.6.5 (Parallel Transport Determines Covariant Differentiation). *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose $\gamma : I \rightarrow M$ is a smooth curve and V is a smooth vector field along γ. For each $t_0 \in I$,*

$$D_t V(t_0) = \lim_{t_1 \rightarrow t_0} \frac{P_{t_1 t_0}^{\gamma} V(t_1) - V(t_0)}{t_1 - t_0}. \tag{4.17}$$

Proof. Let (E_i) be a parallel frame along γ, and write $V(t) = V^i(t)E_i(t)$ for $t \in I$. On the one hand, (4.16) shows that $D_t V(t_0) = \dot{V}^i(t_0) E_i(t_0)$.

On the other hand, for every fixed $t_1 \in I$, the parallel transport of the vector $V(t_1)$ along γ is the constant-coefficient vector field $W(t) = V^i(t_1) E_i(t)$ along γ, so $P^{\gamma}_{t_1 t_0} V(t_1) = V^i(t_1) E_i(t_0)$. Inserting these formulas into (4.17) and taking the limit as $t_1 \to t_0$, we conclude that the right-hand side is also equal to $\dot{V}^i(t_0) E_i(t_0)$. ∎

Corollary 4.6.6 (Parallel Transport Determines the Connection)**.** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose X and Y are smooth vector fields on M. For every $p \in M$,*

$$\left.\nabla_X Y\right|_p = \lim_{h \to 0} \frac{P^{\gamma}_{h0} Y_{\gamma(h)} - Y_p}{h} \tag{4.18}$$

where $\gamma : I \to M$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

Proof. Given $p \in M$ and a smooth curve γ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$, let $V(t)$ denote the vector field along γ determined by Y, so $V(t) = Y_{\gamma(t)}$. By property (iii) of Theorem 4.4.2, $\left.\nabla_X Y\right|_p$ is equal to $D_t V(0)$, so the result follows from Theorem 4.6.5. ∎

A smooth vector or tensor field on M is said to be **parallel** (with respect to ∇) if it is parallel along every smooth curve in M. For example, Exercise 4.6.1 shows that every constant-coefficient vector field on $\mathbb{R}^n$ is parallel.

Proposition 4.6.7. *Suppose M is a smooth manifold with or without boundary, ∇ is a connection in TM, and A is a smooth vector or tensor field on M. Then A is parallel on M if and only if $\nabla A \equiv 0$.*

Proof. [5] Problem 4-12. ∎

Although Theorem 4.6.3 showed that it is always possible to extend a vector at a point to a parallel vector field along any given curve, it may not be possible in general to extend it to a parallel vector field on an open subset of the manifold. The impossibility of finding such extensions is intimately connected with the phenomenon of curvature, which will occupy a major portion of our attention in the second half of the book.

4.7 Pullback Connections

Like vector fields, connections in the tangent bundle cannot be either pushed forward or pulled back by arbitrary smooth maps. However, there is a natural way to pull back such connections by means of a diffeomorphism. In this section we define this operation and enumerate some of its most important properties.

Suppose M and $\widetilde{M}$ are smooth manifolds and $\varphi : M \to \widetilde{M}$ is a diffeomorphism. For a smooth vector field $X \in \mathfrak{X}(M)$, recall that the **pushforward of** X is the unique vector field $\varphi_* X \in \mathfrak{X}(\widetilde{M})$ that satisfies $d\varphi_p(X_p) = (\varphi_* X)_{\varphi(p)}$ for all $p \in M$. (see [4] p.182-183)

Lemma 4.7.1 (Pullback Connections)**.** *Suppose M and $\widetilde{M}$ are smooth manifolds with or without boundary. If $\widetilde{\nabla}$ is a connection in $T\widetilde{M}$ and $\varphi : M \to \widetilde{M}$ is a diffeomorphism, then the map $\varphi^* \widetilde{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by*

$$\left(\varphi^* \widetilde{\nabla}\right)_X Y = \left(\varphi^{-1}\right)_* \left(\widetilde{\nabla}_{\varphi_* X} \left(\varphi_* Y\right)\right) \tag{4.19}$$

*is a connection in TM, called the **pullback of** $\widetilde{\nabla}$ **by** φ.*

Proof. It is immediate from the definition that $\left(\varphi^*\widetilde{\nabla}\right)_X Y$ is linear over $\mathbb{R}$ in Y. To see that it is linear over $C^\infty(M)$ in X, let $f \in C^\infty(M)$, and let $\tilde{f} = f \circ \varphi^{-1}$, so $\varphi_*(fX) = \tilde{f}\varphi_* X$. Then

$$\begin{aligned}
&\left(\varphi^*\tilde{\nabla}\right)_{fX} Y = \left(\varphi^{-1}\right)_* \left(\tilde{\nabla}_{\tilde{f}\varphi_* X}\left(\varphi_* Y\right)\right) \\
&= \left(\varphi^{-1}\right)_* \left(\tilde{f}\widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right)\right) \\
&= f\left(\varphi^*\widetilde{\nabla}\right)_X Y.
\end{aligned}$$

Finally, to prove the product rule in Y, let f and $\tilde{f}$ be as above, and note that an easy result [5] A.7 implies $\left(\varphi_* X\right)(\tilde{f}) = (Xf) \circ \varphi^{-1}$. Thus

$$\begin{aligned}
\left(\varphi^*\tilde{\nabla}\right)_X (fY) &= \left(\varphi^{-1}\right)_* \left(\tilde{\nabla}_{\varphi_* X}\left(\tilde{f}\varphi_* Y\right)\right) \\
&= \left(\varphi^{-1}\right)_* \left(\tilde{f}\widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right) + \left(\varphi_* X\right)(\tilde{f})\varphi_* Y\right) \\
&= f\left(\varphi^*\widetilde{\nabla}\right)_X Y + (Xf)Y.
\end{aligned}$$

■

The next proposition shows that various important concepts defined in terms of connections-covariant derivatives along curves, parallel transport, and geodesics all behave as expected with respect to pullback connections.

Proposition 4.7.2 (Properties of Pullback Connections)**.** *Suppose M and $\widetilde{M}$ are smooth manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a diffeomorphism. Let $\widetilde{\nabla}$ be a connection in $T\widetilde{M}$ and let $\nabla = \varphi^*\widetilde{\nabla}$ be the pullback connection in TM. Suppose $\gamma : I \to M$ is a smooth curve. (a) φ takes covariant derivatives along curves to covariant derivatives along curves: if V is a smooth vector field along γ, then*

$$d\varphi \circ D_t V = \widetilde{D}_t(d\varphi \circ V),$$

where D_t is covariant differentiation along γ with respect to ∇, and $\widetilde{D}_t$ is covariant differentiation along $\varphi \circ \gamma$ with respect to $\tilde{\nabla}$. (b) φ takes geodesics to geodesics: if γ is a ∇-geodesic in M, then $\varphi \circ \gamma$ is a $\widetilde{\nabla}$-geodesic in $\widetilde{M}$. (c) φ takes parallel transport to parallel transport: for every $t_0, t_1 \in I$,

$$d\varphi_{\gamma(t_1)} \circ P^{\gamma}_{t_0 t_1} = P^{\varphi\circ\gamma}_{t_0 t_1} \circ d\varphi_{\gamma(t_0)}.$$

Proof. [5] Problem 4-13. ■

4.8 Problems

Exercise 4.8.1. *([5] 4-6) Let M be a smooth manifold and let ∇ be a connection in TM. Define a map $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by*

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

*(a) Show that τ is a $(1, 2)$-tensor field, called the **torsion tensor of** ∇.*

(b) We say that ∇ is symmetric if its torsion vanishes identically. Show that ∇ is symmetric if and only if its connection coefficients with respect to every coordinate frame are symmetric: $\Gamma^k_{ij} = \Gamma^k_{ji}$. [Warning: They might not be symmetric with respect to other frames.]

(c) Show that ∇ is symmetric if and only if the covariant Hessian $\nabla^2 u$ of every smooth function $u \in C^\infty(M)$ is a symmetric 2-tensor field. (See Example 4.3.8.)

(d) Show that the Euclidean connection $\bar{\nabla}$ on $\mathbb{R}^n$ is symmetric.

Exercise 4.8.2. *([5] 4-9) Let M be a smooth manifold, and let ∇^0 and ∇^1 be two connections on TM.*

(a) Show that ∇^0 and ∇^1 have the same torsion (4.8.1) if and only if their difference tensor is symmetric, i.e., $D(X,Y) = D(Y,X)$ for all X and Y.

(b) Show that ∇^0 and ∇^1 determine the same geodesics if and only if their difference tensor is antisymmetric, i.e., $D(X,Y) = -D(Y,X)$ for all X and Y.

Chapter 5

The Levi-Civita Connection

5.1 The Tangential Connection Revisited

We are eventually going to show that on each Riemannian manifold there is a natural connection that is particularly well suited to computations in Riemannian geometry. Since we get most of our intuition about Riemannian manifolds from studying submanifolds of $\mathbb{R}^n$ with the induced metric, let us start by examining that case.

Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. A geodesic in M should be "as straight as possible." A reasonable way to make this rigorous is to require that the geodesic have no acceleration in directions tangent to the manifold, or in other words that its acceleration vector have zero orthogonal projection onto TM.

The tangential connection $\nabla^\top_X(Y) = \pi^\top \left(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}|_M\right)$ defined in Example 4.2.10 is perfectly suited to this task, because it computes covariant derivatives on M by taking ordinary derivatives in $\mathbb{R}^n$ and projecting them orthogonally to TM.

It is easy to compute covariant derivatives along curves in M with respect to the tangential connection. Suppose $\gamma : I \to M$ is a smooth curve. Then γ can be regarded as either a smooth curve in M or a smooth curve in $\mathbb{R}^n$, and a smooth vector field V along γ that takes its values in TM can be regarded as either a vector field along γ in M or a vector field along γ in $\mathbb{R}^n$. Let $\overline{D}_t V$ denote the covariant derivative of V along γ (as a curve in $\mathbb{R}^n$) with respect to the Euclidean connection $\overline{\nabla}$, and let $D^\top_t V$ denote its covariant derivative along γ (as a curve in M) with respect to the tangential connection $\nabla^\top$. [5] Proposition 5.1 shows a simple relationship between them: $\forall t \in I$, $D^\top_t V(t) = \pi^\top(\overline{D}_t V(t))$. Via plugging the zero connection coefficients of the Euclidean connection on $\mathbb{R}^n$ into (4.11), we see that $\overline{D}_t \gamma'(t) = \gamma''(t)$. Thus, the smooth curve $\gamma : I \to M$ is a geodesic with respect to the tangential connection on M if and only if its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.

Analogs for embedded Riemannian or pseudo-Riemannian manifolds in pseudo-Euclidean space $\mathbb{R}^{r,s}$ are provided in [5] p.117 as well.

5.2 Connections on Abstract Riemannian Manifolds

There is a celebrated (and hard) theorem of John Nash that says that every Riemannian metric on a smooth manifold can be realized as the induced metric of some embedding in a Euclidean space. That theorem was later generalized independently by Robert Greene and Chris J. S. Clarke to pseudo-Riemannian metrics. Thus, in a certain sense, we would lose no generality by studying only submanifolds of Euclidean and pseudo-Euclidean spaces with their induced metrics, for which the tangential connection would suffice. However, when we are trying to understand *intrinsic* properties of a Riemannian manifold, an embedding introduces a great deal of extraneous information, and in some cases actually makes it harder to discern which geometric properties depend only on the metric. Our task in this chapter is to distinguish some important properties of the tangential connection that make sense for connections on an abstract Riemannian or pseudo-Riemannian manifold, and to use them to single out a unique connection in the abstract case.

5.2.1 Metric Connections

The Euclidean connection on $\mathbb{R}^n$ has one very nice property with respect to the Euclidean metric: it satisfies the product rule

$$\overline{\nabla}_X\langle Y, Z\rangle=\left\langle\overline{\nabla}_X Y, Z\right\rangle+\left\langle Y, \overline{\nabla}_X Z\right\rangle, \tag{5.1}$$

as you can verify easily by computing in terms of the standard basis. (In this formula, the left-hand side represents the covariant derivative of the real-valued function $\langle Y, Z\rangle$ regarded as a $(0,0)$-tensor field, which is really just $X\langle Y, Z\rangle$ by virtue of property (ii) of Prop. 4.3.1.) The Euclidean connection has the same property with respect to the pseudo-Euclidean metric on $\mathbb{R}^{r,s}$. It is almost immediate that the tangential connection on a Riemannian or pseudo-Riemannian submanifold satisfies the same product rule, if we now interpret all the vector fields as being tangent to M and interpret the inner products as being taken with respect to the induced metric on M (see Prop. 5.2.3 below).

This property makes sense on an abstract Riemannian or pseudo-Riemannian manifold. Let g be a Riemannian or pseudo-Riemannian metric on a smooth manifold M (with or without boundary). A connection ∇ on TM is said to be **compatible with** g, or to be a **metric connection**, if it satisfies the following product rule for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X\langle Y, Z\rangle=\left\langle\nabla_X Y, Z\right\rangle+\left\langle Y, \nabla_X Z\right\rangle .$$

The next proposition gives several alternative characterizations of compatibility with a metric, any one of which could be used as the definition.

Proposition 5.2.1 (Characterizations of Metric Connections). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be a connection on TM. The following conditions are equivalent:*

(a) ∇ is compatible with $g: \nabla_X\langle Y, Z\rangle=\left\langle\nabla_X Y, Z\right\rangle+\left\langle Y, \nabla_X Z\right\rangle$.

(b) g is parallel with respect to $\nabla: \nabla g \equiv 0$.

(c) In terms of any smooth local frame (E_i), the connection coefficients of ∇ satisfy

$$\Gamma_{ki}^l g_{lj}+\Gamma_{kj}^l g_{il}=E_k\left(g_{ij}\right) . \tag{5.2}$$

(d) If V, W are smooth vector fields along any smooth curve γ, then

$$\frac{d}{dt}\langle V, W\rangle=\left\langle D_t V, W\right\rangle+\left\langle V, D_t W\right\rangle . \tag{5.3}$$

(e) If V, W are parallel vector fields along a smooth curve γ in M, then $\langle V, W\rangle$ is constant along γ.

(f) Given any smooth curve γ in M, every parallel transport map along γ is a linear isometry.

(g) Given any smooth curve γ in M, every orthonormal basis at a point of γ can be extended to a parallel orthonormal frame along γ.

Proof. First we prove (a) $\Leftrightarrow$ (b). By (4.7) and (4.4), the total covariant derivative of the symmetric 2-tensor g is given by

$$(\nabla g)(Y, Z, X)=\left(\nabla_X g\right)(Y, Z)=X(g(Y, Z))-g\left(\nabla_X Y, Z\right)-g\left(Y, \nabla_X Z\right) .$$

This is zero for all X, Y, Z if and only if (5.1) is satisfied for all X, Y, Z. To prove (b) $\Leftrightarrow$ (c), note that Proposition 4.3.5 shows that the components of ∇g in terms of a smooth local frame (E_i) are

$$g_{ij;k} = E_k\left(g_{ij}\right) - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}.$$

These are all zero if and only if (5.2) is satisfied. Next we prove (a) $\Leftrightarrow$ (d). Assume (a), and let V, W be smooth vector fields along a smooth curve $\gamma : I \to M$. Given $t_0 \in I$, in a neighborhood of $\gamma\left(t_0\right)$ we may choose coordinates $\left(x^i\right)$ and write $V = V^i \partial_i$ and $W = W^j \partial_j$ for some smooth functions $V^i, W^j (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$. Applying (5.1) to the extendible vector fields ∂_i, ∂_j, we obtain

$$\begin{aligned} \frac{d}{dt}\langle V, W\rangle &= \frac{d}{dt}\left(V^i W^j \left\langle\partial_i, \partial_j\right\rangle\right) \\ &= \left(\dot{V}^i W^j + V^i \dot{W}^j\right)\left\langle\partial_i, \partial_j\right\rangle + V^i W^j\left(\left\langle\nabla_{\gamma'(t)} \partial_i, \partial_j\right\rangle + \left\langle\partial_i, \nabla_{\gamma'(t)} \partial_j\right\rangle\right) \\ &= \left\langle D_t V, W\right\rangle + \left\langle V, D_t W\right\rangle, \end{aligned}$$

which proves (d). Conversely, if (d) holds, then in particular it holds for extendible vector fields along γ, and then (a) follows from part (iii) of Theorem 4.4.2.

Now we will prove $(\mathrm{d}) \Rightarrow (\mathrm{e}) \Rightarrow (\mathrm{f}) \Rightarrow (\mathrm{g}) \Rightarrow (\mathrm{d})$. Assume first that (d) holds. If V and W are parallel along γ, then (5.3) shows that $\langle V, W\rangle$ has zero derivative with respect to t, so it is constant along γ.

Now assume (e). Let v_0, w_0 be arbitrary vectors in $T_{\gamma(t_0)} M$, and let V, W be their parallel transports along γ, so that $V\left(t_0\right) = v_0, W\left(t_0\right) = w_0, P_{t_0 t_1}^\gamma v_0 = V\left(t_1\right)$, and $P_{t_0 t_1}^\gamma w_0 = W\left(t_1\right)$. Because $\langle V, W\rangle$ is constant along γ, it follows that $\left\langle P_{t_0 t_1}^\gamma v_0, P_{t_0 t_1}^\gamma w_0\right\rangle = \left\langle V\left(t_1\right), W\left(t_1\right)\right\rangle = \left\langle V\left(t_0\right), W\left(t_0\right)\right\rangle = \left\langle v_0, w_0\right\rangle$, so $P_{t_0 t_1}^\gamma$ is a linear isometry.

Next, assuming (f), we suppose $\gamma : I \to M$ is a smooth curve and (b_i) is an orthonormal basis for $T_{\gamma(t_0)} M$, for some $t_0 \in I$. We can extend each b_i by parallel transport to obtain a smooth parallel vector field E_i along γ, and the assumption that parallel transport is a linear isometry guarantees that the resulting n-tuple (E_i) is an orthonormal frame at all points of γ.

Finally, assume that (g) holds, and let (E_i) be a parallel orthonormal frame along γ. Given smooth vector fields V and W along γ, we can express them in terms of this frame as $V = V^i E_i$ and $W = W^j E_j$. The fact that the frame is orthonormal means that the metric coefficients $g_{ij} = \left\langle E_i, E_j\right\rangle$ are constants along γ(± 1 or 0), and the fact that it is parallel means that $D_t V = \dot{V}^i E_i$ and $D_t W = \dot{W}^i E_i$. Thus both sides of (5.3) reduce to the following expression:

$$g_{ij}\left(\dot{V}^i W^j + V^i \dot{W}^j\right).$$

This proves (d). ■

Corollary 5.2.2. *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold with or without boundary, ∇ is a metric connection on M, and $\gamma : I \to M$ is a smooth curve.*

(a) $|\gamma'(t)|$ is constant if and only if $D_t \gamma'(t)$ is orthogonal to $\gamma'(t)$ for all $t \in I$.

(b) If γ is a geodesic, then $|\gamma'(t)|$ is a constant.

Proof. Let $V(t) = W(t) = \gamma'(t)$ in proposition 5.2.1(d). ■

Proposition 5.2.3. *If M is an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the tangential connection on M is compatible with the induced Riemannian or pseudo-Riemannian metric.*

Proof. We will show that $\nabla^\top$ satisfies (5.1). Suppose $X, Y, Z \in \mathfrak{X}(M)$, and let $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ be smooth extensions of them to an open subset of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$. At points of M, we have

$$\begin{aligned}
\nabla_X^\top\langle Y, Z\rangle &= X\langle Y, Z\rangle = \widetilde{X}\langle\widetilde{Y}, \widetilde{Z}\rangle \\
&= \overline{\nabla}_{\widetilde{X}}\langle\widetilde{Y}, \widetilde{Z}\rangle \\
&= \left\langle\overline{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\right\rangle + \left\langle\widetilde{Y}, \overline{\nabla}_{\widetilde{X}}\widetilde{Z}\right\rangle \\
&= \left\langle\pi^\top\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\right), \widetilde{Z}\right\rangle + \left\langle\widetilde{Y}, \pi^\top\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Z}\right)\right\rangle \\
&= \left\langle\nabla_X^\top Y, Z\right\rangle + \left\langle Y, \nabla_X^\top Z\right\rangle,
\end{aligned}$$

where the next-to-last equality follows from the fact that $\widetilde{Z}$ and $\widetilde{Y}$ are tangent to M. ■

5.2.2 Symmetric Connections

It turns out that every abstract Riemannian or pseudo-Riemannian manifold admits many different metric connections (see [5] Problem 5-1), so requiring compatibility with the metric is not sufficient to pin down a unique connection on such a manifold. To do so, we turn to another key property of the tangential connection. Recall the definition of the Euclidean connection. The expression on the right-hand side of that definition is reminiscent of part of the coordinate expression for the Lie bracket:

$$[X, Y] = X\left(Y^i\right)\frac{\partial}{\partial x^i} - Y\left(X^i\right)\frac{\partial}{\partial x^i}.$$

In fact, the two terms in the Lie bracket formula are exactly the coordinate expressions for $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$. Therefore, the Euclidean connection satisfies the following identity for all smooth vector fields X, Y :

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

This expression has the virtue that it is coordinate-independent and makes sense for every connection on the tangent bundle. We say that a connection ∇ on the tangent bundle of a smooth manifold M is **symmetric** if

$$\nabla_X Y - \nabla_Y X \equiv [X, Y] \text{ for all } X, Y \in \mathfrak{X}(M).$$

The symmetry condition can also be expressed in terms of the **torsion tensor** of the connection, which was introduced in Problem 4.8.1; this is the smooth $(1,2)$-tensor field $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Thus a connection ∇ is symmetric if and only if its torsion vanishes identically. It follows from the result of Problem 4.8.1 that a connection is symmetric if and only if its connection coefficients in every coordinate frame satisfy $\Gamma_{ij}^k = \Gamma_{ji}^k$; this is the origin of the term "symmetric."

Proposition 5.2.4. *If M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space, then the tangential connection on M is symmetric.*

Proof. Let M be an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed either with the Euclidean metric or with a pseudoEuclidean metric $\bar{q}^{(r,s)}, r + s = n$. Let $X, Y \in \mathfrak{X}(M)$, and let $\widetilde{X}, \widetilde{Y}$ be smooth extensions of them to an open subset of the ambient space. If $\iota : M \hookrightarrow \mathbb{R}^n$ represents the inclusion map, it follows that X and Y are ι-related to $\widetilde{X}$ and $\widetilde{Y}$, respectively, and thus by the naturality of the Lie bracket ([5] Prop. A.39), $[X,Y]$ is ι-related to $[\widetilde{X},\widetilde{Y}]$. In particular, $[\widetilde{X},\widetilde{Y}]$ is tangent to M, and its restriction to M is equal to $[X,Y]$. Therefore,

$$\begin{aligned}\nabla^{\top}_X Y - \nabla^{\top}_Y X &= \pi^{\top}\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\Big|_M - \overline{\nabla}_{\widetilde{Y}}\widetilde{X}\Big|_M\right)\\ &= \pi^{\top}\left([\widetilde{X},\widetilde{Y}]\Big|_M\right)\\ &= [\widetilde{X},\widetilde{Y}]\Big|_M\\ &= [X,Y].\end{aligned}$$

■

The last two propositions show that if we wish to single out a connection on each Riemannian or pseudo-Riemannian manifold in such a way that it matches the tangential connection when the manifold is presented as an embedded submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$ with the induced metric, then we must require at least that the connection be compatible with the metric and symmetric. It is a pleasant fact that these two conditions are enough to determine a unique connection.

Theorem 5.2.5 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary). There exists a unique connection ∇ on TM that is compatible with g and symmetric. It is called the* ***Levi-Civita connection*** *of g (or also, when g is positive definite, the* ***Riemannian connection****).*

Proof. We prove uniqueness first, by deriving a formula for ∇. Suppose, therefore, that ∇ is such a connection, and let $X,Y,Z \in \mathfrak{X}(M)$. Writing the compatibility equation three times with X,Y,Z cyclically permuted, we obtain

$$\begin{aligned}X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle\\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle\end{aligned}$$

Using the symmetry condition on the last term in each line, this can be rewritten as

$$\begin{aligned}X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X,Z]\rangle\\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y,X]\rangle\\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z,Y]\rangle\end{aligned}$$

Adding the first two of these equations and subtracting the third, we obtain

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle.$$

Finally, solving for $\langle \nabla_X Y, Z\rangle$, we get

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle). \tag{5.4}$$

Now suppose ∇^1 and ∇^2 are two connections on TM that are symmetric and compatible with g. Since the right-hand side of (5.4) does not depend on the connection, it follows that $\left\langle \nabla^1_X Y - \nabla^2_X Y, Z\right\rangle = 0$ for all X,Y,Z. This can happen only if $\nabla^1_X Y = \nabla^2_X Y$ for all X and Y, so $\nabla^1 = \nabla^2$.

To prove existence, we use (5.4), or rather a coordinate version of it. It suffices to prove that such a connection exists in each coordinate chart, for then uniqueness ensures that the connections in different charts agree where they overlap.

Let $\left(U,\left(x^i\right)\right)$ be any smooth local coordinate chart. Applying (5.4) to the coordinate vector fields, whose Lie brackets are zero, we obtain

$$\langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = \frac{1}{2}\left(\partial_i \langle \partial_j, \partial_l\rangle + \partial_j \langle \partial_l, \partial_i\rangle - \partial_l \langle \partial_i, \partial_j\rangle\right). \tag{5.5}$$

Recall the definitions of the metric coefficients and the connection coefficients:

$$g_{ij} = \langle \partial_i, \partial_j\rangle, \quad \nabla_{\partial_i}\partial_j = \Gamma_{ij}^m \partial_m.$$

Inserting these into (5.5) yields

$$\Gamma_{ij}^m g_{ml} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{5.6}$$

Finally, multiplying both sides by the inverse matrix g^{kl} and noting that $g_{ml}g^{kl} = \delta_m^k$, we get

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{5.7}$$

This formula certainly defines a connection in each chart, and it is evident from the formula that $\Gamma_{ij}^k = \Gamma_{ji}^k$, so the connection is symmetric by Problem 4.8.1(b). Thus only compatibility with the metric needs to be checked. Using (5.6) twice, we get

$$\begin{aligned}\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} &= \frac{1}{2}\left(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ki}\right) + \frac{1}{2}\left(\partial_k g_{ji} + \partial_j g_{ki} - \partial_i g_{kj}\right) \\ &= \partial_k g_{ij}\end{aligned}$$

By Proposition 5.2.1 (c), this shows that ∇ is compatible with g. ∎

A bonus of this proof is that it gives us explicit formulas that can be used for computing the Levi-Civita connection in various circumstances.

Corollary 5.2.6 (Formulas for the Levi-Civita Connection)**.** *Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be its Levi-Civita connection.*

(a) IN TERMS OF VECTOR FIELDS: If X, Y, Z are smooth vector fields on M, then

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle)\end{aligned} \tag{5.8}$$

*(This is known as **Koszul's formula**.)*

(b) IN COORDINATES: In any smooth coordinate chart for M, the coefficients of the Levi-Civita connection are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{5.9}$$

(c) IN A LOCAL FRAME: Let (E_i) be a smooth local frame on an open subset $U \subseteq M$, and let $c_{ij}^k : U \to \mathbb{R}$ be the n^3 smooth functions defined by

$$[E_i, E_j] = c_{ij}^k E_k. \tag{5.10}$$

Then the coefficients of the Levi-Civita connection in this frame are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m\right). \tag{5.11}$$

(d) *IN A LOCAL ORTHONORMAL FRAME: If g is Riemannian, (E_i) is a smooth local orthonormal frame, and the functions c_{ij}^k are defined by (5.11), then*

$$\Gamma_{ij}^k = \frac{1}{2}\left(c_{ij}^k - c_{ik}^j - c_{jk}^i\right). \tag{5.12}$$

Proof. We derived (5.8) and (5.9) in the proof of Theorem 5.2.5. To prove (5.11), apply formula (5.8) with $X = E_i, Y = E_j$, and $Z = E_l$, to obtain

$$\begin{aligned}\Gamma_{ij}^q g_{ql} &= \langle \nabla_{E_i} E_j, E_l \rangle \\ &= \frac{1}{2}\left(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m\right).\end{aligned}$$

Multiplying both sides by g^{kl} and simplifying yields (5.11). Finally, under the hypotheses of (d), we have $g_{ij} = \delta_{ij}$, so (5.11) reduces to (5.12) after rearranging and using the fact that c_{ij}^k is antisymmetric in i, j. ■

On every Riemannian or pseudo-Riemannian manifold, we will always use the Levi-Civita connection from now on without further comment. Geodesics with respect to this connection are called **Riemannian** (or **pseudo-Riemannian**) **geodesics**, or simply "geodesics" as long as there is no risk of confusion. The connection coefficients Γ_{ij}^k of the Levi-Civita connection in coordinates, given by (5.9), are called the **Christoffel symbols** of g.

The next proposition shows that these connections are familiar ones in the case of embedded submanifolds of Euclidean or pseudo-Euclidean spaces.

Proposition 5.2.7.

(a) *The Levi-Civita connection on a (pseudo-)Euclidean space is equal to the Euclidean connection.*

(b) *Suppose M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-) Euclidean space. Then the Levi-Civita connection on M is equal to the tangential connection $\nabla^\top$.*

Proof. We observed earlier in this chapter that the Euclidean connection is symmetric and compatible with both the Euclidean metric $\bar{g}$ and the pseudo-Euclidean metrics $\bar{q}^{(r,s)}$, which implies (a). Part (b) then follows from Propositions 5.2.3 and 5.2.4. ■

An important consequence of the definition is that because Levi-Civita connections are defined in coordinate-independent terms, they behave well with respect to isometries. Recall the definition of the pullback of a connection (see Lemma 4.7.1).

Proposition 5.2.8 (Naturality of the Levi-Civita Connection)**.** *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and let ∇ denote the Levi-Civita connection of g and $\widetilde{\nabla}$ that of $\widetilde{g}$. If $\varphi : M \to \widetilde{M}$ is an isometry, then $\varphi^* \widetilde{\nabla} = \nabla$.*

Proof. By uniqueness of the Levi-Civita connection, it suffices to show that the pullback connection $\varphi^* \widetilde{\nabla}$ is symmetric and compatible with g. The fact that φ is an isometry means that for any $X, Y \in \mathfrak{X}(M)$ and $p \in M$,

$$\langle Y_p, Z_p \rangle = \langle d\varphi_p(Y_p), d\varphi_p(Z_p) \rangle = \left\langle (\varphi_* Y)_{\varphi(p)}, (\varphi_* Z)_{\varphi(p)} \right\rangle,$$

where φ^*Y is a vector field, called the pushforward of Y by φ; see [4] p.183. In other words, $\langle Y, Z\rangle = \langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi$. Therefore,

$$\begin{aligned} X\langle Y, Z\rangle &= X\left(\langle \varphi_* Y, \varphi_* Z\rangle \circ \varphi\right) \\ &\xlongequal{\text{[4]Cor.8.21}} \left(\left(\varphi_* X\right)\langle \varphi_* Y, \varphi_* Z\rangle\right)\circ\varphi \\ &\xlongequal{\widetilde{\nabla}\text{ a metric conn.}} \left(\left\langle \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), \varphi_* Z\right\rangle + \left\langle \varphi_* Y, \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Z\right)\right\rangle\right)\circ\varphi \\ &\xlongequal{\text{see below}} \left\langle \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), Z\right\rangle + \left\langle Y, \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Z\right)\right\rangle \\ &\xlongequal{(4.19)} \left\langle \left(\varphi^*\widetilde{\nabla}\right)_X Y, Z\right\rangle + \left\langle Y, \left(\varphi^*\widetilde{\nabla}\right)_X Z\right\rangle, \end{aligned}$$

which shows that the pullback connection is compatible with g. The fourth equality is true in the same manner as $\langle Y, Z\rangle = \langle \varphi_* Y, \varphi_* Z\rangle\circ\varphi$. Specifically, $\left\langle \varphi_*\left(\left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right)\right), \varphi_* Z\right\rangle\circ\varphi = \left\langle \left(\varphi^{-1}\right)_* \widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right), Z\right\rangle$. Symmetry of the pullback connection is proved as follows:

$$\begin{aligned} \left(\varphi^*\widetilde{\nabla}\right)_X Y - \left(\varphi^*\widetilde{\nabla}\right)_Y X &\xlongequal{(4.19)} \left(\varphi^{-1}\right)_*\left(\widetilde{\nabla}_{\varphi_* X}\left(\varphi_* Y\right) - \widetilde{\nabla}_{\varphi_* Y}\left(\varphi_* X\right)\right) \\ &\xlongequal{\widetilde{\nabla}\text{ a sym. conn.}} \left(\varphi^{-1}\right)_*\left[\varphi_* X, \varphi_* Y\right] \\ &= [X, Y] \end{aligned}$$

■

Corollary 5.2.9 (Naturality of Geodesics). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a local isometry. If γ is a geodesic in M, then $\varphi\circ\gamma$ is a geodesic in $\widetilde{M}$.*

Proof. This is an immediate consequence of Proposition 4.7.2, together with the fact that being a geodesic is a local property. ■

Like every connection on the tangent bundle, the Levi-Civita connection induces connections on all tensor bundles.

Proposition 5.2.10. *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold. The connection induced on each tensor bundle by the Levi-Civita connection is compatible with the induced inner product on tensors, in the sense that $X\langle F, G\rangle = \langle \nabla_X F, G\rangle + \langle F, \nabla_X G\rangle$ for every vector field X and every pair of smooth tensor fields $F, G \in \Gamma\left(T^{(k,l)}TM\right)$.*

Proof. Since every tensor field can be written as a sum of tensor products of vector and/or covector fields, it suffices to consider the case in which $F = \alpha_1\otimes\cdots\otimes\alpha_{k+l}$ and $G = \beta_1\otimes\cdots\otimes\beta_{k+l}$, where α_i and β_i are covariant or contravariant 1-tensor fields, as appropriate. In this case, the formula follows from (2.7) by a routine computation. ■

Proposition 5.2.11. *Let (M, g) be an oriented Riemannian manifold. The Riemannian volume form of g is parallel with respect to the Levi-Civita connection.*

Proof. Let $p \in M$ and $v \in T_pM$ be arbitrary, and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Let $(E_1, \ldots, E_n)$ be a parallel oriented orthonormal frame along γ. Since $dV_g\left(E_1, \ldots, E_n\right) \equiv 1$ and $D_tE_i \equiv 0$ along γ, formula (4.4) shows that $\nabla_v\left(dV_g\right) = D_t\left(dV_g\right)\big|_{t=0} = 0$. ■

Proposition 5.2.12. *The musical isomorphisms commute with the total covariant derivative operator: if F is any smooth tensor field with a contravariant ith index position, and $\flat$ represents the operation of lowering the i th index, then*

$$\nabla\left(F^{\flat}\right)=(\nabla F)^{\flat}. \tag{5.13}$$

Similarly, if G has a covariant ith position and $\sharp$ denotes raising the ith index, then

$$\nabla\left(G^{\sharp}\right)=(\nabla G)^{\sharp}. \tag{5.14}$$

Proof. The discussion on subsection 2.3.1 shows that $F^{\flat}=\operatorname{tr}(F\otimes g)$, where the trace is taken on the i th and last indices of $F\otimes g$. Because g is parallel, for every vector field X we have $\nabla_X(F\otimes g)=(\nabla_X F)\otimes g$. Because ∇_X commutes with traces, therefore,

$$\nabla_X\left(F^{\flat}\right)=\nabla_X(\operatorname{tr}(F\otimes g))=\operatorname{tr}\left((\nabla_X F)\otimes g\right)=\left(\nabla_X F\right)^{\flat}.$$

This shows that when X is inserted into the last index position on both sides of (5.13), the results are equal. Since X is arbitrary, this proves (5.13). Because the sharp and flat operators are inverses of each other when applied to the same index position, (5.14) follows by substituting $F=G^{\sharp}$ into (5.13) and applying $\sharp$ to both sides. ■

5.3 Exponential Map

Note that the results in this section are generally true for all connection in TM, not just for the Levi-Civita connection. For simplicity, we restrict attention here to the latter case. We also restrict to manifolds without boundary, in order to avoid complications with geodesics running into a boundary.

The next lemma shows that geodesics with proportional initial velocities are related in a simple way.

Lemma 5.3.1 (Rescaling Lemma)**.** *For every $p\in M, v\in T_pM$, and $c,t\in\mathbb{R}$,*

$$\gamma_{cv}(t)=\gamma_v(ct),$$

whenever either side is defined.

Proof. See [5] Lemma 5.18. ■

The assignment $v\mapsto\gamma_v$ defines a map from TM to the set of geodesics in M. More importantly, by virtue of the rescaling lemma, it allows us to define a map from (a subset of) the tangent bundle to M itself, which sends each line $\{cv\}$ through the origin in T_pM to a geodesic. Define a subset $\mathcal{E}\subseteq TM$, the **domain of the exponential map**, by

$$\mathcal{E}=\left\{v\in TM:\gamma_v\text{ is defined on an interval containing }[0,1]\right\},$$

and then define the **exponential map** $\exp:\mathcal{E}\to M$ by

$$\exp(v)=\gamma_v(1)$$

For each $p\in M$, the **restricted exponential map at** p, denoted by $\exp_p$, is the restriction of exp to the set $\varepsilon_p=\mathcal{E}\cap T_pM$.

The exponential map of a Riemannian manifold should not be confused with the exponential map of a Lie group. To avoid confusion, we always designate the exponential map of a Lie group G by $\exp^G$, and reserve the undecorated notation $\exp$ for the Riemannian exponential map.

The next proposition describes some essential features of the exponential map. Recall that a subset of a vector space V is said to be star-shaped with respect to a point $x \in S$ if for every $y \in S$, the line segment from x to y is contained in S.

Proposition 5.3.2 (Properties of the Exponential Map). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let $\exp : \mathcal{E} \to M$ be its exponential map.*

(a) $\mathcal{E}$ is an open subset of TM containing the image of the zero section, and each set $\varepsilon_p \subseteq T_pM$ is star-shaped with respect to 0 .

(b) For each $v \in TM$, the geodesic γ_v is given by

$$\gamma_v(t) = \exp(tv)$$

for all t such that either side is defined.

(c) The exponential map is smooth.

(d) For each point $p \in M$, the differential $d\left(\exp_p\right)_0 : T_0\left(T_pM\right) \cong T_pM \to T_pM$ is the identity map of T_pM, under the usual identification of $T_0\left(T_pM\right)$ with T_pM.

Proof. Write $n = \dim M$. The rescaling lemma with $t = 1$ says precisely that $\exp(cv) = \gamma_{cv}(1) = \gamma_v(c)$ whenever either side is defined; this is (b). Moreover, if $v \in \mathcal{E}_p$, then by definition γ_v is defined at least on $[0, 1]$. Thus for $0 \leq t \leq 1$, the rescaling lemma says that

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

is defined. Thus, $\{tv : t \in [0, 1]\} \subseteq \mathcal{E}_p \implies$ the segment $[0, v]$ is in $\mathcal{E}_p$. This shows that ε_p is star-shaped with respect to 0.

Next we will show that $\mathcal{E}$ is open and exp is smooth. To do so, we revisit the proof of the theorem of existence and uniqueness theorem for geodesics 4.5.1 and reformulate it in a more invariant way. Let $\left(x^i\right)$ be any smooth local coordinates on an open set $U \subseteq M$, let $\pi : TM \to M$ be the projection, and let $\left(x^i, v^i\right)$ denote the associated natural coordinates for $\pi^{-1}(U) \subseteq TM$. In terms of these coordinates, formula (4.14) defines a smooth vector field G on $\pi^{-1}(U)$. The integral curves of G are the curves $\eta(t) = \left(x^1(t), \ldots, x^n(t), v^1(t), \ldots, v^n(t)\right)$ that satisfy the system of ODEs given by (4.13), which is equivalent to the geodesic equation under the substitution $v^k = \dot{x}^k$, as we observed in the proof of Theorem 4.5.1. Stated somewhat more invariantly, every integral curve of G on $\pi^{-1}(U)$ projects to a geodesic under $\pi : TM \to M$ (which in these coordinates is just $\pi(x, v) = x$); conversely, every geodesic $\gamma(t) = \left(x^1(t), \ldots, x^n(t)\right)$ in U lifts to an integral curve of G in $\pi^{-1}(U)$ by setting $v^i(t) = \dot{x}^i(t)$.

The importance of G stems from the fact that it actually defines a global vector field on the total space of TM, called the **geodesic vector field**. Then the unique C^∞ maximal flow θ obtained from fundamental theorem on flows 1.2.8 is called **geodesic flow**. We could verify that G defines a global vector field by computing the transformation law for the components of G under a change of coordinates and showing that they take the same form in every coordinate chart; but fortunately there is a way to avoid that messy computation. The key observation, to be proved below, is that G acts on a function $f \in C^\infty(TM)$ by

$$Gf(p, v) = G_{(p,v)}f = \left.\frac{d}{dt}\right|_{t=0} f\left(\gamma_v(t), \gamma_v'(t)\right). \tag{5.15}$$

(Here and whenever convenient, we use the notations (p, v) and v interchangeably for an element $v \in T_pM$, depending on whether we wish to emphasize the point at which v is tangent.) Since this formula is independent of coordinates, it shows that the various definitions of G given by (4.14) in different coordinate systems agree.

To prove that G satisfies (5.15), we write the components of the geodesic $\gamma_v(t)$ as $x^i(t)$ and those of its velocity as $v^i(t) = \dot{x}^i(t)$. Using the chain rule and the geodesic equation in the form (4.13), we can write the right-hand side of (5.15) as

$$\left.\left(\frac{\partial f}{\partial x^k}(\overbrace{x(t)}^{=\gamma_v(t)}, \overbrace{v(t)}^{=\gamma_v'(t)})\dot{x}^k(t) + \frac{\partial f}{\partial v^k}(x(t), v(t))\dot{v}^k(t)\right)\right|_{t=0}$$
$$\overset{\gamma_v = x(t)\text{ geodesic} \Leftrightarrow (4.13)}{=\!=\!=} \frac{\partial f}{\partial x^k}(p,v)v^k - \frac{\partial f}{\partial v^k}(p,v)v^i v^j \Gamma^k_{ij}(p)$$
$$\overset{(4.14)}{=\!=} Gf(p,v).$$

The fundamental theorem on flows shows that there exist an open set $\mathcal{D} \subseteq \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a smooth map $\theta : \mathcal{D} \to TM$, such that each curve $\theta^{(p,v)}(t) = \theta(t,(p,v))$ is the unique maximal integral curve of G starting at (p,v), defined on an open interval containing 0.

Now suppose $(p,v) \in \mathcal{E}$. This means that the geodesic γ_v is defined at least on the interval $[0,1]$, and therefore so is the integral curve of G starting at $(p,v) \in TM$. Since $(1,(p,v)) \in \mathcal{D}$, there is a neighborhood of $(1,(p,v))$ in $\mathbb{R} \times TM$ on which the flow of G is defined (Fig. 5.1). In particular, this means that there is a neighborhood of (p,v) on which the flow exists for $t \in [0,1]$, and therefore on which the exponential map is defined. This shows that $\mathcal{E}$ is open.

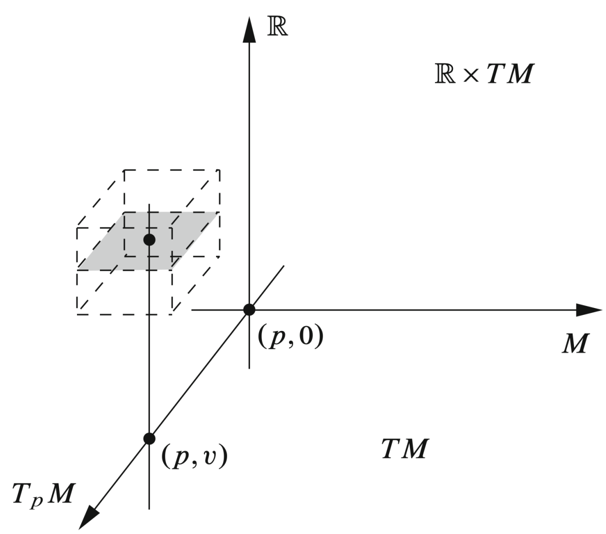


Figure 5.1: $\mathcal{E}$ is open.

Since geodesics are projections of integral curves of G, it follows that the exponential map can be expressed as

$$\exp_p(v) = \gamma_v(1) = \pi \circ \theta(1,(p,v))$$

wherever it is defined, and therefore $\exp_p(v)$ is a smooth function of (p,v). To compute $d\left(\exp_p\right)_0(v)$ for an arbitrary vector $v \in T_pM$, we just need to choose a curve τ in T_pM starting at 0 whose initial velocity is v, and compute the initial velocity of $\exp_p \circ \tau$. A convenient curve is $\tau(t) = tv$, which yields

$$d\left(\exp_p\right)_0(v) = \left.\frac{d}{dt}\right|_{t=0}\left(\exp_p \circ \tau\right)(t) = \left.\frac{d}{dt}\right|_{t=0}\exp_p(tv) = \left.\frac{d}{dt}\right|_{t=0}\gamma_v(t) = v.$$

Thus $d\left(\exp_p\right)_0$ is the identity map. ■

Corollary 5.2.9 on the naturality of geodesics translates into the following important property of the exponential map.

Proposition 5.3.3 (Naturality of the Exponential Map). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a local isometry. Then for every $p \in M$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}_p & \xrightarrow{d\varphi_p} & \widetilde{\mathcal{E}}_{\varphi(p)} \\ {\scriptstyle \exp_p}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\varphi(p)}} \\ M & \xrightarrow[\varphi]{} & \widetilde{M} \end{array}$$

where $\mathcal{E}_p \subseteq T_pM$ and $\widetilde{\mathcal{E}}_{\varphi(p)} \subseteq T_{\varphi(p)}\widetilde{M}$ are the domains of the restricted exponential maps $\exp_p$ (with respect to g) and $\exp_{\varphi(p)}$ (with respect to $\tilde{g}$), respectively.

Exercise 5.3.4. *Prove above proposition*

An important consequence of the naturality of the exponential map is the following proposition, which says that local isometries of connected manifolds are completely determined by their values and differentials at a single point.

Proposition 5.3.5. *Proposition 5.22. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian or pseudo-Riemannian manifolds, with M connected. Suppose $\varphi, \psi : M \to \widetilde{M}$ are local isometries such that for some point $p \in M$, we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

Proof. [5] Problem 5-10. ■

A Riemannian or pseudo-Riemannian manifold (M, g) is said to be **geodesically complete** if every maximal geodesic is defined for all $t \in \mathbb{R}$, or equivalently if the domain of the exponential map is all of TM. It is easy to construct examples of manifolds that are not geodesically complete; for example, n every proper open subset of $\mathbb{R}^n$ with its Euclidean metric or with a pseudo-Euclidean metric, there are geodesics that reach the boundary in finite time. Similarly, on $\mathbb{R}^n$ with the metric $\left(\sigma^{-1}\right)^* g$ obtained from the sphere by stereographic projection, there are geodesics that escape to infinity in finite time.

5.4 Normal Neighborhood and Normal Coordinates

We continue to let (M, g) be a Riemannian or pseudo-Riemannian manifold of dimension n (without boundary). Recall that for every $p \in M$, the restricted exponential map $\exp_p$ maps the open subset $\mathcal{E}_p \subseteq T_pM$ smoothly into M. Because $d\left(\exp_p\right)_0$ is invertible, the inverse function theorem guarantees that there exist a neighborhood V of the origin in T_pM and a neighborhood U of p in M such that $\exp_p : V \to U$ is a diffeomorphism. A neighborhood U of $p \in M$ that is the diffeomorphic image under $\exp_p$ of a star-shaped neighborhood of $0 \in T_pM$ is called a **normal neighborhood of** p.

Every orthonormal basis (b_i) for T_pM determines a basis isomorphism $B : \mathbb{R}^n \to T_pM$ by $B\left(x^1, \ldots, x^n\right) = x^i b_i$. If $U = \exp_p(V)$ is a normal neighborhood of p, we can combine this isomorphism with the exponential map to get a smooth coordinate $\operatorname{map} \varphi = B^{-1} \circ \left(\left.\exp_p\right|_V\right)^{-1} : U \to \mathbb{R}^n :$

$$\begin{array}{ccc} T_pM & \xrightarrow{B^{-1}} & \mathbb{R}^n \\ {\scriptstyle (\exp_p|_V)^{-1}}\uparrow & \nearrow{\scriptstyle \varphi} & \\ U & & \end{array}$$

Such coordinates are called **(Riemannian or pseudo-Riemannian) normal coordinates centered at** p.

Proposition 5.4.1 (Uniqueness of Normal Coordinates). *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, p a point of M, and U a normal neighborhood of p. For every normal coordinate chart on U centered at p, the coordinate basis is orthonormal at p; and for every orthonormal basis (b_i) for T_pM, there is a unique normal coordinate chart (x^i) on U such that $\partial_i|_p = b_i$ for $i = 1, \ldots, n$. In the Riemannian case, any two normal coordinate charts (x^i) and $(\widetilde{x}^j)$ are related by*

$$\tilde{x}^j = A_i^j x^i \tag{5.16}$$

for some (constant) matrix $\left(A_i^j\right) \in \mathrm{O}(n)$.

Proof. Let φ be a normal coordinate chart on U centered at p, with coordinate functions (x^i). By definition, this means that $\varphi = B^{-1} \circ \exp_p^{-1}$, where $B : \mathbb{R}^n \to T_pM$ is the basis isomorphism determined by some orthonormal basis (b_i) for T_pM. Note that $d\varphi_p^{-1} = d\left(\exp_p\right)_0 \circ dB_0 = B$ because $d\left(\exp_p\right)_0$ is the identity and B is linear. Thus $\partial_i|_p = d\varphi_p^{-1}\left(\partial_i|_0\right) = B\left(\partial_i|_0\right) = b_i$, which shows that the coordinate basis is orthonormal at p. Conversely, every orthonormal basis (b_i) for T_pM yields a basis isomorphism B and thus a normal coordinate chart $\varphi = B^{-1} \circ \exp_p^{-1}$, which satisfies $\partial_i|_p = b_i$ by the computation above. If $\widetilde{\varphi} = \widetilde{B}^{-1} \circ \exp_p^{-1}$ is another such chart, then

$$\widetilde{\varphi} \circ \varphi^{-1} = \widetilde{B}^{-1} \circ \exp_p^{-1} \circ \exp_p \circ B = \widetilde{B}^{-1} \circ B,$$

which is a linear isometry of $\mathbb{R}^n$ and therefore has the form (5.16) in terms of standard coordinates on $\mathbb{R}^n$. Since $(\widetilde{x}^j)$ and (x^i) are the same coordinates if and only if $\left(A_i^j\right)$ is the identity matrix, this shows that the normal coordinate chart associated with a given orthonormal basis is unique. ∎

Proposition 5.4.2 (Properties of Normal Coordinates). *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, and let $\left(U, (x^i)\right)$ be any normal coordinate chart centered at $p \in M$.*

(a) The coordinates of p are $(0, \ldots, 0)$.

(b) The components of the metric at p are $g_{ij} = \delta_{ij}$ if g is Riemannian, and $g_{ij} = \pm\delta_{ij}$ otherwise.

(c) For every $v = v^i \partial_i\big|_p \in T_pM$, the geodesic γ_v starting at p with initial velocity v is represented in normal coordinates by the line

$$\gamma_v(t) = \left(tv^1, \ldots, tv^n\right), \tag{5.17}$$

as long as t is in some interval I containing 0 such that $\gamma_v(I) \subseteq U$.

(d) The Christoffel symbols in these coordinates vanish at p.

(e) All of the first partial derivatives of g_{ij} in these coordinates vanish at p.

Proof. Part (a) follows directly from the definition of normal coordinates, and parts (b) and (c) follow from Propositions 5.4.1 and 5.3.2(b), respectively.

To prove (d), let $v = v^i \partial_i\big|_p \in T_pM$ be arbitrary. The geodesic equation (4.12) for $\gamma_v(t) = \left(tv^1, \ldots, tv^n\right)$ simplifies to

$$\Gamma_{ij}^k(tv) v^i v^j = 0.$$

Evaluating this expression at $t = 0$ shows that $\Gamma_{ij}^k(0) v^i v^j = 0$ for every index k and every vector v. In particular, with $v = \partial_a$ for some fixed a, this shows that $\Gamma_{aa}^k = 0$ for each a and k (no summation). Substituting

$v = \partial_a + \partial_b$ and $v = \partial_b - \partial_a$ for any fixed pair of indices a and b and subtracting, we conclude also that $\Gamma^k_{ab} = 0$ at p for all a, b, k. Finally, (e) follows from (d) together with (5.2) in the case $E_k = \partial_k$. ■

Because they are given by the simple formula (5.17), the geodesics starting at p and lying in a normal neighborhood of p are called **radial geodesics**. (But be warned that geodesics that do not pass through p do not in general have a simple form in normal coordinates.)

5.5 Problems

Exercise 5.5.1. *(LeeRM 5-1) Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let ∇ be its Levi-Civita connection. Suppose $\widetilde{\nabla}$ is another connection on TM, and D is the difference tensor between ∇ and $\widetilde{\nabla}$ (Prop.4.2.13). Let D^b denote the covariant 3-tensor field defined by $D^b(X, Y, Z) = \langle D(X, Y), Z\rangle$. Show that $\widetilde{\nabla}$ is compatible with g if and only if D^b is antisymmetric in its last two arguments: $D^\flat(X, Y, Z) = -D^\flat(X, Z, Y)$ for all $X, Y, Z \in \mathfrak{X}(M)$. Conclude that on every Riemannian or pseudo-Riemannian manifold of dimension at least 2, the space of metric connections is an infinite-dimensional affine space.*

Exercise 5.5.2. *(LeeRM 5-8) Let G be a Lie group and g its Lie algebra. Suppose g is a bi-invariant Riemannian metric on G, and $\langle\cdot,\cdot\rangle$ is the corresponding inner product on g (see [5] Prop. 3.12). Let ad: $g \to g\mathfrak{l}(\mathfrak{g})$ denote the adjoint representation of g (see [5] Appendix C).*

(a) Show that $\mathrm{ad}(X)$ is a skew-adjoint endomorphism of g for every $X \in \mathrm{g}$:

$$\langle \mathrm{ad}(X)Y, Z\rangle = -\langle Y, \mathrm{ad}(X)Z\rangle.$$

[Hint: Take the derivative of $\left\langle \mathrm{Ad}\left(\exp^G tX\right)Y, \mathrm{Ad}\left(\exp^G tX\right)Z\right\rangle$ with respect to t at $t = 0$, where $\exp^G$ is the Lie group exponential map of G, and use the fact that $\mathrm{Ad}_* =$ ad.]

(b) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$ whenever X and Y are left-invariant vector fields on G.

(c) Show that the geodesics of g starting at the identity are exactly the oneparameter subgroups. Conclude that under the canonical isomorphism of $\mathrm{g} \cong T_eG$ described in [5] Proposition C.3, the restricted Riemannian exponential map at the identity coincides with the Lie group exponential $\mathrm{map}\exp^G : \mathrm{g} \to G$. (See [5] Prop.C.7.)

(d) Let $\mathbb{R}^+$be the set of positive real numbers, regarded as a Lie group under multiplication. Show that $g = t^{-2}dt^2$ is a bi-invariant metric on $\mathbb{R}^+$, and the restricted Riemannian exponential map at 1 is given by $c\partial/\partial t \mapsto e^c$.

Chapter 6

Geodesics and Distance

6.1 Lengths

To say that $\gamma : I \to M$ is a **smooth curve** is to say that it is smooth as a map from the manifold (with boundary) I to M. If I has one or two endpoints and M has empty boundary, then γ is smooth if and only if it extends to a smooth curve defined on some open interval containing I. (If $\partial M \neq \varnothing$, then smoothness of γ has to be interpreted as meaning that each coordinate representation of γ has a smooth extension to an open interval.) A **curve segment** is a curve whose domain is a compact interval. A smooth curve $\gamma : I \to M$ has a well-defined velocity $\gamma'(t) \in T_{\gamma(t)}M$ for each $t \in I$. We say that γ is a **regular curve** if $\gamma'(t) \neq 0$ for $t \in I$. This implies that γ is an immersion, so its image has no "corners" or "kinks." If M is a smooth manifold with or without boundary, a (continuous) curve segment $\gamma : [a, b] \to M$ is said to be **piecewise regular** if there exists a partition $(a_0, \ldots, a_k)$ of $[a, b]$ such that $\gamma|_{[a_{i-1}, a_i]}$ is a regular curve segment (meaning it is smooth with nonvanishing velocity) for $i = 1, \ldots, k$. For brevity, we refer to a piecewise regular curve segment as an **admissible curve**, and any partition $(a_0, \ldots, a_k)$ such that $\gamma|_{[a_{i-1}, a_i]}$ is smooth for each i an **admissible partition for** γ. (There are many admissible partitions for a given admissible curve, because we can always add more points to the partition.) It is also convenient to consider any map $\gamma : \{a\} \to M$ whose domain is a single real number to be an admissible curve.

Suppose γ is an admissible curve and $(a_0, \ldots, a_k)$ is an admissible partition for it. At each of the intermediate partition points $a_1, \ldots, a_{k-1}$, there are two one-sided velocity vectors, which we denote by $\gamma'\left(a_i^-\right) = \lim_{t \nearrow a_i} \gamma'(t)$, $\gamma'\left(a_i^+\right) = \lim_{t \searrow a_i} \gamma'(t)$. They are both nonzero, but they need not be equal.

If $\gamma : I \to M$ is a smooth curve, we define a **reparametrization of** γ to be a curve of the form $\tilde{\gamma} = \gamma \circ \varphi : I' \to M$, where $I' \subseteq \mathbb{R}$ is another interval and $\varphi : I' \to I$ is a diffeomorphism. Because intervals are connected, φ is either strictly increasing or strictly decreasing on I'. We say that $\tilde{\gamma}$ is a **forward reparametrization** if φ is increasing, and a **backward reparametrization** if it is decreasing.

For an admissible curve $\gamma : [a, b] \to M$, we define a **reparametrization of** γ a little more broadly, as a curve of the form $\tilde{\gamma} = \gamma \circ \varphi$, where $\varphi : [c, d] \to [a, b]$ is a homeomorphism for which there is a partition $(c_0, \ldots, c_k)$ of $[c, d]$ such that the restriction of φ to each subinterval $[c_{i-1}, c_i]$ is a diffeomorphism onto its image.

If $\gamma : [a, b] \to M$ is an admissible curve, we define the length of γ to be

$$L_g(\gamma) = \int_a^b \left|\gamma'(t)\right|_g dt$$

The integrand is bounded and continuous everywhere on $[a, b]$ except possibly at the finitely many points where γ is not smooth, so this integral is well defined.

Proposition 6.1.1 (Properties of Lengths)**.** *Suppose (M, g) is a Riemannian manifold with or without boundary, and $\gamma : [a, b] \to M$ is an admissible curve.*

(a) ADDITIVITY OF LENGTH: If $a < c < b$, then $L_g(\gamma) = L_g\left(\gamma|_{[a,c]}\right) + L_g\left(\gamma|_{[c,b]}\right)$.

(b) PARAMETER INDEPENDENCE: If $\tilde{\gamma}$ is a reparametrization of γ, then $L_g(\gamma) = L_g(\widetilde{\gamma})$.

(c) ISOMETRY INVARIANCE: If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds (with or without boundary) and $\varphi : M \to \widetilde{M}$ is a local isometry, then $L_g(\gamma) = L_{\tilde{g}}(\varphi \circ \gamma)$.

Exercise 6.1.2. *Prove above proposition.*

Suppose $\gamma : [a, b] \to M$ is an admissible curve. The **arc-length function of** γ is the function $s : [a, b] \to \mathbb{R}$ defined by

$$s(t) = L_g\left(\gamma|_{[a,t]}\right) = \int_a^t |\gamma'(u)|_g \, du.$$

It is continuous everywhere, and it follows from the fundamental theorem of calculus that it is smooth wherever γ is, with derivative $s'(t) = |\gamma'(t)|$. For this reason, if $\gamma : I \to M$ is any smooth curve (not necessarily a curve segment), we define the **speed of** γ at any time $t \in I$ to be the scalar $|\gamma'(t)|$. We say that γ is a **unit-speed curve** if $|\gamma'(t)| = 1$ for all t, and a **constant-speed curve** if $|\gamma'(t)|$ is constant. If γ is a piecewise smooth curve, we say that γ has unit speed if $|\gamma'(t)| = 1$ wherever γ is smooth. If $\gamma : [a, b] \to M$ is a unit-speed admissible curve, then its arc-length function has the simple form $s(t) = t - a$. If, in addition, its parameter interval is of the form $[0, b]$ for some $b > 0$, then the arc-length function is $s(t) = t$. For this reason, a unit-speed admissible curve whose parameter interval is of the form $[0, b]$ is said to be **parametrized by arc length**.

Proposition 6.1.3. *Suppose (M, g) is a Riemannian manifold with or without boundary.*

(a) Every regular curve in M has a unit-speed forward reparametrization.

(b) Every admissible curve in M has a unique forward reparametrization by arc length.

Proof. See [5] Proposition 2.49. ■

6.2 Riemannian Distance

Suppose (M, g) is a connected Riemannian manifold with or without boundary. For each pair of points $p, q \in M$, we define the **Riemannian distance from** p **to** q, denoted by $d_g(p, q)$, to be the infimum of the lengths of all admissible curves from p to q. The following proposition guarantees that $d_g(p, q)$ is a well-defined nonnegative real number for each $p, q \in M$.

Proposition 6.2.1. *If M is a connected smooth manifold (with or without boundary), then any two points of M can be joined by an admissible curve.*

For convenience, if (M, g) is a disconnected Riemannian manifold, we also let $d_g(p, q)$ denote the Riemannian distance from p to q, provided that p and q lie in the same connected component of M. (See also Problem 2-30.)

Proposition 6.2.2 (Isometry Invariance of the Riemannian Distance Function)**.** *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is an isometry. Then $d_{\tilde{g}}(\varphi(x), \varphi(y)) = d_g(x, y)$ for all $x, y \in M$.*

Remark 6.2.3. Note that unlike lengths of curves, Riemannian distances are not necessarily preserved by local isometries. ♠

Theorem 6.2.4 (Riemannian Manifolds as Metric Spaces). *Let (M, g) be a connected Riemannian manifold with or without boundary. With the distance function d_g, M is a metric space whose metric topology is the same as the given manifold topology.*

Proof. See [5] Theorem 2.55. ■

Thanks to the preceding theorem, it makes sense to apply all the concepts of the theory of metric spaces to a connected Riemannian manifold (M, g). For example, we say that M is **(metrically) complete** if every Cauchy sequence in M converges. A subset $A \subseteq M$ is **bounded** if there is a constant C such that $d_g(p, q) \leq C$ for all $p, q \in A$; if this is the case, the **diameter of** A is the smallest such constant:

$$\operatorname{diam}(A) = \sup \{d_g(p, q) : p, q \in A\}.$$

Since every compact metric space is bounded, every compact connected Riemannian manifold with or without boundary has finite diameter. (Note that the unit sphere with the Riemannian distance determined by the round metric has diameter π, not 2, since the Riemannian distance between antipodal points is π.)

6.3 Geodesics and Minimizing Curves

Let (M, g) be a Riemannian manifold. An admissible curve γ in M is said to be a **minimizing curve** if $L_g(\gamma) \leq L_g(\tilde{\gamma})$ for every admissible curve $\tilde{\gamma}$ with the same endpoints. When M is connected, it follows from the definition of the Riemannian distance that γ is minimizing if and only if $L_g(\gamma)$ is equal to the distance between its endpoints.

Our first goal in this section is to show that all minimizing curves are geodesics. To do so, we will think of the length function L_g as a functional on the set of all admissible curves in M with fixed starting and ending points. (Real-valued functions whose domains are themselves sets of functions are typically called **functionals**.) Our project is to search for minima of this functional.

6.3.1 Families of Curves

Given intervals $I, J \subseteq \mathbb{R}$, a continuous map $\Gamma : J \times I \to M$ is called a **one-parameter family of curves**. Such a family defines two collections of curves in M: the **main curves** $\Gamma_s(t) = \Gamma(s, t)$ defined for $t \in I$ by holding s constant, and the **transverse curves** $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined for $s \in J$ by holding t constant.

If such a family Γ is smooth (or at least continuously differentiable), we denote the velocity vectors of the main and transverse curves by

$$\partial_t \Gamma(s, t) = (\Gamma_s)'(t) \in T_{\Gamma(s,t)}M; \quad \partial_s \Gamma(s, t) = \Gamma^{(t)\prime}(s) \in T_{\Gamma(s,t)}M.$$

Each of these is an example of a **vector field along** Γ, which is a continuous map $V : J \times I \to TM$ such that $V(s, t) \in T_{\Gamma(s,t)}M$ for each (s, t).

The families of curves that will interest us most in this chapter are of the following type. A one-parameter family Γ is called an **admissible family of curves** if (i) its domain is of the form $J \times [a, b]$ for some open interval J; (ii) there is a partition $(a_0, \ldots, a_k)$ of $[a, b]$ such that Γ is smooth on each rectangle of the form $J \times [a_{i-1}, a_i]$; and (iii) $\Gamma_s(t) = \Gamma(s, t)$ is an admissible curve for each $s \in J$ (Fig. 6.1). Every such partition is called an admissible partition for the family.

If $\gamma : [a, b] \to M$ is a given admissible curve, a **variation of** γ is an admissible family of curves $\Gamma : J \times [a, b] \to M$ such that J is an open interval containing 0 and $\Gamma_0 = \gamma$. It is called a **proper variation** if in addition, all of the main curves have the same starting and ending points: $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all $s \in J$.

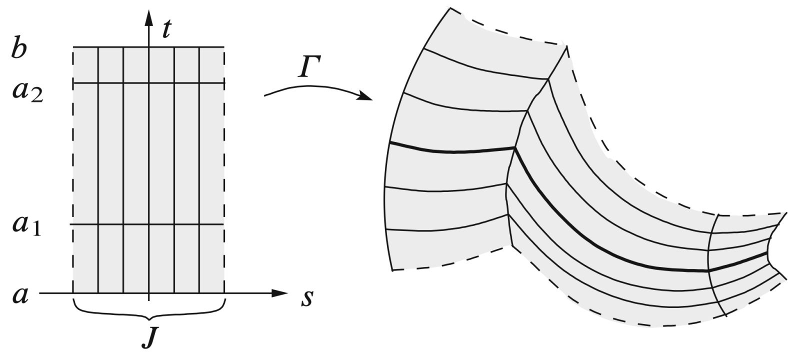


Figure 6.1: Admissble family of curves.

In the case of an admissible family, the transverse curves are smooth on J for each t, but the main curves are in general only piecewise regular. Thus the velocity vector fields $\partial_s\Gamma$ and $\partial_t\Gamma$ are smooth on each rectangle $J \times [a_{i-1}, a_i]$, but not generally on the whole domain.

We can say a bit more about $\partial_s\Gamma$, though. If Γ is an admissible family, a **piecewise smooth vector field along** Γ is a (continuous) vector field along Γ whose restriction to each rectangle $J \times [a_{i-1}, a_i]$ is smooth for some admissible partition $(a_0, \ldots, a_k)$ for Γ. In fact, $\partial_s\Gamma$ is always such a vector field. To see that it is continuous on the whole domain $J \times [a, b]$, note on the one hand that for each $i = 1, \ldots, k-1$, the values of $\partial_s\Gamma$ along the set $J \times \{a_i\}$ depend only on the values of Γ on that set, since the derivative is taken only with respect to the s variable; on the other hand, $\partial_s\Gamma$ is continuous (in fact smooth) on each subrectangle $J \times [a_{i-1}, a_i]$ and $J \times [a_i, a_{i+1}]$, so the right-hand and left-hand limits at $t = a_i$ must be equal. Therefore $\partial_s\Gamma$ is always a piecewise smooth vector field along Γ. (However, $\partial_t\Gamma$ is typically not continuous at $t = a_i$.)

If Γ is a variation of γ, the **variation field of** Γ is the piecewise smooth vector field $V(t) = \partial_s\Gamma(0, t)$ along γ. We say that a vector field V along γ is **proper** if $V(a) = 0$ and $V(b) = 0$; it follows easily from the definitions that the variation field of every proper variation is itself proper.

Lemma 6.3.1. *If γ is an admissible curve and V is a piecewise smooth vector field along γ, then V is the variation field of some variation of γ. If V is proper, the variation can be taken to be proper as well.*

Proof. Suppose γ and V satisfy the hypotheses, and set $\Gamma(s, t) = \exp_{\gamma(t)}(sV(t))$. By compactness of $[a, b]$, there is some positive ε such that Γ is defined on $(-\varepsilon, \varepsilon) \times [a, b]$. By composition, Γ is smooth on $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ for each subinterval $[a_{i-1}, a_i]$ on which V is smooth, and it is continuous on its whole domain. By the properties of the exponential map, the variation field of Γ is V. Moreover, if $V(a) = 0$ and $V(b) = 0$, the definition gives $\Gamma(s, a) \equiv \gamma(a)$ and $\Gamma(s, b) \equiv \gamma(b)$, so Γ is proper. ■

If V is a piecewise smooth vector field along Γ, we can compute the covariant derivative of V either along the main curves (at points where V is smooth) or along the transverse curves; the resulting vector fields along Γ are denoted by D_tV and D_sV respectively.

A key ingredient in the proof that minimizing curves are geodesics is the symmetry of the Levi-Civita connection. It enters into our proofs in the form of the following lemma. (Although we state and use this lemma only for the Levi-Civita connection, the proof shows that it is actually true for every symmetric connection in TM.)

Lemma 6.3.2 (Symmetry Lemma)**.** *Let $\Gamma : J \times [a, b] \to M$ be an admissible family of curves in a Riemannian manifold. On every rectangle $J \times [a_{i-1}, a_i]$ where Γ is smooth, $D_s(\partial_t\Gamma) = D_t(\partial_s\Gamma)$.*

Proof. This is a local question, so we may compute in local coordinates (x^i) around a point $\Gamma(s_0, t_0)$. Writing the components of Γ as $\Gamma(s,t) = \left(x^1(s,t), \ldots, x^n(s,t)\right)$, we have

$$\partial_t \Gamma = \frac{\partial x^k}{\partial t}\partial_k; \quad \partial_s \Gamma = \frac{\partial x^k}{\partial s}\partial_k.$$

Then, using the coordinate formula (4.11) for covariant derivatives along curves, we obtain

$$D_s \partial_t \Gamma = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma^k_{ji}\right)\partial_k; \; D_t \partial_s \Gamma = \left(\frac{\partial^2 x^k}{\partial t \partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma^k_{ji}\right)\partial_k.$$

Now, the lemma follows from the following

$$\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma^k_{ji} = \frac{\partial x^j}{\partial s}\frac{\partial x^i}{\partial t}\Gamma^k_{ji} \xlongequal{i \leftrightarrow j} \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma^k_{ij} \xlongequal{\text{Problem 4.8.1}} \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma^k_{ji}$$

■

6.3.2 Minimizing Curves are Geodesics

We can now compute an expression for the derivative of the length functional along a variation of a curve. Traditionally, the derivative of a functional on a space of maps is called its **first variation**.

Theorem 6.3.3 (First Variation Formula). *Let (M,g) be a Riemannian manifold. Suppose $\gamma : [a,b] \to M$ is a unit-speed admissible curve, $\Gamma : J \times [a,b] \to M$ is a variation of γ, and V is its variation field (Fig. 6.3). Then $L_g(\Gamma_s)$ is a smooth function of s, and*

$$\left.\frac{d}{ds}\right|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t\gamma'\rangle\, dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle + \langle V(b), \gamma'(b)\rangle - \langle V(a), \gamma'(a)\rangle, \tag{6.1}$$

where $(a_0, \ldots, a_k)$ is an admissible partition for V, and for each $i = 1, \ldots, k-1$, $\Delta_i\gamma' = \gamma'\left(a_i^+\right) - \gamma'\left(a_i^-\right)$ is the "jump" in the velocity vector field γ' at a_i (Fig.6.2). In particular, if Γ is a proper variation, then

$$\left.\frac{d}{ds}\right|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t\gamma'\rangle\, dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle. \tag{6.2}$$

Figure 6.2: $\Delta_i\gamma'$ is the "jump" in γ' at a_i

Proof. On every rectangle $J \times [a_{i-1}, a_i]$ where Γ is smooth, since the integrand in $L_g(\Gamma_s)$ is smooth and the domain of integration is compact, we can differentiate under the integral sign as many times as we wish.

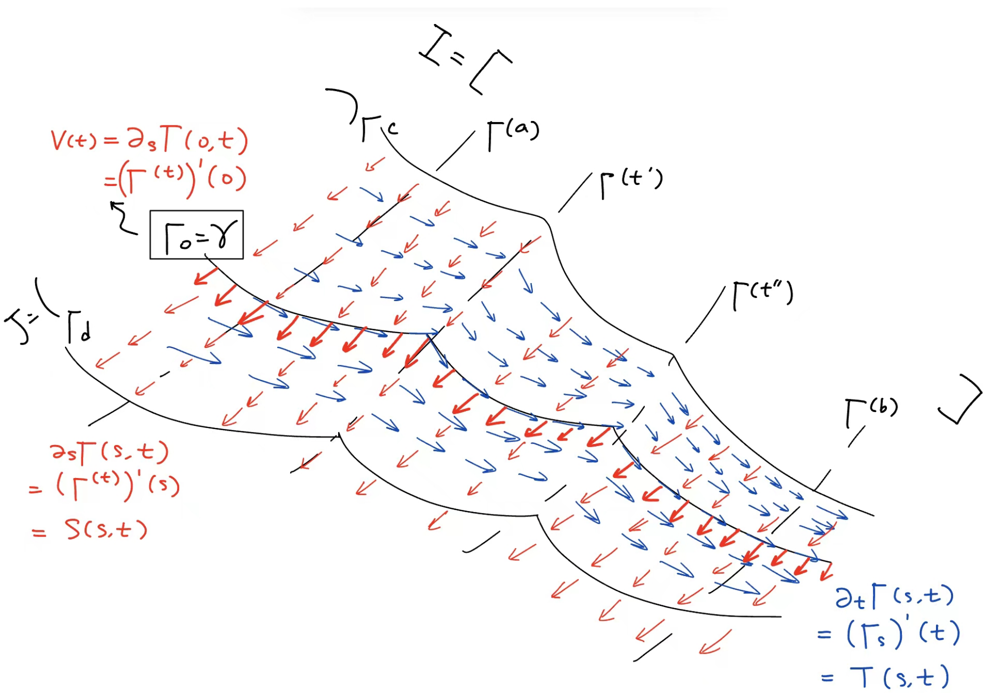


Figure 6.3: Vector fields V, S, and T

Because $L_g(\Gamma_s)$ is a finite sum of such integrals, it follows that it is a smooth function of s. For brevity, let us introduce the notations (see the blue and red v.f. in Fig. 6.3)

$$T(s,t) = \partial_t \Gamma(s,t), \quad S(s,t) = \partial_s \Gamma(s,t).$$

Differentiating on the interval $[a_{i-1}, a_i]$ yields

$$\begin{aligned} \frac{d}{ds} L_g\left(\Gamma_s|_{[a_{i-1},a_i]}\right) &= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s} \langle T, T\rangle^{1/2} dt \\ &\overset{(5.3)}{=\!=} \int_{a_{i-1}}^{a_i} \frac{1}{2} \langle T, T\rangle^{-1/2} 2 \langle D_s T, T\rangle \, dt \\ &\overset{\text{Lemma 6.3.2}}{=\!=} \int_{a_{i-1}}^{a_i} \frac{1}{|T|} \langle D_t S, T\rangle \, dt \end{aligned} \tag{6.3}$$

where we have used the symmetry lemma in the last line. Setting $s = 0$ and noting that $S(0,t) = V(t)$ and $T(0,t) = \gamma'(t)$ (which has length 1 given by assumption of the theorem), we get

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} L_g\left(\Gamma_s|_{[a_{i-1},a_i]}\right) &= \int_{a_{i-1}}^{a_i} \langle D_t V, \gamma'\rangle \, dt \\ &\overset{(5.3)}{=\!=} \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt}\langle V, \gamma'\rangle - \langle V, D_t\gamma'\rangle\right) dt \\ &\overset{\text{FTC}}{=\!=} \left\langle V(a_i), \gamma'(a_i^-)\right\rangle - \left\langle V(a_{i-1}), \gamma'(a_{i-1}^+)\right\rangle - \int_{a_{i-1}}^{a_i} \langle V, D_t\gamma'\rangle \, dt. \end{aligned}$$

Finally, summing over i, we obtain (6.1). ■

Because every admissible curve has a unit-speed parametrization and length is independent of parametrization, the requirement in the above proposition that γ be of unit speed is not a real restriction, but rather just a computational convenience.

Theorem 6.3.4. *In a Riemannian manifold, every minimizing curve is a geodesic when it is given a unit-speed parametrization.*

Proof. Suppose $\gamma : [a, b] \to M$ is minimizing and of unit speed (so that we can use previous theorem), and $(a_0, \ldots, a_k)$ is an admissible partition for γ. If Γ is any proper variation of γ, then $L_g(\Gamma_s)$ is a smooth function of s that achieves its minimum at $s = 0$ (we are given that γ is minimizing), so it follows from elementary calculus that $d(L_g(\Gamma_s))/ds = 0$ when $s = 0$. Since every proper vector field along γ is the variation field of some proper variation (Lemma 6.3.1), the right-hand side of (6.2) must vanish for every such V.

First we show that $D_t\gamma' = 0$ on each subinterval $[a_{i-1}, a_i]$, so γ is a "broken geodesic." Choose one such interval, and let $\varphi \in C^\infty(\mathbb{R})$ be a bump function such that $\varphi > 0$ on (a_{i-1}, a_i) and $\varphi = 0$ elsewhere. Then (6.2) with $V = \varphi D_t\gamma'$ (which is proper and we can thus apply the last sentence of the first paragraph) becomes

$$0 = -\int_{a_{i-1}}^{a_i} \varphi \left|D_t\gamma'\right|^2 dt \qquad (*)$$

Since the integrand is nonnegative and $\varphi > 0$ on (a_{i-1}, a_i), this shows that $D_t\gamma' = 0$ on each such subinterval.

Next we need to show that $\Delta_i\gamma' = 0$ for each i between 0 and k, which is to say that γ has no corners. For each such i, we can use a bump function in a coordinate chart to construct a piecewise smooth vector field V along γ such that $V(a_i) = \Delta_i\gamma'$ and $V(a_j) = 0$ for $j \neq i$. Then (6.2) reduces to $-\left|\Delta_i\gamma'\right|^2 = 0$, so $\Delta_i\gamma' = 0$ for each i.

Finally, since the two one-sided velocity vectors of γ match up at each a_i, it follows from uniqueness of geodesics that $\gamma|_{[a_i, a_{i+1}]}$ is the continuation of the geodesic $\gamma|_{[a_{i-1}, a_i]}$, and therefore γ is smooth. ■

The preceding proof has an enlightening geometric interpretation. Under the assumption that $D_t\gamma' \neq 0$, the first variation with $V = \varphi D_t\gamma'$ is negative (RHS of $(*)$), which shows that deforming γ in the direction of its acceleration vector field (since $\varphi > 0$) decreases its length (Fig. 6.4). Similarly, the length of a broken geodesic γ is decreased by deforming it in the direction of a vector field V such that $V(a_i) = \Delta_i\gamma'$ (Fig. 6.5). Geometrically, this corresponds to "rounding the corner."

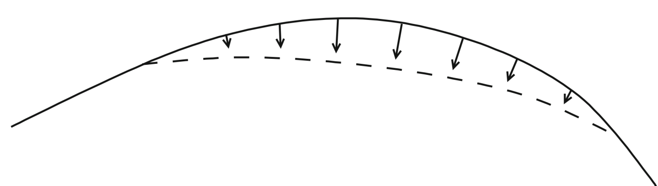


Figure 6.4: Deforming in the direction of the acceleration.

The first variation formula actually tells us a bit more than is claimed in Theorem 6.3.4. In proving that γ is a geodesic, we did not use the full strength of the assumption that the length of Γ_s takes a minimum when $s = 0$; we only used the fact that its derivative is zero. We say that an admissible curve γ is a **critical point of** L_g if for every proper variation Γ_s of γ, the derivative of $L_g(\Gamma_s)$ with respect to s is zero at $s = 0$. Therefore we can strengthen Theorem 6.3.4 in the following way.

Corollary 6.3.5. *A unit-speed admissible curve γ is a critical point for L_g if and only if it is a geodesic.*

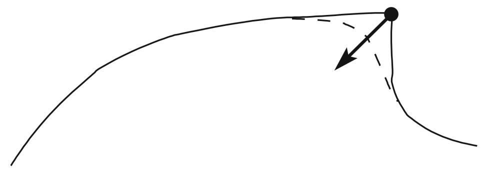


Figure 6.5: Rounding the corner

Proof. If γ is a critical point, the proof of Theorem 6.3.4 goes through without modification to show that γ is a geodesic. Conversely, if γ is a geodesic, then the first term on the right-hand side of (6.2) vanishes by the geodesic equation, and the second term vanishes because γ' has no jumps. ■

The geodesic equation $D_t\gamma' = 0$ thus characterizes the critical points of the length functional. In general, the equation that characterizes critical points of a functional on a space of maps is called the **variational equation** or the **Euler-Lagrange equation** of the functional. Many interesting equations in differential geometry arise as variational equations.

6.3.3 Geodesics Are Locally Minimizing

Next we turn to the converse of Theorem 6.3.4. It is easy to see that the literal converse is not true, because not every geodesic segment is minimizing. For example, every geodesic segment on $\mathbb{S}^n$ that goes more than halfway around the sphere is not minimizing, because the other portion of the same great circle is a shorter curve segment between the same two points. For that reason, we concentrate initially on local minimization properties of geodesics.

As usual, let (M, g) be a Riemannian manifold. A regular (or piecewise regular) curve $\gamma : I \to M$ is said to be **locally minimizing** if every $t_0 \in I$ has a neighborhood $I_0 \subseteq I$ such that whenever $a, b \in I_0$ with $a < b$, the restriction of γ to $[a, b]$ is minimizing.

Lemma 6.3.6. *Every minimizing admissible curve segment is locally minimizing.*

Exercise 6.3.7. *Prove the preceding lemma.*

Our goal in this section is to show that geodesics are locally minimizing. The proof will be based on a careful analysis of the geodesic equation in Riemannian normal coordinates.

If ε is a positive number such that $\exp_p$ is a diffeomorphism from the ball $B_\varepsilon(0) \subseteq T_pM$ to its image (where the radius of the ball is measured with respect to the norm defined by g_p), then the image set $\exp_p\left(B_\varepsilon(0)\right)$ is a normal neighborhood of p, called a **geodesic ball in** M, or sometimes an **open geodesic ball** for clarity.

Also, if the closed ball $\bar{B}_\varepsilon(0)$ is contained in an open set $V \subseteq T_pM$ on which $\exp_p$ is a diffeomorphism onto its image, then $\exp_p\left(\bar{B}_\varepsilon(0)\right)$ is called a **closed geodesic ball**, and $\exp_p\left(\partial B_\varepsilon(0)\right)$ is called a **geodesic sphere**. Given such a V, by compactness there exists $\varepsilon' > \varepsilon$ such that $B_{\varepsilon'}(0) \subseteq V$, so every closed geodesic ball is contained in an open geodesic ball of larger radius. In Riemannian normal coordinates centered at p, the open and closed geodesic balls and geodesic spheres centered at p are just the coordinate balls and spheres.

Suppose U is a normal neighborhood of $p \in M$. Given any normal coordinates $\left(x^i\right)$ on U centered at p, define the **radial distance function** $r : U \to \mathbb{R}$ by

$$r(x) = \sqrt{\left(x^1\right)^2 + \cdots + \left(x^n\right)^2}, \tag{6.4}$$

and the **radial vector field** on $U\backslash\{p\}$, denoted by ∂_r, by

$$\partial_r = \frac{x^i}{r(x)} \frac{\partial}{\partial x^i} \tag{6.5}$$

In Euclidean space, $r(x)$ is the distance to the origin, and ∂_r is the unit vector field pointing radially outward from the origin. (The notation is suggested by the fact that ∂_r is a coordinate derivative in polar or spherical coordinates.)

Lemma 6.3.8. *In every normal neighborhood U of $p \in M$, the radial distance function and the radial vector field are well defined, independently of the choice of normal coordinates. Both r and ∂_r are smooth on $U\backslash\{p\}$, and r^2 is smooth on all of U.*

Proof. Proposition 5.4.1 shows that any two normal coordinate charts on U are related by $\widetilde{x}^i = A^i_j x^j$ for some orthogonal matrix $\left(A^i_j\right)$, and a straightforward computation shows that both r and ∂_r are invariant under such coordinate changes. The smoothness statements follow directly from the coordinate formulas. ∎

The crux of the proof that geodesics are locally minimizing is the following deceptively simple geometric lemma.

Theorem 6.3.9 (The Gauss Lemma)**.** *Let (M, g) be a Riemannian manifold, let U be a geodesic ball centered at $p \in M$, and let ∂_r denote the radial vector field on $U\backslash\{p\}$. Then ∂_r is a unit vector field orthogonal to the geodesic spheres in $U\backslash\{p\}$.*

Proof. We will work entirely in normal coordinates $\left(x^i\right)$ on U centered at p, using the properties of normal coordinates described in Proposition 5.4.2.

Let $q \in U\backslash\{p\}$ be arbitrary, with coordinate representation $\left(q^1, \ldots, q^n\right)$, and let $b = r(q) = \sqrt{\left(q^1\right)^2 + \cdots + \left(q^n\right)^2}$, where r is the radial distance function defined by (6.5). It follows that $\left.\partial_r\right|_q$ has the coordinate representation

$$\left.\partial_r\right|_q = \frac{q^i}{b} \left.\frac{\partial}{\partial x^i}\right|_q .$$

Let $v = \left.v^i \partial_i\right|_p \in T_p M$ be the tangent vector at p with components $v^i = q^i / b$. By Proposition 5.4.2(c), the radial geodesic with initial velocity v is given in these coordinates by

$$\gamma_v(t) = \left(t v^1, \ldots, t v^n\right) .$$

It satisfies $\gamma_v(0) = p, \gamma_v(b) = q$, and $\gamma_v'(b) = \left.v^i \partial_i\right|_q = \left.\partial_r\right|_q$. Because g_p is equal to the Euclidean metric in these coordinates, we have

$$\left|\gamma_v'(0)\right|_g = |v|_g = \sqrt{\left(v^1\right)^2 + \cdots + \left(v^n\right)^2} = \frac{1}{b} \sqrt{\left(q^1\right)^2 + \cdots + \left(q^n\right)^2} = 1,$$

so v is a unit vector, and thus γ_v is a unit-speed geodesic. It follows that $\left.\partial_r\right|_q = \gamma_v'(b)$ is also a unit vector.

To prove that ∂_r is orthogonal to the geodesic spheres let q, b, and v be as above, and let $\Sigma_b = \exp_p\left(\partial B_b(0)\right)$ be the geodesic sphere containing q. In these coordinates, Σ_b is the set of points satisfying the equation $\left(x^1\right)^2 + \cdots + \left(x^n\right)^2 = b^2$. Let $w \in T_q M$ be any vector tangent to Σ_b at q. We need to show that $\left\langle w, \left.\partial_r\right|_q \right\rangle_g = 0$.

Choose a smooth curve $\sigma : (-\varepsilon, \varepsilon) \to \Sigma_b$ satisfying $\sigma(0) = q$ and $\sigma'(0) = w$, and write its coordinate representation in $\left(x^i\right)$-coordinates as $\sigma(s) = \left(\sigma^1(s), \ldots, \sigma^n(s)\right)$. The fact that $\sigma(s)$ lies in Σ_b for all s means that

$$\left(\sigma^1(s)\right)^2 + \cdots + \left(\sigma^n(s)\right)^2 = b^2 .$$

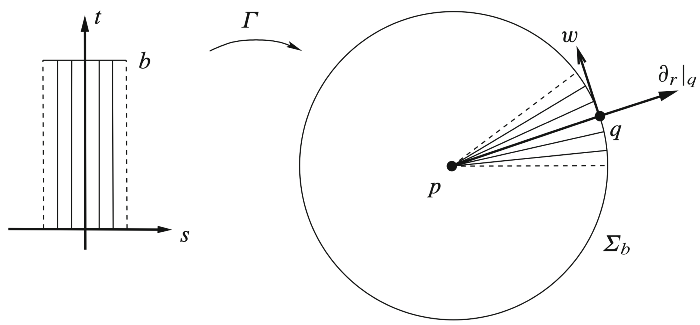


Figure 6.6: Proof of Gauss lemma.

Define a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [0, b] \to U$ (Fig.6.6) by

$$\Gamma(s,t) = \left(\frac{t}{b}\sigma^1(s), \ldots, \frac{t}{b}\sigma^n(s)\right).$$

For each $s \in (-\varepsilon, \varepsilon), \Gamma_s$ is a geodesic by Proposition 5.4.2(c). Its initial velocity is $\Gamma_s'(0) = (1/b)\sigma^i(s)\partial_i \mid p$, which is a unit vector by (6.6) and the fact that g_p is the Euclidean metric in coordinates; thus each Γ_s is a unit-speed geodesic. As before, let $S = \partial_s\Gamma$ and $T = \partial_t\Gamma$. It follows from the definitions that

$$\begin{aligned}
S(0,0) &= \left.\frac{d}{ds}\right|_{s=0} \Gamma_s(0) = 0; \\
T(0,0) &= \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v; \\
S(0,b) &= \left.\frac{d}{ds}\right|_{s=0} \sigma(s) = w; \\
T(0,b) &= \left.\frac{d}{dt}\right|_{t=b} \gamma_v(t) = \gamma_v'(b) = \left.\partial_r\right|_q.
\end{aligned}$$

Therefore $\langle S, T\rangle$ is zero when $(s,t) = (0,0)$ and equal to $\left\langle w, \left.\partial_r\right|_q\right\rangle$ when $(s,t) = (0,b)$, so to prove the theorem it suffices to show that $\langle S, T\rangle$ is independent of t. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\langle S,T\rangle &= \langle D_t S, T\rangle + \langle S, D_t T\rangle && \text{(compatibility with the metric)} \\
&= \langle D_s T, T\rangle + \langle S, D_t T\rangle && \text{(symmetry lemma)} \\
&= \langle D_s T, T\rangle + 0 && \text{(each } \Gamma_s \text{ is a geodesic)} \\
&= \frac{1}{2}\frac{\partial}{\partial s}|T|^2 = 0 && (|T| = |\Gamma_s'| \equiv 1 \text{ for all } (s,t)).
\end{aligned} \tag{6.6}$$

This proves the theorem. ■

We will use the Gauss lemma primarily in the form of the next corollary.

Corollary 6.3.10. *Let U be a geodesic ball centered at $p \in M$, and let r and ∂_r be the radial distance and radial vector field as defined by (6.4) and (6.5). Then $\operatorname{grad} r = \partial_r$ on $U \backslash \{p\}$.*

Proof. By the result of Problem 2.5.8, it suffices to show that ∂_r is orthogonal to the level sets of r and $\partial_r(r) \equiv |\partial_r|_g^2$. The first claim follows directly from the Gauss lemma, and the second from the fact that $\partial_r(r) \equiv 1$ by direct computation in normal coordinates, which in turn is equal to $|\partial_r|_g^2$ by the Gauss lemma. ■

Here is the payoff: our first step toward proving that geodesics are locally minimizing. Note that this is not yet the full strength of the theorem we are aiming for, because it shows only that for each point on a geodesic, sufficiently small segments of the geodesic starting at that point are minimizing. We will remove this restriction after a little more work below.

Proposition 6.3.11. *Let (M, g) be a Riemannian manifold. Suppose $p \in M$ and q is contained in a geodesic ball around p. Then (up to reparametrization) the radial geodesic from p to q is the unique minimizing curve in M from p to q.*

Proof. Choose $\varepsilon > 0$ such that $\exp_p(B_\varepsilon(0))$ is a geodesic ball containing q. Let $\gamma : [0, c] \to M$ be the radial geodesic from p to q parametrized by arc length, and write $\gamma(t) = \exp_p(tv)$ for some unit vector $v \in T_pM$. Then $L_g(\gamma) = c$, since γ has unit speed.

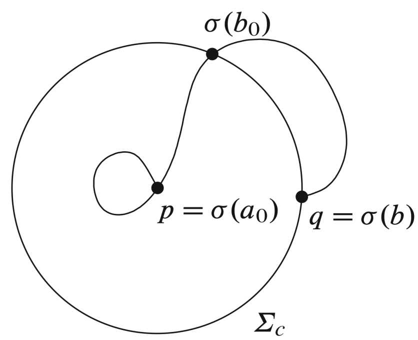


Figure 6.7: Radial geodesics are minimizing.

To show that γ is minimizing, we need to show that every other admissible curve from p to q has length at least c. Let $\sigma : [0, b] \to M$ be an arbitrary admissible curve from p to q, which we may assume to be parametrized by arc length as well. Let $a_0 \in [0, b]$ denote the last time that $\sigma(t) = p$, and $b_0 \in [0, b]$ the first time after a_0 that $\sigma(t)$ meets the geodesic sphere Σ_c of radius c around p (Fig.6.7). Then the composite function $r \circ \sigma$ is continuous on $[a_0, b_0]$ and piecewise smooth in (a_0, b_0), so we can apply the fundamental theorem of calculus to conclude that

$$\begin{aligned} r(\sigma(b_0)) - r(\sigma(a_0)) &= \int_{a_0}^{b_0} \frac{d}{dt} r(\sigma(t)) dt = \int_{a_0}^{b_0} dr(\sigma'(t))\, dt \\ &= \int_{a_0}^{b_0} \left\langle \operatorname{grad} r|_{\sigma(t)}, \sigma'(t) \right\rangle dt \le \int_{a_0}^{b_0} |\operatorname{grad} r|_{\sigma(t)}||\sigma'(t)|\, dt \\ &= \int_{a_0}^{b_0} |\sigma'(t)|\, dt = L_g\left(\sigma|_{[a_0, b_0]}\right) \le L_g(\sigma) \end{aligned} \tag{6.7}$$

Thus $L_g(\sigma) \ge r(\sigma(b_0)) - r(\sigma(a_0)) = c$, so γ is minimizing. Now suppose $L_g(\sigma) = c$. Then $b = c$, and both inequalities in (6.7) are equalities. Because we assume that σ is a unit-speed curve, the second of these equalities implies that $a_0 = 0$ and $b_0 = b = c$, since otherwise the segments of σ before $t = a_0$ and after $t = b_0$ would contribute positive lengths. The first equality then implies that the nonnegative expression $|\operatorname{grad} r|_{\sigma(t)}||\sigma'(t)| - \left\langle \operatorname{grad} r|_{\sigma(t)}, \sigma'(t) \right\rangle$ is identically zero on $[0, b]$, which is possible only if

$\sigma'(t)$ is a positive multiple of grad $r|_{\sigma(t)}$ for each t. Since we assume that σ has unit speed, we must have $\sigma'(t) = \operatorname{grad} r|_{\sigma(t)} = \partial_r|_{\sigma(t)}$. Thus σ and γ are both integral curves of ∂_r passing through q at time $t = c$, so $\sigma = \gamma$. ∎

The next two corollaries show how radial distance functions, balls, and spheres in normal coordinates are related to their global metric counterparts.

Corollary 6.3.12. *Let (M, g) be a connected Riemannian manifold and $p \in M$. Within every open or closed geodesic ball around p, the radial distance function $r(x)$ defined by (6.4) is equal to the Riemannian distance from p to x in M.*

Proof. Since every closed geodesic ball is contained in an open geodesic ball of larger radius, we need only consider the open case. If x is in the open geodesic ball $\exp_p(B_c(0))$, the radial geodesic γ from p to x is minimizing by Proposition 6.3.11. Since its velocity is equal to ∂_r, which is a unit vector in both the g-norm and the Euclidean norm in normal coordinates, the g-length of γ is equal to its Euclidean length, which is $r(x)$. ∎

Corollary 6.3.13. *In a connected Riemannian manifold, every open or closed geodesic ball is also an open or closed metric ball of the same radius, and every geodesic sphere is a metric sphere of the same radius.*

Proof. Let (M, g) be a Riemannian manifold, and let $p \in M$ be arbitrary. First, let $V = \exp_p\left(\bar{B}_c(0)\right) \subseteq M$ be a closed geodesic ball of radius $c > 0$ around p. Suppose q is an arbitrary point of M. If $q \in V$, then Corollary 6.3.12 shows that q is also in the closed metric ball of radius c. Conversely, suppose $q \notin V$. Let S be the geodesic sphere $\exp_p(\partial B_c(0))$. The complement of S is the disjoint union of the open sets $\exp_p(B_c(0))$ and $M \setminus \exp_p\left(\bar{B}_c(0)\right)$, and hence disconnected. Thus if $\gamma : [a, b] \to M$ is any admissible curve from p to q, there must be a time $t_0 \in (a, b)$ when $\gamma(t_0) \in S$, and then Corollary 6.3.12 shows that the length of $\gamma|_{[a,t_0]}$ must be at least c. Since $\gamma|_{[t_0,b]}$ must have positive length, it follows that $d_g(p, q) > c$, so q is not in the closed metric ball of radius c around p.

Next, let $W = \exp_p(B_c(0))$ be an open geodesic ball of radius c. Since W is the union of all closed geodesic balls around p of radius less than c, and the open metric ball of radius c is similarly the union of all closed metric metric balls of smaller radii, the result of the preceding paragraph shows that W is equal to the open metric ball of radius c.

Finally, if $S = \exp_p(\partial B_c(0))$ is a geodesic sphere of radius c, the arguments above show that S is equal to the closed metric ball of radius c minus the open metric ball of radius c, which is exactly the metric sphere of radius c. ∎

The last corollary suggests a simplified notation for geodesic balls and spheres in M. From now on, we will use the notations $B_c(p) = \exp_p(B_c(0)), \bar{B}_c(p) = \exp_p\left(\bar{B}_c(0)\right)$, and $S_c(p) = \exp_p(\partial B_c(0))$ for open and closed geodesic balls and geodesic spheres, which we now know are also open and closed metric balls and spheres. (To avoid confusion, we refrain from using this notation for other metric balls and spheres unless explicitly stated.)

6.4 Uniformly Normal Neighborhoods

6.5 Completeness

6.5.1 Closed Geodesics

6.6 Problems

Chapter 7

Curvature

Recall that a Riemannian manifold is said to be **flat** if it is locally isometric to a Euclidean space, that is, if every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric. Similarly, a pseudo-Riemannian manifold is flat if it is locally isometric to a pseudo-Euclidean space. For Euclidean connection on $\mathbb{R}^n$, we see that $\overline{\nabla}_X \overline{\nabla}_Y Z = XY\left(Z^k\right)\partial_k$, $\overline{\nabla}_Y \overline{\nabla}_X Z = YX\left(Z^k\right)\partial_k$, and $\left(XY\left(Z^k\right) - YX\left(Z^k\right)\right)\partial_k = \overline{\nabla}_{[X,Y]}Z$ due to Example 4.8. Thus, the following relation holds for all vector fields X, Y, Z defined on an open subset of $\mathbb{R}^n$:

$$\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z = \overline{\nabla}_{[X,Y]}Z.$$

We say that a connection ∇ on a smooth manifold M satisfies the **flatness criterion** if whenever X, Y, Z are smooth vector fields defined on an open subset of M, the following identity holds:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]}Z. \tag{7.1}$$

Example 7.0.1. The metric on the n-torus induced by the embedding in $\mathbb{R}^{2n}$ given in Example 2.2.10 is flat, because each point has a coordinate neighborhood in which the metric is Euclidean. ♣

Proposition 7.0.2. *If (M, g) is a flat Riemannian or pseudo-Riemannian manifold, then its Levi-Civita connection satisfies the flatness criterion.*

Proof. We just showed that the Euclidean connection on $\mathbb{R}^n$ satisfies (7.1). By naturality (see Proposition 5.2.8), the Levi-Civita connection on every manifold that is locally isometric to a Euclidean or pseudo-Euclidean space must also satisfy the same identity. ■

7.1 Curvature Tensor

Motivated by the computation in the preceding section, we make the following definition. Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and define a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \tag{7.2}$$

Proposition 7.1.1. *The map R defined above is multilinear over $C^\infty(M)$, and thus defines a (1,3)-tensor field on M.*

Proof. The map R is obviously multilinear over $\mathbb{R}$. For $f \in C^\infty(M)$,

$$\begin{aligned} R(X, fY)Z &= \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\ &\overset{\text{prop.1.2.4}}{=\!=\!=} \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{f[X,Y]+(Xf)Y} Z \\ &\overset{\text{defn.4.2.1}}{=\!=\!=} (Xf)\nabla_Y Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z \\ &= fR(X,Y)Z. \end{aligned}$$

The same proof shows that R is linear over $C^\infty(M)$ in X, because $R(X,Y)Z = -R(Y,X)Z$ from the definition. The remaining case to be checked is linearity over $C^\infty(M)$ in Z: using definition of connection and Lie Bracket, we see

$$\begin{aligned} R(X,Y)fZ &= \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ \\ &= \nabla_X (f\nabla_Y Z + YfZ) - \nabla_Y (f\nabla_X Z + XfZ) - f\nabla_{[X,Y]} Z - [X,Y]fZ \\ &= f\nabla_X \nabla_Y Z + Xf\nabla_Y Z + Yf\nabla_X Z + X(Yf)Z \\ &\quad - f\nabla_Y \nabla_X Z - Yf\nabla_X Z - Xf\nabla_Y Z - Y(Xf)Z \\ &\quad - f\nabla_{[X,Y]} Z - [X,Y]fZ \\ &= fR(X,Y)Z \end{aligned}$$

By the tensor characterization lemma 1.1.18, the fact that R is multilinear over $C^\infty(M)$ implies that it is a $(1,3)$-tensor field (R takes in three vectors and output one vector, so $R \in \mathcal{L}(V,V,V;V) \cong V \otimes V^* \otimes V^* \otimes V^* = T^{(1,3)}(V)$.) ■

Thanks to this proposition, for each pair of vector fields $X, Y \in \mathfrak{X}(M)$, the map $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X,Y)Z$ is a smooth bundle endomorphism of TM (see [4] 10.29), called the **curvature endomorphism determined by X and Y**. The tensor field R itself is called the **(Riemann) curvature endomorphism** or the $(1,3)$**-curvature tensor** or the **Riemann curvature tensor of the second kind** (Some authors call it simply the curvature tensor, but we reserve that name instead for another closely related tensor field, defined below.)

As a $(1,3)$-tensor field, the curvature endomorphism can be written in terms of any local frame with one upper and three lower indices. We adopt the convention that the last index is the contravariant (upper) one. (This is contrary to our default assumption that covector arguments come first.) Thus, for example, the curvature endomorphism can be written in terms of local coordinates (x^i) as

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

where the coefficients $R_{ijk}{}^l$ are defined by

$$R(\partial_i, \partial_j)\, \partial_k = R_{ijk}{}^l \partial_l. \tag{7.3}$$

We explain this a bit: looking back at proposition 1.1.5, we write above equation really to mean that for the $R \in T^{(1,3)}TM$,

$$\Psi(R)\,(\partial_i, \partial_j, \partial_k) = R_{ijk}{}^l \partial_l,$$

or

$$\tau(R(\,\cdot\,, \partial_i, \partial_j, \partial_k)) = R_{ijk}{}^l \partial_l.$$

Since τ is an isomorphism whose inverse sends a vector to its evaluation map $\bar{v}$, showing this equality is exactly showing that

$$\overline{R_{ijk}{}^l \partial_l} = R(\,\cdot\,, \partial_i, \partial_j, \partial_k),$$

but $\overline{R_{ijk}{}^l \partial_l}(dx^m) = \sum_l R_{ijk}{}^l dx^m(\partial_l) = R_{ijk}{}^m$ and $R(\,\cdot\,, \partial_i, \partial_j, \partial_k)(dx^m) = R(dx^m, \partial_i, \partial_j, \partial_k) = R_{ijk}{}^m$.

The next proposition shows how to compute the components of R in coordinates.

Proposition 7.1.2. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. In terms of any smooth local coordinates, the components of the $(1,3)$-curvature tensor are given by*

$$R_{ijk}{}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l}. \tag{7.4}$$

Proof.

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \nabla_{[\partial_i, \partial_j]} \partial_k \\
&\overset{[4](8.10)}{=\!=\!=} \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k \\
&\overset{(4.2)}{=\!=} \nabla_{\partial_i} (\Gamma_{jk}^{m} \partial_m) - \nabla_{\partial_j} (\Gamma_{ik}^{m} \partial_m) \\
&\overset{\Gamma\text{'s are functions}}{=\!=\!=\!=\!=} \Gamma_{jk}^{m} \nabla_{\partial_i} \partial_m + \partial_i \Gamma_{jk}^{m} \partial_m - \Gamma_{ik}^{m} \nabla_{\partial_j} \partial_m - \partial_j \Gamma_{ik}^{m} \partial_m \\
&= \Gamma_{jk}^{m} \Gamma_{im}^{l} \partial_l + \partial_i \Gamma_{jk}^{l} \partial_l - \Gamma_{ik}^{m} \Gamma_{jm}^{l} \partial_l - \partial_j \Gamma_{ik}^{l} \partial_l \\
&= [\partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l} \partial_l
\end{aligned}$$

The characterization (7.3) then concludes. ■

Importantly for our purposes, the curvature endomorphism also measures the failure of second covariant derivatives along families of curves to commute. Given a smooth one-parameter family of curves $\Gamma : J \times I \to M$, recall from previous chapter that the velocity fields $T(s,t) = \partial_t \Gamma(s,t) = (\Gamma_s)'(t)$ and $S(s,t) = \partial_s \Gamma(s,t) = \Gamma^{(t)\prime}(s)$ are smooth vector fields along Γ.

Proposition 7.1.3. *Suppose (M, g) is a smooth Riemannian or pseudo-Riemannian manifold and $\Gamma : J \times I \to M$ is a smooth one-parameter family of curves in M. Then for every smooth vector field V along Γ,*

$$D_s D_t V - D_t D_s V = R\left(\partial_s \Gamma, \partial_t \Gamma\right) V. \tag{7.5}$$

Proof. This is a local question, so for each $(s,t) \in J \times I$, we can choose smooth coordinates $\left(x^i\right)$ defined on a neighborhood of $\Gamma(s,t)$ and write

$$\Gamma(s,t) = \left(\gamma^1(s,t), \ldots, \gamma^n(s,t)\right), \quad V(s,t) = V^j(s,t) \partial_j\big|_{\Gamma(s,t)}.$$

The product rule for covariant derivatives along curves yields

$$D_t V = \frac{\partial V^i}{\partial t} \partial_i + V^i D_t \partial_i.$$

Therefore, applying product rule again, we get

$$D_s D_t V = \frac{\partial^2 V^i}{\partial s \partial t} \partial_i + \frac{\partial V^i}{\partial t} D_s \partial_i + \frac{\partial V^i}{\partial s} D_t \partial_i + V^i D_s D_t \partial_i.$$

Interchanging s and t and subtracting, we see that all the terms except the last cancel:

$$D_s D_t V - D_t D_s V = V^i \left(D_s D_t \partial_i - D_t D_s \partial_i\right). \tag{7.6}$$

Now we need to compute the commutator in parentheses. For brevity, let us write

$$S = \partial_s \Gamma = \frac{\partial \gamma^k}{\partial s} \partial_k; \quad T = \partial_t \Gamma = \frac{\partial \gamma^j}{\partial t} \partial_j.$$

Because ∂_i is extendible,

$$D_t \partial_i = \nabla_T \partial_i = \frac{\partial \gamma^j}{\partial t} \nabla_{\partial_j} \partial_i,$$

and therefore, because $\nabla_{\partial_j}\partial_i$ is also extendible,

$$\begin{aligned} D_s D_t \partial_i &= D_s\left(\frac{\partial \gamma^j}{\partial t}\nabla_{\partial_j}\partial_i\right) \\ &= \frac{\partial^2\gamma^j}{\partial s\partial t}\nabla_{\partial_j}\partial_i + \frac{\partial \gamma^j}{\partial t}\nabla_S\left(\nabla_{\partial_j}\partial_i\right) \\ &= \frac{\partial^2\gamma^j}{\partial s\partial t}\nabla_{\partial_j}\partial_i + \frac{\partial \gamma^j}{\partial t}\frac{\partial \gamma^k}{\partial s}\nabla_{\partial_k}\nabla_{\partial_j}\partial_i. \end{aligned}$$

Interchanging $s \leftrightarrow t$ and $j \leftrightarrow k$ and subtracting, we find that the first terms cancel, and we get

$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \frac{\partial \gamma^j}{\partial t}\frac{\partial \gamma^k}{\partial s}\left(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i\right) \\ &= \frac{\partial \gamma^j}{\partial t}\frac{\partial \gamma^k}{\partial s} R\left(\partial_k, \partial_j\right)\partial_i = R(S,T)\partial_i \end{aligned}$$

Finally, inserting this into (7.6) yields the result. ■

For many purposes, the information contained in the curvature endomorphism is much more conveniently encoded in the form of a covariant 4-tensor. We define the **(Riemann) curvature tensor (of the first kind)** to be the $(0,4)$-tensor field $Rm = R^\flat$ (also denoted by $Riem$ by some authors) obtained from the $(1,3)$-curvature tensor R by lowering its last index. Its action on vector fields is given by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle_g \tag{7.7}$$

(Thr LHS is $R^\flat(X,Y,Z,W) = R(X,Y,Z,W^\flat)$; what this really means is that for $R \in L(V,V,V;V)$ given by (7.2), $\Phi(R)(X,Y,Z,W^\flat)$ =RHS. This is true as $\Phi(R)(X,Y,Z,W^\flat) = W^\flat(R(X,Y,Z)) = \widehat{g}(W)(R(X,Y)Z) = g(W, R(X,Y)Z) = g(R(X,Y)Z,W)$.) In terms of any smooth local coordinates it is written

$$Rm = R_{ijkl}dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $R_{ijkl} = g_{lm}R_{ijk}{}^m$ (see Example 2.3.4). Thus (7.4) yields

$$R_{ijkl} = g_{lm}\left(\partial_i\Gamma_{jk}^m - \partial_j\Gamma_{ik}^m + \Gamma_{jk}^p\Gamma_{ip}^m - \Gamma_{ik}^p\Gamma_{jp}^m\right). \tag{7.8}$$

It is appropriate to note here that there is much variation in the literature with respect to index positions in the definitions of the curvature endomorphism and curvature tensor. While almost all authors define the $(1,3)$-curvature tensor as we have, there are a few whose definition is the negative of ours. There is much less agreement on the definition of the $(0,4)$-curvature tensor: whichever definition is chosen for the curvature endomorphism, you will see the curvature tensor defined as in (7.7) but with various permutations of (X,Y,Z,W) on the right-hand side. After applying the symmetries of the curvature tensor that we will prove later in this chapter, however, all of the definitions agree up to sign. There are various arguments to support one choice or another; we have made a choice that makes equation (7.7) easy to remember. You just have to be careful when you begin reading any book or article to determine the author's sign convention.

The next proposition gives one reason for our interest in the curvature tensor.

Proposition 7.1.4. *The curvature tensor is a local isometry invariant: if (M,g) and $(\widetilde{M},\widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi: M \to \widetilde{M}$ is a local isometry, then $\varphi^*\widetilde{Rm} = Rm$.*

Exercise 7.1.5. *Prove above proposition.*

7.2 Flat Manifolds

To give a qualitative geometric interpretation to the curvature tensor, we will show that it is precisely the obstruction to being locally isometric to Euclidean (or pseudo-Euclidean) space. (In next chapter, after we have developed more machinery, we will be able to give a far more detailed quantitative interpretation.) The crux of the proof is the following lemma.

Lemma 7.2.1. *Suppose M is a smooth manifold, and ∇ is any connection on M satisfying the flatness criterion. Given $p \in M$ and any vector $v \in T_pM$, there exists a parallel vector field V on a neighborhood of p such that $V_p = v$.*

Proof. Let $p \in M$ and $v \in T_pM$ be arbitrary, and let $\left(x^1, \ldots, x^n\right)$ be any smooth coordinates for M centered at p. By shrinking the coordinate neighborhood if necessary, we may assume that the image of the coordinate map is an open cube $C_\varepsilon = \left\{x : \left|x^i\right| < \varepsilon, i = 1, \ldots, n\right\}$. We use the coordinate map to identify the coordinate domain with C_ε.

Begin by parallel transporting v along the x^1-axis; then from each point on the x^1-axis, parallel transport along the coordinate line parallel to the x^2-axis; then successively parallel transport along coordinate lines parallel to the x^3 through x^n axes (Fig. 7.2). The result is a vector field V defined in C_ε. The fact that V is smooth follows from by an inductive application of Theorem 1.2.8 to vector fields of the form $\left.W_k\right|_{(x,v)} = \partial/\partial x^k - v^i \Gamma_{ki}^j(x) \partial/\partial v^j$ on $C_\varepsilon \times \mathbb{R}^n$; the details are left as an exercise.

Since $\nabla_X V$ is linear over $C^\infty(M)$ in X, to show that V is parallel, it suffices to show that $\nabla_{\partial_i} V = 0$ for each $i = 1, \ldots, n$. By construction, $\nabla_{\partial_1} V = 0$ on the x^1-axis, $\nabla_{\partial_2} V = 0$ on the $\left(x^1, x^2\right)$-plane, and in general $\nabla_{\partial_k} V = 0$ on the slice $M_k \subseteq C_\varepsilon$ defined by $x^{k+1} = \cdots = x^n = 0$. We will prove the following fact by induction on k :

$$\nabla_{\partial_1} V = \cdots = \nabla_{\partial_k} V = 0 \quad \text{on } M_k.$$

For $k = 1$, this is true by construction, and for $k = n$, it means that V is parallel on the whole cube C_ε. So assume that (7.9) holds for some k. By construction, $\nabla_{\partial_{k+1}} V = 0$ on all of M_{k+1}, and for $i \leq k$, the inductive hypothesis shows that $\nabla_{\partial_i} V = 0$ on the hyperplane $M_k \subseteq M_{k+1}$. Since $\left[\partial_{k+1}, \partial_i\right] = 0$, the flatness criterion gives

$$\nabla_{\partial_{k+1}}\left(\nabla_{\partial_i} V\right) = \nabla_{\partial_i}\left(\nabla_{\partial_{k+1}} V\right) = 0 \quad \text{on } M_{k+1}.$$

This shows that $\nabla_{\partial_i} V$ is parallel along the x^{k+1}-curves starting on M_k. Since $\nabla_{\partial_i} V$ vanishes on M_k and the zero vector field is the unique parallel transport of zero, we conclude that $\nabla_{\partial_i} V$ is zero on each x^{k+1}-curve. Since every point of M_{k+1} is on one of these curves, it follows that $\nabla_{\partial_i} V = 0$ on all of M_{k+1}. This completes the inductive step to show that V is parallel. ■

Exercise 7.2.2. *Prove that the vector field V constructed in the preceding proof is smooth.*

Theorem 7.2.3. *A Riemannian or pseudo-Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. One direction is immediate: Proposition 7.0.2 showed that the Levi-Civita connection of a flat metric satisfies the flatness criterion, so its curvature endomorphism is identically zero, which implies that the curvature tensor is also zero.

Now suppose (M, g) has vanishing curvature tensor. This means that the curvature endomorphism vanishes as well, so the Levi-Civita connection satisfies the flatness criterion. We begin by showing that g shares one important property with Euclidean and pseudo-Euclidean metrics: it admits a parallel orthonormal frame in a neighborhood of each point.

Let $p \in M$, and choose an orthonormal basis $\left(b_1, \ldots, b_n\right)$ for T_pM. In the pseudo-Riemannian case, we may assume that the basis is in standard order (with positive entries before negative ones in the matrix

$g_{ij} = g_p(b_i, b_j)$). Lemma 7.2.1 shows that there exist parallel vector fields $E_1, \ldots, E_n$ on a neighborhood U of p such that $E_i|_p = b_i$ for each $i = 1, \ldots, n$. Because parallel transport preserves inner products, the vector fields (E_j) are orthonormal (and hence linearly independent) in all of U. Because the Levi-Civita connection is symmetric, we have

$$[E_i, E_j] \xlongequal{\text{symmetric conn.}} \nabla_{E_i} E_j - \nabla_{E_j} E_i = \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) E_k \xlongequal{\text{Pb.4.8.1}} 0.$$

Thus the vector fields $(E_1, \ldots, E_n)$ form a commuting orthonormal frame on U. The canonical form theorem for commuting vector fields ([4] proposition **??**) shows that there are coordinates $\left(y^1, \ldots, y^n\right)$ on a (possibly smaller) neighborhood of p such that $E_i = \partial/\partial y^i$ for $i = 1, \ldots, n$. In any such coordinates, $g_{ij} = g(\partial_i, \partial_j) = g(E_i, E_j) = \pm\delta_{ij}$, so the map $y = \left(y^1, \ldots, y^n\right)$ is an isometry from a neighborhood of p to an open subset of the appropriate Euclidean or pseudo-Euclidean space. ■

Using similar ideas, we can give a more precise interpretation of the meaning of the curvature tensor: it is a measure of the extent to which parallel transport around a small rectangle fails to be the identity map.

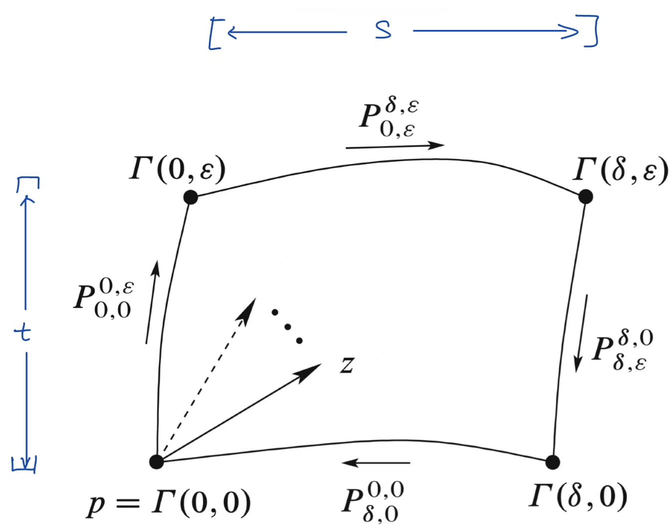


Figure 7.1: The curvature endomorphism and parallel transport around a closed loop.

Theorem 7.2.4. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold; let I be an open interval containing 0; let $\Gamma : I \times I \to M$ be a smooth one-parameter family of curves; and let $p = \Gamma(0,0), x = \partial_s\Gamma(0,0)$, and $y = \partial_t\Gamma(0,0)$ (see Fig.6.3). For any $s_1, s_2, t_1, t_2 \in I$, let $P_{s_1,t_1}^{s_1,t_2} : T_{\Gamma(s_1,t_1)}M \to T_{\Gamma(s_1,t_2)}M$ denote parallel transport along the curve $\Gamma_{s_1}|_{[t_1,t_2]} : t \mapsto \Gamma(s_1, t)$ from time t_1 to time t_2, and let $P_{s_1,t_1}^{s_2,t_1} : T_{\Gamma(s_1,t_1)}M \to T_{\Gamma(s_2,t_1)}M$ denote parallel transport along the curve $\Gamma^{(t_1)}|_{[s_1,s_2]} : s \mapsto \Gamma(s, t_1)$ from time s_1 to time s_2. (See Fig.7.1) Then for every $z \in T_pM$,*

$$R(x,y)z = \lim_{\delta,\varepsilon\to 0} \frac{P_{\delta,0}^{0,0} \circ P_{\delta,\varepsilon}^{\delta,0} \circ P_{0,\varepsilon}^{\delta,\varepsilon} \circ P_{0,0}^{0,\varepsilon}(z) - z}{\delta\varepsilon}. \tag{7.9}$$

Proof. Define a vector field Z along Γ by first parallel transporting z along the curve $t \mapsto \Gamma(0,t)$, and then for each t, parallel transporting $Z(0,t)$ along the curve $s \mapsto \Gamma(s,t)$. The resulting vector field along Γ is smooth by another application of Theorem 1.2.8 as in the proof of lemma 7.2.1; and by construction, it satisfies $D_tZ(0,t) = 0$ for all $t \in I$, and $D_sZ(s,t) = 0$ for all $(s,t) \in I \times I$. Proposition 7.1.3 shows that

$$R(x,y)z = D_sD_tZ(0,0) - D_tD_sZ(0,0) = D_sD_tZ(0,0).$$

Thus we need only show that $D_s D_t Z(0,0)$ is equal to the limit on the right-hand side of (7.10). From Theorem 4.6.5, we have

$$(D_t Z)(s,0) = \lim_{\varepsilon \to 0} \frac{P^{s,0}_{s,\varepsilon}(Z(s,\varepsilon)) - Z(s,0)}{\varepsilon}, \tag{7.10}$$

$$(D_s(D_t Z))(0,0) = \lim_{\delta \to 0} \frac{P^{0,0}_{\delta,0}(D_t Z(\delta,0)) - D_t Z(0,0)}{\delta}. \tag{7.11}$$

Evaluating (7.10) first at $s = \delta$ and then at $s = 0$, and inserting the resulting expressions into (7.11), we obtain

$$(D_s(D_t Z))(0,0) = \lim_{\delta,\varepsilon \to 0} \frac{P^{0,0}_{\delta,0} \circ P^{\delta,0}_{\delta,\varepsilon}(Z(\delta,\varepsilon)) - P^{0,0}_{\delta,0}(Z(\delta,0)) - P^{0,0}_{0,\varepsilon}(Z(0,\varepsilon)) + Z(0,0)}{\delta\varepsilon}. \tag{7.12}$$

Here we have used the fact that parallel transport is linear, so the ε-limit can be pulled past $P^{0,0}_{\delta,0}$.

Now, the fact that Z is parallel along $t \mapsto \Gamma(0,t)$ and along all of the curves $s \mapsto \Gamma(s,t)$ implies

$$P^{0,0}_{\delta,0}(Z(\delta,0)) = P^{0,0}_{0,\varepsilon}(Z(0,\varepsilon)) = Z(0,0) = z$$
$$Z(\delta,\varepsilon) = P^{\delta,\varepsilon}_{0,\varepsilon}(Z(0,\varepsilon)) = P^{\delta,\varepsilon}_{0,\varepsilon} \circ P^{0,\varepsilon}_{0,0}(z).$$

Inserting these relations into (7.12) yields (7.9). ■

7.3 Symmetries of the Curvature Tensor

The curvature tensor on a Riemannian or pseudo-Riemannian manifold has a number of symmetries besides the obvious skew-symmetry in its first two arguments.

Proposition 7.3.1 (Symmetries of the Curvature Tensor)**.** *Let (M,g) be a Riemannian or pseudo-Riemannian manifold. The $(0,4)$-curvature tensor of g has the following symmetries for all vector fields W, X, Y, Z:*

(a) $Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$.

(b) $Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$.

(c) $Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$.

(d) $Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0$.

Remark 7.3.2. Before we begin the proof, a few remarks are in order. First, as the proof will show, (a) is a trivial consequence of the definition of the curvature endomorphism; (b) follows from the compatibility of the Levi-Civita connection with the metric; (d) follows from the symmetry of the connection; and (c) follows from (a), (b), and (d). The identity in (d) is called the **algebraic Bianchi identity** (or more traditionally but less informatively, the **first Bianchi identity**). It is easy to show using (a)-(d) that a three-term sum obtained by cyclically permuting any three arguments of Rm is also zero. Finally, it is useful to record the form of these symmetries in terms of components with respect to any basis:

(a') $R_{ijkl} = -R_{jikl}$.

(b') $R_{ijkl} = -R_{ijlk}$.

(c') $R_{ijkl} = R_{klij}$

(d') $R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

♠

Proof. Identity (a) is immediate from the definition of the curvature tensor, because $R(W,X)Y = -R(X,W)Y$. To prove (b), it suffices to show that $Rm(W,X,Y,Y) = 0$ for all Y, denoted as identity $(\star)$ for then (b) follows from the expansion of $Rm(W,X,Y+Z,Y+Z) = 0$:

$$\begin{aligned}
0 &\overset{(\star)}{=\!=} Rm(W,X,Y+Z,Y+Z) \\
&= \langle R(W,X)(Y+Z), Y+Z\rangle_g \\
&= \langle R(W,X)Y + R(W,X)Z, Y+Z\rangle_g \\
&= \langle R(W,X)\underline{Y,Y}\rangle_g + \langle R(W,X)Y,Z\rangle_g + \langle R(W,X)Z,Y\rangle_g + \langle R(W,X)\underline{Z,Z}\rangle_g \\
&\overset{(\star)}{=\!=} \langle R(W,X)Y,Z\rangle_g + \langle R(W,X)Z,Y\rangle_g \\
\Longrightarrow &\ \langle R(W,X)Y,Z\rangle_g = -\langle R(W,X)Z,Y\rangle_g, \text{ or } Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)
\end{aligned}$$

we now show $(\star)$: the compatibility with the metric gives

$$\begin{aligned}
&\nabla_X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\Longrightarrow &\ \underbrace{\nabla_X\langle Y,Y\rangle}_{X|Y|^2} = \langle \nabla_X Y, Y\rangle + \langle Y, \nabla_X Y\rangle = 2\langle \nabla_X Y, Y\rangle \qquad (*)
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\Longrightarrow &\ \underbrace{\nabla_W\langle \nabla_X Y,Z\rangle}_{W\langle \nabla_X Y, Z\rangle} = \langle \nabla_W\nabla_X Y, Z\rangle + \langle \nabla_X Y, \nabla_W Z\rangle \qquad (**)
\end{aligned}$$

Thus,

$$\begin{aligned}
WX|Y|^2 &\overset{(*)(**)}{=\!=\!=} W\left(2\left\langle \nabla_X Y, Y\right\rangle\right) = 2\left\langle \nabla_W\nabla_X Y, Y\right\rangle + 2\left\langle \nabla_X Y, \nabla_W Y\right\rangle; \\
XW|Y|^2 &\overset{(*)(**)}{=\!=\!=} X\left(2\left\langle \nabla_W Y, Y\right\rangle\right) = 2\left\langle \nabla_X\nabla_W Y, Y\right\rangle + 2\left\langle \nabla_W Y, \nabla_X Y\right\rangle; \\
[W,X]|Y|^2 &\overset{(*)}{=\!=} 2\left\langle \nabla_{[W,X]} Y, Y\right\rangle.
\end{aligned}$$

When we subtract the second and third equations from the first, the left-hand side is zero. The terms $2\left\langle \nabla_X Y, \nabla_W Y\right\rangle$ and $2\left\langle \nabla_W Y, \nabla_X Y\right\rangle$ cancel on the right-hand side, giving

$$\begin{aligned}
0 &= 2\left\langle \nabla_W\nabla_X Y, Y\right\rangle - 2\left\langle \nabla_X\nabla_W Y, Y\right\rangle - 2\left\langle \nabla_{[W,X]} Y, Y\right\rangle \\
&= 2\langle R(W,X)Y,Y\rangle \\
&= 2Rm(W,X,Y,Y).
\end{aligned}$$

Next we prove (d). From the definition of Rm, this will follow immediately from

$$R(W,X)Y + R(X,Y)W + R(Y,W)X = 0.$$

Using the definition of R and the symmetry of the connection, the left-hand side expands to

$$\begin{aligned}
&\left(\nabla_W\nabla_X Y - \nabla_X\nabla_W Y - \nabla_{[W,X]} Y\right) \\
&+ \left(\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]} W\right) \\
&+ \left(\nabla_Y\nabla_W X - \nabla_W\nabla_Y X - \nabla_{[Y,W]} X\right) \\
&= \nabla_W\left(\nabla_X Y - \nabla_Y X\right) + \nabla_X\left(\nabla_Y W - \nabla_W Y\right) + \nabla_Y\left(\nabla_W X - \nabla_X W\right) \\
&- \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X \\
&= \nabla_W[X,Y] + \nabla_X[Y,W] + \nabla_Y[W,X] \\
&- \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X \\
&= [W,[X,Y]] + [X,[Y,W]] + [Y,[W,X]].
\end{aligned}$$

This is zero by the Jacobi identity (see property 1.2.4).

Finally, we show that identity (c) follows from the other three. Writing the algebraic Bianchi identity four times with indices cyclically permuted gives

$$\begin{aligned} Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) &= 0 \\ Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) &= 0 \\ Rm(Y,Z,W,X) + Rm(Z,W,Y,X) + Rm(W,Y,Z,X) &= 0 \\ Rm(Z,W,X,Y) + Rm(W,X,Z,Y) + Rm(X,Z,W,Y) &= 0. \end{aligned}$$

Now add up all four equations. Applying (b) four times makes all the terms in the first two columns cancel. Then applying (a) and (b) in the last column yields $2Rm(Y,W,X,Z) - 2Rm(X,Z,Y,W) = 0$, which is equivalent to (c). ■

There is one more identity that is satisfied by the covariant derivatives of the curvature tensor on every Riemannian manifold. Classically, it was called the second Bianchi identity, but modern authors tend to use the more informative name differential Bianchi identity.

Proposition 7.3.3 (Differential Bianchi Identity). *The total covariant derivative of the curvature tensor satisfies the following identity:*

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(X,Y,V,W,Z) + \nabla Rm(X,Y,W,Z,V) = 0. \tag{7.13}$$

In components, this is

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0. \tag{7.14}$$

Proof. First of all, we show that by the symmetries of Rm, (7.13) is equivalent to

$$\nabla Rm(Z,V,X,Y,W) + \nabla Rm(V,W,X,Y,Z) + \nabla Rm(W,Z,X,Y,V) = 0. \tag{7.15}$$

For example,

$$\begin{aligned} &\nabla Rm(X,Y,Z,V,W) = (\nabla_W Rm)(X,Y,Z,V) \\ &\overset{(4.4)}{=\!=} W(Rm(X,Y,Z,V)) - Rm(\nabla_W X,Y,Z,V) - Rm(X,\nabla_W Y,Z,V) - Rm(X,Y,\nabla_W Z,V) - Rm(X,Y,Z,\nabla_W V) \\ &\overset{\text{prop.7.3.1}}{=\!=} \underbrace{-W(Rm(Z,V,X,Y))}_{=-\nabla_W(\underbrace{Rm(Z,V,X,Y)}_{\text{a function}})} - \underbrace{Rm(\nabla_W X,Y,Z,V)}_{(1)} - \underbrace{Rm(X,\nabla_W Y,Z,V)}_{(2)} - \underbrace{Rm(X,Y,\nabla_W Z,V)}_{(3)} - \underbrace{Rm(X,Y,Z,\nabla_W V)}_{(4)} \end{aligned}$$

and

$$\begin{aligned} &\nabla Rm(Z,V,X,Y,W) \\ &= W(Rm(Z,V,X,Y)) - \overbrace{Rm(\nabla_W Z,V,X,Y)}^{(3)} - \overbrace{Rm(Z,\nabla_W V,X,Y)}^{(4)} - \overbrace{Rm(Z,V,\nabla_W X,Y)}^{(1)} - \overbrace{Rm(Z,V,X,\nabla_W Y)}^{(2)} \end{aligned}$$

Equation (7.15) be proved by a long and tedious computation, but there is a standard shortcut for such calculations in Riemannian geometry that makes our task immeasurably easier. To prove that (7.15) holds at a particular point p, it suffices by multilinearity to prove the formula when X,Y,Z,V,W are basis vectors with respect to some frame. The shortcut consists in choosing a special frame for each point p to simplify the computations there.

Let p be an arbitrary point, let (x^i) be normal coordinates centered at p, and let X,Y,Z,V,W be arbitrary coordinate basis vector fields. These vectors satisfy two properties that simplify our computations enormously: (1) their commutators vanish identically, since $[\partial_i, \partial_j] \equiv 0$; and (2) their covariant derivatives vanish at p, since $\Gamma_{ij}^k(p) = 0$ (Prop.5.4.2(d)).

Using these facts and the compatibility of the connection with the metric, the first term in (7.15) evaluated at p becomes

$$\begin{aligned}
\left(\nabla_W Rm\right)(Z,V,X,Y) &= \nabla_W(Rm(Z,V,X,Y)) \\
&= \nabla_W\langle R(Z,V)X,Y\rangle \\
&= \nabla_W\left\langle \nabla_Z\nabla_V X - \nabla_V\nabla_Z X - \nabla_{[Z,V]}X, Y\right\rangle \\
&= \langle \nabla_W\nabla_Z\nabla_V X - \nabla_W\nabla_V\nabla_Z X, Y\rangle .
\end{aligned}$$

Write this equation three times, with the vector fields W, Z, V cyclically permuted. Summing all three gives

$$\begin{aligned}
&\nabla Rm(Z,V,X,Y,W) + \nabla Rm(V,W,X,Y,Z) + \nabla Rm(W,Z,X,Y,V) \\
&= \langle \nabla_W\nabla_Z\nabla_V X - \nabla_W\nabla_V\nabla_Z X \\
&\quad + \nabla_Z\nabla_V\nabla_W X - \nabla_Z\nabla_W\nabla_V X \\
&\quad + \nabla_V\nabla_W\nabla_Z X - \nabla_V\nabla_Z\nabla_W X, Y\rangle \\
&= \langle R(W,Z)\left(\nabla_V X\right) + R(Z,V)\left(\nabla_W X\right) + R(V,W)\left(\nabla_Z X\right), Y\rangle \\
&= 0,
\end{aligned}$$

where the last line follows because $\nabla_V X = \nabla_W X = \nabla_Z X = 0$ at p. ■

7.4 The Ricci Identities

The curvature endomorphism also appears as the obstruction to commutation of total covariant derivatives. Recall that if F is any smooth tensor field of type (k,l), then its second covariant derivative $\nabla^2 F = \nabla(\nabla F)$ is a smooth $(k,l+2)$-tensor field, and for vector fields X and Y, the notation $\nabla^2_{X,Y}F$ denotes $\nabla^2 F(\ldots,Y,X)$. Given vector fields X and Y, let $R(X,Y)^*: T^*M \to T^*M$ denote the **dual map to** $R(X,Y)$, defined by

$$\left(R(X,Y)^*\eta\right)(Z) = \eta(R(X,Y)Z).$$

Theorem 7.4.1 (Ricci Identities)**.** *On a Riemannian or pseudo-Riemannian manifold M, the second total covariant derivatives of vector and tensor fields satisfy the following identities. If Z is a smooth vector field,*

$$\nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = R(X,Y)Z. \tag{7.16}$$

If β is a smooth 1-form,

$$\nabla^2_{X,Y}\beta - \nabla^2_{Y,X}\beta = -R(X,Y)^*\beta. \tag{7.17}$$

And if B is a smooth (k,l)-tensor field,

$$\begin{aligned}
&\left(\nabla^2_{X,Y}B - \nabla^2_{Y,X}B\right)\left(\omega^1,\ldots,\omega^k,V_1,\ldots,V_l\right) \\
&= B\left(R(X,Y)^*\omega^1,\omega^2,\ldots,\omega^k,V_1,\ldots,V_l\right) + \cdots \\
&+ B\left(\omega^1,\ldots,\omega^{k-1},R(X,Y)^*\omega^k,V_1,\ldots,V_l\right) \\
&- B\left(\omega^1,\ldots,\omega^k,R(X,Y)V_1,V_2,\ldots,V_l\right) - \cdots \\
&- B\left(\omega^1,\ldots,\omega^k,V_1,\ldots,V_{l-1},R(X,Y)V_l\right)
\end{aligned} \tag{7.18}$$

for all covector fields ω^i and vector fields V_j. In terms of any local frame, the component versions of these formulas read

$$Z^i;pq - Z^i;qp = -R_{pqm}{}^i Z^m, \tag{7.19}$$

$$\beta_{j;pq} - \beta_{j;qp} = R_{pqj}{}^m\beta_m, \tag{7.20}$$

$$\begin{aligned}
B^{i_1\ldots i_k}_{j_1\ldots j_l;pq} - B^{i_1\ldots i_k}_{j_1\ldots j_l;qp} = &- R_{pqm}{}^{i_1}B^{mi_2\ldots i_k}_{j_1\ldots j_l} - \cdots - R_{pqm}{}^{i_k}B^{i_1\ldots i_k m}_{j_1\ldots j_l} \\
&+ R_{pqj_1}{}^m B^{i_1\ldots i_k}_{mj_2\ldots j_l} + \cdots + R_{pqj_l}{}^m B^{i_1\ldots i_k}_{j_1\ldots j_{l-1}m}.
\end{aligned} \tag{7.21}$$

Proof. For any tensor field B and vector fields X, Y, Proposition 4.3.7 implies

$$\begin{aligned}\nabla^2_{X,Y}B - \nabla^2_{Y,X}B &= \nabla_X\nabla_Y B - \nabla_{(\nabla_X Y)}B - \nabla_Y\nabla_X B + \nabla_{(\nabla_Y X)}B \\ &= \nabla_X\nabla_Y B - \nabla_Y\nabla_X B - \nabla_{[X,Y]}B,\end{aligned} \tag{7.22}$$

where the last equality follows from the symmetry of the connection. In particular, this holds when $B = Z$ is a vector field, so (7.16) follows directly from the definition of the curvature endomorphism. Next we prove (7.17). Using (4.6) repeatedly, we compute

$$\begin{aligned}(\nabla_X\nabla_Y\beta)(Z) &= X((\nabla_Y\beta)(Z)) - (\nabla_Y\beta)(\nabla_X Z) \\ &= X(Y(\beta(Z)) - \beta(\nabla_Y Z)) - (\nabla_Y\beta)(\nabla_X Z) \\ &= XY(\beta(Z)) - (\nabla_X\beta)(\nabla_Y Z) - \beta(\nabla_X\nabla_Y Z) - (\nabla_Y\beta)(\nabla_X Z).\end{aligned} \tag{7.23}$$

Reversing the roles of X and Y, we get

$$(\nabla_Y\nabla_X\beta)(Z) = YX(\beta(Z)) - (\nabla_Y\beta)(\nabla_X Z) - \beta(\nabla_Y\nabla_X Z) - (\nabla_X\beta)(\nabla_Y Z), \tag{7.24}$$

and applying (4.6) one more time yields

$$\left(\nabla_{[X,Y]}\beta\right)(Z) = [X,Y](\beta(Z)) - \beta\left(\nabla_{[X,Y]}Z\right). \tag{7.25}$$

Now subtract (7.24) and (7.25) from (7.23): all but three of the terms cancel, yielding

$$\begin{aligned}\left(\nabla_X\nabla_Y\beta - \nabla_Y\nabla_X\beta - \nabla_{[X,Y]}\beta\right)(Z) &= -\beta\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right) \\ &= -\beta(R(X,Y)Z),\end{aligned}$$

which is equivalent to (7.17). Next consider the action of $\nabla^2_{X,Y} - \nabla^2_{Y,X}$ on an arbitrary tensor product:

$$\begin{aligned}&\left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)(F\otimes G) \\ &= \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)(F\otimes G) \\ &= \nabla_X\nabla_Y F\otimes G + \nabla_Y F\otimes\nabla_X G + \nabla_X F\otimes\nabla_Y G + F\otimes\nabla_X\nabla_Y G \\ &\quad - \nabla_Y\nabla_X F\otimes G - \nabla_X F\otimes\nabla_Y G - \nabla_Y F\otimes\nabla_X G - F\otimes\nabla_Y\nabla_X G \\ &\quad - \nabla_{[X,Y]}F\otimes G - F\otimes\nabla_{[X,Y]}G \\ &= \left(\nabla^2_{X,Y}F - \nabla^2_{Y,X}F\right)\otimes G + F\otimes\left(\nabla^2_{X,Y}G - \nabla^2_{Y,X}G\right).\end{aligned}$$

A simple induction using this relation together with (7.16) and (7.17) shows that for all smooth vector fields $W_1, \ldots, W_k$ and 1-forms $\eta^1, \ldots, \eta^l$,

$$\begin{aligned}&\left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)\left(W_1\otimes\cdots\otimes W_k\otimes\eta^1\otimes\cdots\otimes\eta^l\right) \\ &= (R(X,Y)W_1)\otimes W_2\otimes\cdots\otimes W_k\otimes\eta^1\otimes\cdots\otimes\eta^l + \cdots \\ &+ W_1\otimes\cdots\otimes W_{k-1}\otimes(R(X,Y)W_k)\otimes\eta^1\otimes\cdots\otimes\eta^l \\ &+ W_1\otimes\cdots\otimes W_k\otimes\left(-R(X,Y)^*\eta^1\right)\otimes\eta^2\otimes\cdots\otimes\eta^l + \cdots \\ &+ W_1\otimes\cdots\otimes W_k\otimes\eta^1\otimes\cdots\otimes\eta^{l-1}\otimes\left(-R(X,Y)^*\eta^l\right).\end{aligned}$$

Since every tensor field can be written as a sum of tensor products of vector fields and 1-forms, this implies (7.18). Finally, the component formula (7.21) follows by applying (7.18) to

$$\left(\nabla^2_{E_q,E_p}B - \nabla^2_{E_p,E_q}B\right)\left(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}, E_{j_1}, \ldots, E_{j_l}\right),$$

where (E_j) and (ε^i) represent a local frame and its dual coframe, respectively, and using

$$\begin{aligned}R(E_q,E_p)E_j &= R_{qpj}{}^m E_m = -R_{pqj}{}^m E_m, \\ R(E_q,E_p)^*\varepsilon^i &= R_{qpm}{}^i\varepsilon^m = -R_{pqm}{}^i\varepsilon^m.\end{aligned}$$

The other two component formulas are special cases of (7.21). ■

7.5 Ricci and Scalar Curvature

Suppose (M, g) is an n-dimensional Riemannian or pseudo-Riemannian manifold. Because 4-tensors are so complicated, it is often useful to construct simpler tensors that summarize some of the information contained in the curvature tensor. The most important such tensor is the **Ricci curvature** or **Ricci tensor**, denoted by Rc (or often Ric in the literature), which is the covariant 2-tensor field defined as the trace of the curvature endomorphism on its first and last indices. That is, $Rc = C_3^1(R)$ where C_3^1 is the unique linear mapping from $T^{(1,3)}(TM)$ to $T^{(0,2)}(TM)$ such that

$$\omega_1 \otimes \omega_2 \otimes \omega_3 \otimes v_1 \mapsto \langle \omega_1, v_1 \rangle \omega_2 \otimes \omega_3$$

(note that we didn't write $v_1 \otimes \omega_1 \otimes \omega_2 \otimes \omega_3$ because we want to be aligned with the convention that the contravariant index is placed at last for Riemannian endomorphism; as in definition 2.3.3 the order of covariant and contravariant is assumed to be dropped.) Now, since

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

we see that C_3^1 sends R to

$$R_{ijk}{}^l \langle dx^i, \partial_l \rangle dx^j \otimes dx^k = R_{ijk}{}^l \delta_{il} dx^j \otimes dx^k = R_{pjk}{}^p dx^j \otimes dx^k = \overbrace{R_{kij}{}^k dx^i \otimes dx^j}^{Rc}.$$

The components of Rc are usually denoted as R_{ij}, so above equation implies

$$R_{ij} = R_{kij}{}^k$$

Proposition 7.5.1.

(1) For vector fields X, Y,

$$Rc(X, Y) = \operatorname{tr}(Z \mapsto R(Z, X)Y).$$

(2) For orthonormal basis (E_i), we have

$$\operatorname{tr}(Z \mapsto R(Z, X)Y) = \sum_i \langle R(E_i, X)Y, E_i \rangle_g$$

(3) $R_{ij} = g^{km} R_{kijm}$.

Proof. (1): We denote $Z \mapsto R(Z, X)Y$ as the operator $A \in \operatorname{End}(TM)$. Then $f = \Phi(A) \in T^{(1,1)}(TM)$ is defined by

$$f(W, Z) = \Phi(A)(W, Z) = W(R(Z, X)Y)$$

To get the trace of $f = \Phi(A)$, we compute $f(dx^i, \partial_j)$:

$$f_j^i = f(dx^i, \partial_j) = dx^i(R(\partial_j, X^k\partial_k)(Y^m\partial_m)) \overset{(7.3)}{=\!=} dx^i \left(R_{jkm}{}^l X^k Y^m \partial_l\right) = R_{jkm}{}^i X^k Y^m.$$

Thus, the trace of f is

$$\sum_i f_i^i = R_{ikm}{}^i X^k Y^m = R_{kij}{}^k X^i Y^j$$

which is the same as $R_{kij}{}^k dx^i \otimes dx^j(X, Y) = Rc(X, Y)$.

(2) In general, for $f \in \operatorname{End}(V)$,

$$\operatorname{tr}(f) = \sum_i \langle E_i, f(E_i) \rangle.$$

That's because $f(E_i) = \sum_j f_{ji} E_j$ where (f_{ij}) is the matrix of f, and $\langle E_i, f(E_i)\rangle = \sum_j f_{ji}\langle E_i, E_j\rangle = \sum_j f_{ji}\delta_{ij} = f_{\mathrm{II}}$.

(3) It is known that the components of Riemann curvature tensor satisfies $R_{ijkl} = g_{lm} R_{ijk}{}^m$. Thus

$$\begin{aligned} g^{km} R_{kijm} &= g^{km} g_{mp} R_{kij}{}^p \\ &\overset{(2.5)}{=\!=} \delta^k_p R_{kij}{}^p \\ &= R_{kij}{}^k = R_{ij} \end{aligned}$$

■

The **scalar curvature** is the function S pointewise defined as the trace of the Ricci tensor:

$$S = \operatorname{tr}_g Rc = R_i{}^i = g^{ij} R_{ij}.$$

where we used observation 2.3.5. Note that $(Rc_p)^\sharp(v,\omega) = Rc_p(v,\omega^\sharp)$ and $S_p = \operatorname{tr}((Rc_p)^\sharp)$ (note that it is the last index, or the second covariant, that is raised, so we write (v,w) instead of (ω, v); just as in definition 2.3.3, the order of covariant and contravariant is assumed to be dropped).

Lemma 7.5.2. *The Ricci curvature is a symmetric 2-tensor field. It can be expressed in any of the following ways:*

$$R_{ij} = R_{kij}{}^k = R_{ik}{}^k{}_j = -R_{ki}{}^k{}_j = -R_{ikj}{}^k.$$

Proof. To show $R_{ij} = R_{ik}{}^k{}_j$, we use Example 2.3.4. By the symmetry of Riemann curvature tensor we obtain

$$R_{ik}{}^k{}_j = g^{km} R_{ikmj} = g^{km}(-R_{kimj}) = g^{km}(-(-R_{kijm})) = g^{km} R_{kijm} \overset{prop.7.5(3)}{=\!=\!=} R_{ij}$$

Similarly,

$$-R_{ki}{}^k{}_j = -g^{km} R_{kimj} = g^{km} R_{kijm} = R_{ij},$$

and

$$-R_{ikj}{}^k = -g^{km} R_{ikjm} = g^{km} R_{kijm} = R_{ij}.$$

■

It is sometimes useful to decompose the Ricci tensor into a multiple of the metric and a complementary piece with zero trace. Define the **traceless Ricci tensor of** g as the following symmetric 2-tensor:

$$\overset{\circ}{Rc} = Rc - \frac{1}{n} S g.$$

Proposition 7.5.3. *Let (M,g) be a Riemannian or pseudo-Riemannian n-manifold. Then $\operatorname{tr}_g Rc \equiv 0$, and the Ricci tensor decomposes orthogonally as*

$$Rc = \overset{\circ}{Rc} + \frac{1}{n} S g. \tag{7.26}$$

Therefore, in the Riemannian case,

$$|Rc|^2_g = |Rc|^2_g + \frac{1}{n} S^2 \tag{7.27}$$

Remark 7.5.4. The statement about norms, and others like it that we will prove below, works only in the Riemannian case because of the additional absolute value signs required to compute norms in the pseudo-Riemannian case. The pseudo-Riemannian analogue would be $\langle Rc, Rc\rangle_g = \langle Rc, Rc\rangle_g + \frac{1}{n} S^2$, but this is not as useful. ♠

Proof. Note that in every local frame, we have

$$\operatorname{tr}_g g = g_{ij} g^{ji} = \delta_i^i = n.$$

It then follows directly from the definition of $\overset{\circ}{Rc}$ that $\operatorname{tr}_g \overset{\circ}{Rc} \equiv 0$ and (7.26) holds:

$$\operatorname{tr}_g \overset{\circ}{Rc} = \operatorname{tr}_g(Rc - \frac{1}{n}Sg) \xlongequal{\text{linearity}} \operatorname{tr}_g Rc - \frac{1}{n}S \operatorname{tr}_g g = S - \frac{1}{n}Sn = 0$$

where we again note that S is a function and S_p is thus only a scalar. The fact that the decomposition is orthogonal follows easily from the fact that for every symmetric 2-tensor h, we have

$$\langle h, g \rangle = g^{ik} g^{jl} h_{ij} g_{kl} = g^{ij} h_{ij} = \operatorname{tr}_g h,$$

and therefore $\langle \overset{\circ}{Rc}, g \rangle = \operatorname{tr}_g \overset{\circ}{Rc} = 0$. Finally, (7.27) follows from (7.26) and the fact that $\langle g, g \rangle = \operatorname{tr}_g g = n$. ■

The next proposition, which follows directly from the differential Bianchi identity, expresses some important relationships among the covariant derivatives of the various curvature tensors. To express it concisely, it is useful to introduce another operator on tensor fields. If T is a smooth 2-tensor field on a Riemannian or pseudo-Riemannian manifold, we define the **exterior covariant derivative of** T to be the 3-tensor field DT defined by

$$(DT)(X, Y, Z) = -(\nabla T)(X, Y, Z) + (\nabla T)(X, Z, Y).$$

In terms of components, this is

$$(DT)_{ijk} = -T_{ij;k} + T_{ik;j}$$

(This operator is a generalization of the ordinary exterior derivative of a 1-form, which can be expressed in terms of the total covariant derivative by $(d\eta)(Y, Z) = -(\nabla\eta)(Y, Z) + (\nabla\eta)(Z, Y)$ by the result of [5] Problem 5-13. The exterior covariant derivative can be generalized to other types of tensors as well, but this is the only case we need.)

Proposition 7.5.5 (Contracted Bianchi Identities). *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. The covariant derivatives of the Riemann, Ricci, and scalar curvatures of g satisfy the following identities:*

$$\operatorname{tr}_g(\nabla Rm) = -D(Rc), \tag{7.28}$$

$$\operatorname{tr}_g(\nabla Rc) = \frac{1}{2} dS, \tag{7.29}$$

where the trace in each case is on the first and last indices. In components, this is

$$R_{ijkl;}{}^{i} = R_{jk;l} - R_{jl;k}, \tag{7.30}$$

$$R_{il;}{}^{i} = \frac{1}{2} S_{;l}. \tag{7.31}$$

Proof. Start with the component form (7.14) of the differential Bianchi identity, raise the index m, and then contract on the indices i, m to obtain (7.30). (Note that covariant differentiation commutes with contraction by Proposition 4.3.1 and with the musical isomorphisms by Proposition 5.2.12, so it does not matter whether the indices that are raised and contracted come before or after the semicolon.) Then do the same with the indices j, k and simplify to obtain (7.31). The coordinate-free formulas (7.28) and (7.29) follow by expanding everything out in components. ■

It is important to note that if the sign convention chosen for the curvature tensor is the opposite of ours, then the Ricci tensor must be defined as the trace of Rm on the first and third (or second and fourth) indices. (The trace on the first two or last two indices is always zero by antisymmetry.) The definition is chosen so that the Ricci and scalar curvatures have the same meaning for everyone, regardless of the conventions chosen for the full curvature tensor. So, for example, if a manifold is said to have positive scalar curvature, there is no ambiguity as to what is meant.

A Riemannian or pseudo-Riemannian metric is said to be an **Einstein metric** if its Ricci tensor is a constant multiple of the metric-that is,

$$Rc = \lambda g \quad \text{for some constant } \lambda. \tag{7.32}$$

This equation is known as the **Einstein equation**. As the next proposition shows, for connected manifolds of dimension greater than 2, it is not necessary to assume that λ is constant; just assuming that the Ricci tensor is a function times the metric is sufficient.

Proposition 7.5.6 (Schur's Lemma)**.** *Suppose (M, g) is a connected Riemannian or pseudo-Riemannian manifold of dimension $n \geq 3$ whose Ricci tensor satisfies $Rc = fg$ for some smooth real-valued function f. Then f is constant and g is an Einstein metric.*

Proof. Proof. Taking traces of both sides of $Rc = fg$ shows that $f = \frac{1}{n}S$, so the traceless Ricci tensor is identically zero. It follows that $\mathring{Rc} \equiv 0$. Because the covariant derivative of the metric is zero, this implies the following equation in any coordinate chart:

$$0 = R_{ij;k} - \frac{1}{n} S_{;k} g_{ij}$$

Tracing this equation on i and k, and comparing with the contracted Bianchi identity (7.31), we conclude that

$$0 = \frac{1}{2} S_{;j} - \frac{1}{n} S_{;j}$$

Because $n \geq 3$, this implies $S_{;j} = 0$. But $S_{;j}$ is the component of $\nabla S = dS$, so connectedness of M implies that S is constant and thus so is f. ■

Corollary 7.5.7. *If (M, g) is a connected Riemannian or pseudo-Riemannian manifold of dimension $n \geq 3$, then g is Einstein if and only if $\mathring{Rc} = 0$.*

Proof. Suppose first that g is an Einstein metric with $Rc = \lambda g$. Taking traces of both sides, we find that $\lambda = \frac{1}{n}S$, and therefore $\mathring{Rc} = Rc - \lambda g = 0$. Conversely, if $\mathring{Rc} = 0$, Schur's lemma implies that g is Einstein. ■

7.6 The Second Fundamental Form

Suppose (M, g) is a Riemannian submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Recall that this means that M is a submanifold of $\widetilde{M}$ endowed with the induced metric $g = \iota_M^* \widetilde{g}$ (where $\iota_M : M \hookrightarrow \widetilde{M}$ is the inclusion map). We will study the relationship between the geometry of M and that of $\widetilde{M}$. We assume that $(\widetilde{M}, \widetilde{g})$ is a Riemannian or pseudo-Riemannian manifold of dimension m, and (M, g) is an embedded n dimensional Riemannian submanifold of $\widetilde{M}$. For other cases, see [5] p.226 for more explanation.

Our first main task is to compare the Levi-Civita connection of M with that of $\widetilde{M}$. The starting point for doing so is the orthogonal decomposition of sections of the ambient tangent bundle $T\widetilde{M}\big|_M$ into tangential and orthogonal components. Just as we did for submanifolds of $\mathbb{R}^n$, we define orthogonal projection maps called **tangential** and **normal projections**:

$$\pi^\top : T\widetilde{M}\Big|_M \to TM,$$
$$\pi^\perp : T\widetilde{M}\Big|_M \to NM.$$

In terms of an adapted orthonormal frame $(E_1, \ldots, E_m)$ for M in $\widetilde{M}$, these are just the usual projections onto $\operatorname{span}(E_1, \ldots, E_n)$ and $\operatorname{span}(E_{n+1}, \ldots, E_m)$ respectively, so both projections are smooth bundle homomorphisms (i.e., they are linear on fibers and map smooth sections to smooth sections). If X is a section of $T\widetilde{M}\Big|_M$, we often use the shorthand notations $X^\top = \pi^\top X$ and $X^\perp = \pi^\perp X$ for its tangential and normal projections.

If X, Y are vector fields in $\mathfrak{X}(M)$, we can extend them to vector fields on an open subset of $\widetilde{M}$ (still denoted by X and Y), apply the ambient covariant derivative operator $\widetilde{\nabla}$, and then decompose at points of M to get

$$\widetilde{\nabla}_X Y = \left(\widetilde{\nabla}_X Y\right)^\top + \left(\widetilde{\nabla}_X Y\right)^\perp. \tag{7.33}$$

We wish to interpret the two terms on the right-hand side of this decomposition. Let us focus first on the normal component. We define the **second fundamental form** of M to be the map $\mathrm{II} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$ (read "two") given by

$$\mathrm{II}(X, Y) = \left(\widetilde{\nabla}_X Y\right)^\perp,$$

where X and Y are extended arbitrarily to an open subset of $\widetilde{M}$. Since $\pi^\perp$ maps smooth sections to smooth sections, $\mathrm{II}(X, Y)$ is a smooth section of NM.

The term **first fundamental form**, by the way, was originally used to refer to the induced metric g on M. Although that usage has mostly been replaced by more descriptive terminology, we seem unfortunately to be stuck with the name "second fundamental form." The word "form" in both cases refers to bilinear form, not differential form.

Proposition 7.6.1 (Properties of the Second Fundamental Form)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and let $X, Y \in \mathfrak{X}(M)$.*

(a) $\mathrm{II}(X, Y)$ is independent of the extensions of X and Y to an open subset of $\widetilde{M}$.

(b) $\mathrm{II}(X, Y)$ is bilinear over $C^\infty(M)$ in X and Y.

(c) $\mathrm{II}(X, Y)$ is symmetric in X and Y.

(d) The value of $\mathrm{II}(X, Y)$ at a point $p \in M$ depends only on X_p and Y_p.

Proof. Proof. Choose particular extensions of X and Y to a neighborhood of M in $\widetilde{M}$, and for simplicity denote the extended vector fields also by X and Y. We begin by proving that $\mathrm{II}(X, Y)$ is symmetric in X and Y when defined in terms of these extensions. The symmetry of the connection $\widetilde{\nabla}$ implies

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = \left(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X\right)^\perp = [X, Y]^\perp.$$

Since X and Y are tangent to M at all points of M, so is their Lie bracket (Cor.1.2.6). Therefore $[X, Y]^\perp = 0$, so II is symmetric.

Because $\widetilde{\nabla}_X Y\Big|_p$ depends only on X_p, it follows that the value of $\mathrm{II}(X, Y)$ at p depends only on X_p, and in particular is independent of the extension chosen for X. Because $\widetilde{\nabla}_X Y$ is linear over $C^\infty(\widetilde{M})$ in X, and every $f \in C^\infty(M)$ can be extended to a smooth function on a neighborhood of M in $\widetilde{M}$, it follows that $\mathrm{II}(X, Y)$ is linear over $C^\infty(M)$ in X. By symmetry, the same claims hold for Y. ■

As a consequence of the preceding proposition, for every $p \in M$ and all vectors $v, w \in T_pM$, it makes sense to interpret $\mathrm{II}(v, w)$ as the value of $\mathrm{II}(V, W)$ at p, where V and W are any vector fields on M such that $V_p = v$ and $W_p = W$, and we will do so from now on without further comment.

The following theorem shows that for the normal part of the decomposition, we have a relationship similar to the Euclidean case: $(\widetilde{\nabla}_X Y)^\top = \nabla_X Y$.

Theorem 7.6.2 (The Gauss Formula). *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. If $X, Y \in \mathfrak{X}(M)$ are extended arbitrarily to smooth vector fields on a neighborhood of M in $\widetilde{M}$, the following formula holds along M :*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)$$

The Gauss formula can also be used to compare intrinsic and extrinsic covariant derivatives along curves. If $\gamma : I \to M$ is a smooth curve and X is a vector field along γ that is everywhere tangent to M, then we can regard X as either a vector field along γ in $\widetilde{M}$ or a vector field along γ in M. We let $\widetilde{D}_t X$ and $D_t X$ denote its covariant derivatives along γ as a curve in $\widetilde{M}$ and as a curve in M, respectively. The next corollary shows how the two covariant derivatives are related.

Corollary 7.6.3 (The Gauss Formula Along a Curve). *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and $\gamma : I \to M$ is a smooth curve. If X is a smooth vector field along γ that is everywhere tangent to M, then*

$$\widetilde{D}_t X = D_t X + \mathrm{II}\left(\gamma', X\right).$$

Although the second fundamental form is defined in terms of covariant derivatives of vector fields tangent to M, it can also be used to evaluate extrinsic covariant derivatives of normal vector fields, as the following proposition shows. To express it concisely, we introduce one more notation. For each normal vector field $N \in \Gamma(NM)$, we obtain a scalar-valued symmetric bilinear form $\mathrm{II}_N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ by

$$\mathrm{II}_N(X, Y) = \langle N, \mathrm{II}(X, Y)\rangle. \tag{7.34}$$

Let $W_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denote the self-adjoint linear map associated with this bilinear form, characterized by

$$\langle W_N(X), Y\rangle = \mathrm{II}_N(X, Y) = \langle N, \mathrm{II}(X, Y)\rangle. \tag{7.35}$$

The map W_N is called the **Weingarten map in the direction of** N. Because the second fundamental form is bilinear over $C^\infty(M)$, it follows that W_N is linear over $C^\infty(M)$ and thus defines a smooth bundle homomorphism from TM to itself.

Proposition 7.6.4 (The Weingarten Equation). *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For every $X \in \mathfrak{X}(M)$ and $N \in \Gamma(NM)$, the following equation holds:*

$$\left(\tilde{\nabla}_X N\right)^\top = -W_N(X) \tag{7.36}$$

when N is extended arbitrarily to an open subset of $\widetilde{M}$.

In addition to describing the difference between the intrinsic and extrinsic connections, the second fundamental form plays an even more important role in describing the difference between the curvature tensors of $\widetilde{M}$ and M. The explicit formula, also due to Gauss, is given in the following theorem.

Theorem 7.6.5 (The Gauss Equation)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For all $W, X, Y, Z \in \mathfrak{X}(M)$, the following equation holds:*

$$\widetilde{Rm}(W, X, Y, Z) = Rm(W, X, Y, Z) - \langle \mathrm{II}(W, Z), \mathrm{II}(X, Y)\rangle + \langle \mathrm{II}(W, Y), \mathrm{II}(X, Z)\rangle.$$

There is one other fundamental submanifold equation, which relates the normal part of the ambient curvature endomorphism to derivatives of the second fundamental form. We will not have need for it, but we include it here for completeness. To state it, we need to introduce a connection on the normal bundle of a Riemannian submanifold.

If (M, g) is a Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, the normal connection $\nabla^{\perp} : \mathfrak{X}(M) \times \Gamma(NM) \to \Gamma(NM)$ is defined by

$$\nabla \overset{\perp}{X} N = \left(\widetilde{\nabla}_X N\right)^{\perp},$$

where N is extended to a smooth vector field on a neighborhood of M in $\widetilde{M}$.

Proposition 7.6.6. *If (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, then $\nabla^{\perp}$ is a well-defined connection in NM, which is compatible with $\tilde{g}$ in the sense that for any two sections N_1, N_2 of NM and every $X \in \mathfrak{X}(M)$, we have*

$$X \langle N_1, N_2\rangle = \left\langle \nabla \frac{\perp}{X} N_1, N_2 \right\rangle + \left\langle N_1, \nabla_X^{\perp} N_2 \right\rangle.$$

Exercise 7.6.7. *Prove the preceding proposition.*

We need the normal connection primarily to make sense of tangential covariant derivatives of the second fundamental form. To do so, we make the following definitions. Let $F \to M$ denote the bundle whose fiber at each point $p \in M$ is the set of bilinear maps $T_pM \times T_pM \to N_pM$. It is easy to check that F is a smooth vector bundle over M, and that smooth sections of F correspond to smooth maps $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$ that are bilinear over $C^{\infty}(M)$, such as the second fundamental form. Define a connection ∇^F in F as follows: if B is any smooth section of F, let $\nabla_X^F B$ be the smooth section of F defined by

$$\left(\nabla_X^F B\right)(Y, Z) = \nabla_X^{\perp}(B(Y, Z)) - B\left(\nabla_X Y, Z\right) - B\left(Y, \nabla_X Z\right).$$

Exercise 7.6.8. *Prove that ∇^F is a connection in F.*

Now we are ready to state the last of the fundamental equations for submanifolds.

Theorem 7.6.9 (The Codazzi Equation)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For all $W, X, Y \in \mathfrak{X}(M)$, the following equation holds:*

$$(\widetilde{R}(W, X)Y)^{\perp} = \left(\nabla_W^F \mathrm{II}\right)(X, Y) - \left(\nabla_X^F \mathrm{II}\right)(W, Y). \tag{7.37}$$

7.6.1 Curvature of Curve

By studying the curvatures of curves, we can give a more geometric interpretation of the second fundamental form. Suppose (M, g) is a Riemannian or pseudoRiemannian manifold, and $\gamma : I \to M$ is a smooth unit-speed curve in M. We define the **(geodesic) curvature of** γ as the length of the acceleration vector field, which is the function $\kappa : I \to \mathbb{R}$ given by

$$\kappa(t) = \left|D_t \gamma'(t)\right|.$$

If γ is an arbitrary regular curve in a Riemannian manifold (not necessarily of unit speed), we first find a unit-speed reparametrization $\tilde{\gamma} = \gamma \circ \varphi$, and then define the curvature of γ at t to be the curvature of $\tilde{\gamma}$ at $\varphi^{-1}(t)$.

In a pseudo-Riemannian manifold, the same approach works, but we have to restrict the definition to curves γ such that $|\gamma'(t)|$ is everywhere nonzero. [5] Problem 8-6 gives a formula that can be used in the Riemannian case to compute the geodesic curvature directly without explicitly finding a unit-speed reparametrization.

From the definition, it follows that a smooth unit-speed curve has vanishing geodesic curvature if and only if it is a geodesic, so the geodesic curvature of a curve can be regarded as a quantitative measure of how far it deviates from being a geodesic. If $M = \mathbb{R}^n$ with the Euclidean metric, the geodesic curvature agrees with the notion of curvature introduced in advanced calculus courses.

Now suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian or pseudo-Riemannian manifold and (M, g) is a Riemannian submanifold. Every regular curve $\gamma : I \to M$ has two distinct geodesic curvatures: its **intrinsic curvature** κ as a curve in M, and its **extrinsic curvature** $\widetilde{\kappa}$ as a curve in $\widetilde{M}$. The second fundamental form can be used to compute the relationship between the two.

Proposition 7.6.10 (Geometric Interpretation of II)**.** *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g}), p \in M$, and $v \in T_pM$.*

(a) $\mathrm{II}(v, v)$ *is the $\tilde{g}$-acceleration at p of the g-geodesic γ_v.*

(b) If v is a unit vector, then $|\,\mathrm{II}(v, v)|$ is the $\tilde{g}$-curvature of γ_v at p.

Note that the second fundamental form is completely determined by its values of the form $\mathrm{II}(v, v)$ for unit vectors v, by the following lemma.

Lemma 7.6.11. *Suppose V is an inner product space, W is a vector space, and $B, B' : V \times V \to W$ are symmetric and bilinear. If $B(v, v) = B'(v, v)$ for every unit vector $v \in V$, then $B = B'$.*

Because the intrinsic and extrinsic accelerations of a curve are usually different, it is generally not the case that a $\widetilde{g}$-geodesic that starts tangent to M stays in M; just think of a sphere in Euclidean space, for example. A Riemannian submanifold (M, g) of $(\widetilde{M}, \widetilde{g})$ is said to be **totally geodesic** if every $\widetilde{g}$-geodesic that is tangent to M at some time t_0 stays in M for all t in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Proposition 7.6.12. *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, The following are equivalent:*

(a) M is totally geodesic in $\widetilde{M}$.

(b) Every g-geodesic in M is also a $\tilde{g}$-geodesic in $\widetilde{M}$.

(c) The second fundamental form of M vanishes identically.

7.7 Hypersurfaces

Now we specialize the preceding considerations to the case in which M is a hypersurface (i.e., a submanifold of codimension 1) in $\widetilde{M}$. Throughout this section, our default assumption is that (M, g) is an embedded n-dimensional Riemannian submanifold of an $(n+1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. (The analogous formulas in the pseudo-Riemannian case are a little different; see [5] Problem 8-19.)

In this situation, at each point of M there are exactly two unit normal vectors. In terms of any local adapted orthonormal frame $(E_1, \ldots, E_{n+1})$, the two choices are $\pm E_{n+1}$. In a small enough neighborhood of each point of M, therefore, we can always choose a smooth unit normal vector field along M.

If both M and $\widetilde{M}$ are orientable, we can use an orientation to pick out a global smooth unit normal vector field along all of M. In general, though, this might or might not be possible. Since all of our computations in this chapter are local, we will always assume that we are working in a small enough neighborhood that a smooth unit normal field exists. We will address as we go along the question of how various quantities depend on the choice of normal vector field.

7.7.1 The Scalar Second Fundamental Form and the Shape Operator

Having chosen a distinguished smooth unit normal vector field N on the hypersurface $M \subseteq \widetilde{M}$, we can replace the vector-valued second fundamental form II by a simpler scalar-valued form. The **scalar second fundamental form of** M is the symmetric covariant 2-tensor field $h \in \Gamma\left(\Sigma^2 T^* M\right)$ defined by $h = \mathrm{II}_N$ (see (7.34)); in other words,

$$h(X, Y) = \langle N, \mathrm{II}(X, Y)\rangle. \tag{7.38}$$

Using the Gauss formula $\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)$ and noting that $\nabla_X Y$ is orthogonal to N, we can rewrite the definition as

$$h(X, Y) = \left\langle N, \widetilde{\nabla}_X Y\right\rangle. \tag{7.39}$$

Also, since N is a unit vector spanning NM at each point, the definition of h is equivalent to

$$\mathrm{II}(X, Y) = h(X, Y)N. \tag{7.40}$$

Note that replacing N by $-N$ multiplies h by -1, so the sign of h depends on which unit normal is chosen; but h is otherwise independent of the choices.

The choice of unit normal field also determines a Weingarten map $W_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by (**??**); in the case of a hypersurface, we use the notation $s = W_N$ and call it the **shape operator of** M. Alternatively, we can think of s as the $(1,1)$-tensor field on M obtained from h by raising an index. It is characterized by

$$\langle sX, Y\rangle = h(X, Y) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Because h is symmetric, s is a self-adjoint endomorphism of TM, that is,

$$\langle sX, Y\rangle = \langle X, sY\rangle \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

As with h, the sign of s depends on the choice of N.

Remark 7.7.1. We can think of s as the $(1,1)$-tensor field on M obtained from h by raising an index. In fact, by mimicing Example 2.3.4 and using (2.5), we see that

$$\begin{aligned}
&h_i{}^j = (h^\sharp)_i{}^j = g^{jl} h_{il} \\
\Longrightarrow\ & h^\flat = h_i{}^j dx^i \otimes \partial_j = g^{il} h_{il} dx^i \otimes \partial_j \\
\Longrightarrow\ & s(X) = \Psi(h^\sharp)(X) = \Psi(g^{il} h_{il} dx^i \otimes \partial_j)(X) \\
&\quad = g^{il} h_{il} [\Psi(dx^i \otimes \partial_j)](X) = g^{il} h_{il} dx^i(X) \partial_j \\
&\quad = g^{il} h_{il} X^i \partial_j \\
\Longrightarrow\ & \langle sX, Y\rangle = \langle g^{jl} h_{il} X^i \partial_j, Y^k \partial_k\rangle = g^{il} g_{jk} h_{il} X^i Y^k \\
&\quad = \delta_k^l h_{il} X^i Y^k = h_{ik} X^i Y^k = h(X, Y)
\end{aligned}$$

where Ψ is the isomorphism from $T^{(1,1)}(TM)$ to $\mathrm{End}(TM)$. ♠

In terms of the tensor fields h and s, the formulas of the last section can be rewritten somewhat more simply. For this purpose, we will use the **Kulkarni-Nomizu product** of symmetric 2-tensors h, k:

$$\begin{aligned}
h \owedge k(w, x, y, z) = {} & h(w, z)k(x, y) + h(x, y)k(w, z) \\
& - h(w, y)k(x, z) - h(x, z)k(w, y),
\end{aligned}$$

and the **exterior covariant derivative** of a smooth symmetric 2-tensor field T is

$$(DT)(x, y, z) = -(\nabla T)(x, y, z) + (\nabla T)(x, z, y).$$

Theorem 7.7.2 (Fundamental Equations for a Hypersurface). *Suppose (M, g) is a Riemannian hypersurface in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and N is a smooth unit normal vector field along M.*

(a) THE GAUSS FORMULA FOR A HYPERSURFACE: If $X, Y \in \mathfrak{X}(M)$ are extended to an open subset of $\widetilde{M}$, then

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N.$$

(b) THE GAUSS FORMULA FOR A CURVE IN A HYPERSURFACE: If $\gamma : I \to M$ is a smooth curve and $X : I \to TM$ is a smooth vector field along γ, then

$$\widetilde{D}_t X = D_t X + h(\gamma', X) N.$$

(c) The WEINGARTEN EQUATION FOR A HYPERSURFACE: For every $X \in \mathfrak{X}(M)$,

$$\tilde{\nabla}_X N = -sX$$

(d) The GAUSS EQUATION FOR A HYPERSURFACE: For all $W, X, Y, Z \in \mathfrak{X}(M)$,

$$\widetilde{Rm}(W, X, Y, Z) = Rm(W, X, Y, Z) - \frac{1}{2}(h \owedge h)(W, X, Y, Z).$$

(e) THE CODAZZI EQUATION FOR A HYPERSURFACE: For all $W, X, Y \in \mathfrak{X}(M)$,

$$\widetilde{Rm}(W, X, Y, N) = (Dh)(Y, W, X).$$

7.7.2 Principal Curvatures

At every point $p \in M$, we have seen that the shape operator s is a self-adjoint linear endomorphism of the tangent space T_pM. To analyze such an operator, we recall some linear-algebraic facts about self-adjoint endomorphisms.

Lemma 7.7.3. *Suppose V is a finite-dimensional inner product space and $s : V \to V$ is a self-adjoint linear endomorphism. Let C denote the set of unit vectors in V. There is a vector $v_0 \in C$ where the function $v \mapsto \langle sv, v\rangle$ achieves its maximum among elements of C, and every such vector is an eigenvector of s with eigenvalue $\lambda_0 = \langle sv_0, v_0\rangle$.*

Proposition 7.7.4 (Finite-Dimensional Spectral Theorem). *Suppose V is a finitedimensional inner product space and $s : V \to V$ is a self-adjoint linear endomorphism. Then V has an orthonormal basis of s-eigenvectors, and all of the eigenvalues are real.*

Proof. The proof is by induction on $n = \dim V$. The $n = 1$ result is easy, so assume that the theorem holds for some $n \geq 1$ and suppose $\dim V = n + 1$. Above lemma shows that s has a unit eigenvector b_0 with a real eigenvalue λ_0. Let $B \subseteq V$ be the span of b_0. Since $s(B) \subseteq B$, self-adjointness of s implies $s\left(B^\perp\right) \subseteq B^\perp$. The inductive hypothesis applied to $s|_B \perp$ implies that $B^\perp$ has an orthonormal basis $(b_1, \ldots, b_n)$ of s-eigenvectors with real eigenvalues, and then $(b_0, b_1, \ldots, b_n)$ is the desired basis of V. ■

Applying this proposition to the shape operator $s : T_pM \to T_pM$, we see that s has real eigenvalues $\kappa_1, \ldots, \kappa_n$, and there is an orthonormal basis $(b_1, \ldots, b_n)$ for T_pM consisting of s-eigenvectors, with $sb_i = \kappa_i b_i$ for each i (no summation). In this basis, both h and s are represented by diagonal matrices, and h has the expression

$$h(v, w) = \kappa_1 v^1 w^1 + \cdots + \kappa_n v^n w^n.$$

The eigenvalues of s at a point $p \in M$ are called the **principal curvatures of** M **at** p, and the corresponding eigenspaces are called the principal directions. The principal curvatures all change sign if we reverse the normal vector, but the principal directions and principal curvatures are otherwise independent of the choice of coordinates or bases.

There are two combinations of the principal curvatures that play particularly important roles for hypersurfaces. The **Gaussian curvature** is defined as $K = \det(s)$, and the **mean curvature** as $H = (1/n)\operatorname{tr}(s) = (1/n)\operatorname{tr}_g(h)$. Since the determinant and trace of a linear endomorphism are basis-independent, these are well defined once a unit normal is chosen. In terms of the principal curvatures, they are

$$K = \kappa_1\kappa_2\cdots\kappa_n, \quad H = \frac{1}{n}\left(\kappa_1 + \cdots + \kappa_n\right),$$

as can be seen by expressing s in terms of an orthonormal basis of eigenvectors. If N is replaced by $-N$, then H changes sign, while K is multiplied by $(-1)^n$.

7.7.3 Hypersurfaces in Euclidean Space

Now we specialize even further, to hypersurfaces in Euclidean space. In this section, we assume that $M \subseteq \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold with the induced Riemannian metric. The Euclidean metric will be denoted as usual by $\bar{g}$, and covariant derivatives and curvatures associated with $\bar{g}$ will be indicated by a bar. The induced metric on M will be denoted by g.

In this setting, because $\overline{Rm} \equiv 0$, the Gauss and Codazzi equations take even simpler forms:

$$\frac{1}{2}h \owedge h = Rm, \tag{7.41}$$

$$Dh = 0, \tag{7.42}$$

or in terms of a local frame for M,

$$h_{il}h_{jk} - h_{ik}h_{jl} = R_{ijkl}, \tag{7.43}$$

$$h_{ij;k} - h_{ik;j} = 0. \tag{7.44}$$

In particular, this means that the Riemann curvature tensor of a hypersurface in $\mathbb{R}^{n+1}$ is completely determined by the second fundamental form. A symmetric 2-tensor field that satisfies $Dh = 0$ is called a Codazzi tensor, so $Dh = 0$ can be expressed succinctly by saying that h is a Codazzi tensor.

Exercise 7.7.5. *Show that a smooth 2-tensor field h on a Riemannian manifold is a Codazzi tensor if and only if both h and ∇h are symmetric.*

The equations $\frac{1}{2}h \owedge h = Rm$ and $Dh = 0$ can be viewed as compatibility conditions for the existence of an embedding or immersion into Euclidean space with prescribed first and second fundamental forms. If (M, g) is a Riemannian n-manifold and h is a given smooth symmetric 2-tensor field on M, then Theorem 8.13 shows that these two equations are necessary conditions for the existence of an isometric immersion $M \to \mathbb{R}^{n+1}$ for which h is the scalar second fundamental form. (Note that an immersion is locally an embedding, so the theorem applies in a neighborhood of each point.) It is a remarkable fact that the Gauss and Codazzi equations are actually sufficient, at least locally. A sketch of a proof of this fact, called the fundamental theorem of hypersurface theory, can be found in [Pet16, pp. 108-109].

In the setting of a hypersurface $M \subseteq \mathbb{R}^{n+1}$, we can give some very concrete geometric interpretations of the quantities we have defined so far. We begin with curves. For every unit vector $v \in T_pM$, let $\gamma = \gamma_v : I \to M$ be the g-geodesic in M with initial velocity v. Then the Gauss formula shows that the ordinary Euclidean acceleration of γ at 0 is $\gamma''(0) = \bar{D}_t\gamma'(0) = h(v, v)N_p$. Thus $|h(v, v)|$ is the Euclidean curvature of γ at 0, and $h(v, v) = \langle\gamma''(0), N_p\rangle > 0$ if and only if $\gamma''(0)$ points in the same direction as N_p. In other words, $h(v, v)$ is positive if γ is curving in the direction of N_p, and negative if it is curving away from N_p.

Proposition 7.7.6. *Suppose $\gamma : I \to \mathbb{R}^m$ is a unit-speed curve, $t_0 \in I$, and $\kappa(t_0) \neq 0$.*

(a) There is a unique unit-speed parametrized circle $c : \mathbb{R} \to \mathbb{R}^m$, called the ***osculating circle*** *at $\gamma(\boldsymbol{t_0})$, with the property that c and γ have the same position, velocity, and acceleration at $t = t_0$.*

(b) The Euclidean curvature of γ at t_0 is $\kappa(t_0) = 1/R$, where R is the radius of the osculating circle.

Proof. An easy geometric argument shows that every circle in $\mathbb{R}^m$ with center q and radius R has a unit-speed parametrization of the form

$$c(t) = q + R\cos\left(\frac{t - t_0}{R}\right) v + R\sin\left(\frac{t - t_0}{R}\right) w,$$

where (v, w) is a pair of orthonormal vectors in $\mathbb{R}^m$. By direct computation, such a parametrization satisfies

$$c(t_0) = q + Rv, \quad c'(t_0) = w, \quad c''(t_0) = -\frac{1}{R} v.$$

Thus if we put

$$R = \frac{1}{|\gamma''(t_0)|} = \frac{1}{\kappa(t_0)}, \quad v = -R\gamma''(t_0), \quad w = \gamma'(t_0), \quad q = \gamma(t_0) - Rv$$

we obtain a circle satisfying the required conditions, and its radius is equal to $1/\kappa(t_0)$ by construction. Uniqueness is left as an exercise. ■

Exercise 7.7.7. *Complete the proof of the preceding proposition by proving uniqueness of the osculating circle.*

7.7.4 Computations in Euclidean Space

When we wish to compute the invariants of a Euclidean hypersurface $M \subseteq \mathbb{R}^{n+1}$, it is usually unnecessary to go to all the trouble of computing Christoffel symbols. Instead, it is usually more effective to use either a defining function or a parametrization to compute the scalar second fundamental form, and then use (**??**) to compute the curvature. Here we describe several contexts in which this computation is not too hard.

Usually the computations are simplest if the hypersurface is presented in terms of a local parametrization. Suppose $M \subseteq \mathbb{R}^{n+1}$ is a smooth embedded hypersurface, and let $X : U \to \mathbb{R}^{n+1}$ be a smooth local parametrization of M. The coordinates $\left(u^1, \ldots, u^n\right)$ on $U \subseteq \mathbb{R}^n$ thus give local coordinates for M. The coordinate vector fields $\partial_i = \partial/\partial u^i$ push forward to vector fields $dX(\partial_i)$ on M, which we can view as sections of the restricted tangent bundle $T\mathbb{R}^{n+1}\big|_M$, or equivalently as $\mathbb{R}^{n+1}$-valued functions. If we think of $X(u) = \left(X^1(u), \ldots, X^{n+1}(u)\right)$ as a vector-valued function of u, these vectors can be written as

$$dX_u(\partial_i) = \partial_i X(u) = \left(\partial_i X^1(u), \ldots, \partial_i X^{n+1}(u)\right).$$

For simplicity, write $X_i = \partial_i X$. Once these vector fields are computed, a unit normal field can be computed as follows: Choose any coordinate vector field $\partial/\partial x^{j_0}$ that is not contained in span $(X_1, \ldots, X_n)$ (there will always be one, at least in a neighborhood of each point). Then apply the Gram-Schmidt algorithm to the local frame $\left(X_1, \ldots, X_n, \partial/\partial x^{j_0}\right)$ along M to obtain an adapted orthonormal frame $(E_1, \ldots, E_{n+1})$. The two choices of unit normal are $N = \pm E_{n+1}$.

The next proposition gives a formula for the second fundamental form that is often easy to use for computation.

Proposition 7.7.8. *Suppose $M \subseteq \mathbb{R}^{n+1}$ is an embedded hypersurface, $X : U \to M$ is a smooth local parametrization of $M, (X_1, \ldots, X_n)$ is the local frame for TM determined by X, and N is a unit normal field on M. Then the scalar second fundamental form is given by*

$$h\left(X_i, X_j\right) = \left\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, N \right\rangle.$$

Here is another approach. When it is practical to write down a smooth vector field $N = N^i \partial_i$ on an open subset of $\mathbb{R}^{n+1}$ that restricts to a unit normal vector field along M, then the shape operator can be computed straightforwardly using the Weingarten equation and observing that the Euclidean covariant derivatives of N are just ordinary directional derivatives in Euclidean space. Thus for every vector $X = X^j \partial_j$ tangent to M, we have

$$sX = -\bar{\nabla}_X N = -\sum_{i,j=1}^{n+1} X^j \left(\partial_j N^i\right) \partial_i$$

One common way to produce such a smooth vector field is to work with a local defining function for M : Recall that this is a smooth real-valued function defined on some open subset $U \subseteq \mathbb{R}^{n+1}$ such that $U \cap M$ is a regular level set of F (see [5] Prop. A.27). The definition ensures that grad F (the gradient of F with respect to $\bar{g}$) is nonzero on some neighborhood of $M \cap U$, so a convenient choice for a unit normal vector field along M is

$$N = \frac{\operatorname{grad} F}{|\operatorname{grad} F|}$$

Here is an application.

Example 7.7.9 (Shape Operators of Spheres)**.** The function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(x) = |x|^2$ is a smooth defining function for each sphere $\mathbb{S}^n(R)$. The gradient of this function is grad $F = 2\sum_i x^i \partial_i$, which has length $2R$ along $\mathbb{S}^n(R)$. The smooth vector field

$$N = \frac{1}{R} \sum_{i=1}^{n+1} x^i \partial_i$$

thus restricts to a unit normal along $\mathbb{S}^n(R)$. (It is the outward pointing normal.) The shape operator is now easy to compute:

$$sX = -\frac{1}{R} \sum_{i,j=1}^{n+1} X^j \left(\partial_j x^i\right) \partial_i = -\frac{1}{R} X.$$

Therefore $s = (-1/R)$ Id. The principal curvatures, therefore, are all equal to $-1/R$, and it follows that the mean curvature is $H = -1/R$ and the Gaussian curvature is $(-1/R)^n$. ♣

For surfaces in $\mathbb{R}^3$, either of the above methods can be used. When a parametrization X is given, the normal vector field is particularly easy to compute: because X_1 and X_2 span the tangent space to M at each point, their cross product is a nonzero normal vector, so one choice of unit normal is

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

7.7.5 The Gaussian Curvature of a Surface Is Intrinsic

Because the Gaussian and mean curvatures are defined in terms of a particular embedding of M into $\mathbb{R}^{n+1}$, there is little reason to suspect that they have much to do with the intrinsic Riemannian geometry of (M, g). The next exercise illustrates the fact that the mean curvature has no intrinsic meaning.

Exercise 7.7.10. *Let $M_1 \subseteq \mathbb{R}^3$ be the plane $\{z=0\}$, and let $M_2 \subseteq \mathbb{R}^3$ be the cylinder $\left\{x^2+y^2=1\right\}$. Show that M_1 and M_2 are locally isometric, but the former has mean curvature zero, while the latter has mean curvature $\pm\frac{1}{2}$, depending on which normal is chosen.*

The amazing discovery made by Gauss was that the Gaussian curvature of a surface in $\mathbb{R}^3$ is actually an intrinsic invariant of the Riemannian manifold (M, g). He was so impressed with this discovery that he called it Theorema Egregium, Latin for "excellent theorem."

Theorem 7.7.11 (Gauss's Theorema Egregium). *Suppose (M,g) is an embedded 2-dimensional Riemannian submanifold of $\mathbb{R}^3$. For every $p \in M$, the Gaussian curvature of M at p is equal to one-half the scalar curvature of g at p, and thus the Gaussian curvature is a local isometry invariant of (M, g).*

Motivated by the Theorema Egregium, for an abstract Riemannian 2-manifold (M, g), not necessarily embedded in $\mathbb{R}^3$, we define the Gaussian curvature to be $K=\frac{1}{2} S$, where S is the scalar curvature. If M is a Riemannian submanifold of $\mathbb{R}^3$, then the Theorema Egregium shows that this new definition agrees with the original definition of K as the determinant of the shape operator.

Corollary 7.7.12. *If (M, g) is a Riemannian 2-manifold, the following relationships hold:*

$$
Rm=\frac{1}{2} K g \owedge g, \quad Rc=K g, \quad S=2 K .
$$

7.8 Sectional Curvature

Now, finally, we can give a quantitative geometric interpretation to the curvature tensor in dimensions higher than 2 . Suppose M is a Riemannian n-manifold (with $n \geq 2$), p is a point of M, and $V \subseteq T_p M$ is a star-shaped neighborhood of zero on which $\exp_p$ is a diffeomorphism onto an open set $U \subseteq M$. Let Π be any 2dimensional linear subspace of $T_p M$. Since $\Pi \cap V$ is an embedded 2-dimensional submanifold of V, it follows that $S_{\Pi}=\exp_p(\Pi \cap V)$ is an embedded 2-dimensional submanifold of $U \subseteq M$ containing p (Fig. 8.5), called the plane section determined by Π. Note that S_{Π} is just the set swept out by geodesics whose initial velocities lie in Π, and $T_p S_{\Pi}$ is exactly Π.

We define the sectional curvature of Π, denoted by $\sec(\Pi)$, to be the intrinsic Gaussian curvature at p of the surface S_{Π} with the metric induced from the embedding $S_{\Pi} \subseteq M$. If (v, w) is any basis for Π, we also use the notation $\sec(v, w)$ for $\sec(\Pi)$.

The next theorem shows how to compute the sectional curvatures in terms of the curvature of (M, g). To make the formula more concise, we introduce the following notation. Given vectors v, w in an inner product space V, we set

$$
|v \wedge w|=\sqrt{|v|^2|w|^2-\langle v, w\rangle^2}
$$

It follows from the Cauchy-Schwarz inequality that $|v \wedge w| \geq 0$, with equality if and only if v and w are linearly dependent, and $|v \wedge w|=1$ when v and w are orthonormal.

Proposition 7.8.1 (Formula for the Sectional Curvature). *Let (M, g) be a Riemannian manifold and $p \in M$. If v, w are linearly independent vectors in $T_p M$, then the sectional curvature of the plane spanned by v and w is given by*

$$
\sec(v, w)=\frac{Rm_p(v, w, w, v)}{|v \wedge w|^2} \tag{7.45}
$$

Exercise 7.8.2. *Suppose (M, g) is a Riemannian manifold and $\widetilde{g}=\lambda g$ for some positive constant λ. Use Theorem 7.30 to prove that for every $p \in M$ and plane $\Pi \subseteq T_p M$, the sectional curvatures of Π with respect to $\widetilde{g}$ and g are related by $\widetilde{\sec}(\Pi)=\lambda^{-1} \sec(\Pi)$.*

The formula for the sectional curvature shows that one important piece of quantitative information provided by the curvature tensor is that it encodes the sectional curvatures of all plane sections. It turns out, in fact, that this is all of the information contained in the curvature tensor: as the following proposition shows, the sectional curvatures completely determine the curvature tensor.

Proposition 7.8.3. *Suppose R_1 and R_2 are algebraic curvature tensors on a finitedimensional inner product space V. Iffor every pair of linearly independent vectors $v, w \in V$,*

$$\frac{R_1(v,w,w,v)}{|v \wedge w|^2} = \frac{R_2(v,w,w,v)}{|v \wedge w|^2}$$

then $R_1 = R_2$.

Proposition 7.8.4 (Geometric Interpretation of Ricci and Scalar Curvatures)**.** *Let (M,g) be a Riemannian n-manifold and $p \in M$.*

(a) For every unit vector $v \in T_pM, Rc_p(v,v)$ is the sum of the sectional curvatures of the 2-planes spanned by $(v,b_2),\ldots,(v,b_n)$, where $(b_1,\ldots,b_n)$ is any orthonormal basis for T_pM with $b_1 = v$.

(b) The scalar curvature at p is the sum of all sectional curvatures of the 2-planes spanned by ordered pairs of distinct basis vectors in any orthonormal basis.

Proof. Given any unit vector $v \in T_pM$, let $(b_1,\ldots,b_n)$ be as in the hypothesis. Then $Rc_p(v,v)$ is given by

$$Rc_p(v,v) = R_{11}(p) = R_{k11}{}^{k}(p) = \sum_{k=1}^{n} Rm_p\left(b_k,b_1,b_1,b_k\right) = \sum_{k=2}^{n} \sec\left(b_1,b_k\right)$$

For the scalar curvature, we let $(b_1,\ldots,b_n)$ be any orthonormal basis for T_pM, and compute

$$\begin{aligned} S(p) = R_j{}^{j}(p) = \sum_{j=1}^{n} Rc_p\left(b_j,b_j\right) &= \sum_{j,k=1}^{n} Rm_p\left(b_k,b_j,b_j,b_k\right) \\ = \sum_{j\neq k} \sec\left(b_j,b_k\right). \end{aligned}$$

■

One consequence of this proposition is that if (M,g) is a Riemannian manifold in which all sectional curvatures are positive, then the Ricci and scalar curvatures are both positive as well. The analogous statement holds if “positive” is replaced by “negative,” “nonpositive,” or “nonnegative.”

If the opposite sign convention is chosen for the curvature tensor, then the righthand side of formula (7.45) has to be adjusted accordingly, with $Rm_p(v,w,v,w)$ taking the place of $Rm_p(v,w,w,v)$. This is so that whatever sign convention is chosen for the curvature tensor, the notion of positive or negative sectional, Ricci, or scalar curvature has the same meaning for everyone.

7.8.1 Sectional Curvatures of the Model Spaces

7.9 Problems

Chapter 8

Laplacian on Riemannian Manifolds

8.1 Basic Examples

[8] chapter 1 Basic Examples

8.2 Hilbert Spaces Associated to a Compact Riemannian Manifold

[1] chapter 2 section 1

8.3 Some Canonical Differential Operators on a Riemannian Manifold

[1] chapter 2 section 2

8.4 Heat Kernel

[8] chapter 3

8.5 Atiyah-Singer Index Theorem

[8] chapter 4

Chapter 9

Jacobi Fields

Chapter 10

Curvature and Topology

Chapter 11

Appendix

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