Measures, Dimensions and Analytic Capacity

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April 4, 2024

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Contents

Chapter 1

Measures and Integrations

We shall work in a metric space X with a metric d, although most of the measure theory presented here goes through in more general settings.

The closed and open balls with centre $x \in X$ and radius $r, 0 < r < \infty$, are denoted by

$$B(x,r) = \{y \in X : d(x,y) \leq r\}$$
$$U(x,r) = \{y \in X : d(x,y) < r\}$$

In $\mathbb{R}^n$ we also set

$$B(r) = B(0,r), U(r) = U(0,r), S(x,r) = \partial B(x,r) \text{ and } S(r) = S(0,r).$$

The diameter of a non-empty subset A of X is

$$d(A) = \sup\{d(x,y) : x, y \in A\}.$$

We agree $d(\varnothing) = 0$. If $x \in X$ and A and B are non-empty subsets of X, the distance from x to A and the distance between A and B are, respectively,

$$d(x,A) = \inf\{d(x,y) : y \in A\},$$
$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

For $\varepsilon > 0$ the closed ε-neighbourhood of A is

$$A(\varepsilon) = \{x \in X : d(x,A) \leq \varepsilon\}.$$

1.1 Measures

A measure for us will be a non-negative, monotonic, subadditive set function vanishing for the empty set.

Definition 1.1.1. A set function $\mu : \{A : A \subset X\} \to [0,\infty] = \{t : 0 \leq t \leq \infty\}$ is called a **measure** if
(1) $\mu(\varnothing) = 0$,

(2) $\mu(A) \leq \mu(B) \quad$ whenever $A \subset B \subset X$,

(3) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i\right) \quad$ whenever $A_1, A_2, \cdots \subset X$.

Usually in measure theory a measure means a non-negative countably additive set function defined on some σ-algebra of subsets of X, which need not be the whole power set $\{A : A \subset X\}$. However, considering measures in the sense of Definition 1.1.1 is a convenience rather than a restriction. That is, if ν is a countably additive non-negative set function on a σ-algebra $\mathcal{A}$ of subsets of X, it can be extended to a measure ν^* on X (in the sense of Definition 1.1.1) by

$$\nu^*(A) = \inf\{\nu(B) : A \subset B \in \mathcal{A}\}.$$

Exercise 1.1.2. Show that ν^* defined above is a measure agreeing with ν on $\mathcal{A}$, and, moreover, that ν^* is Borel regular if $\mathcal{A}$ is contained in the family of Borel sets.

On the other hand, a measure μ gives a countably additive set function when restricted to the σ-algebra of μ measurable sets.

Definition 1.1.3. A set $A \subset X$ is μ **measurable** if

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A) \quad \text{for all } E \subset X.$$

We collect the well-known basic properties of measurable sets in the following theorem.

Theorem 1.1.4. Let μ be a measure on X and let $\mathcal{M}$ be the family of all μ measurable subsets of X.

(1) $\mathcal{M}$ is a σ-algebra, that is,

(i) $\varnothing \in \mathcal{M}$ and $X \in \mathcal{M}$,

(ii) if $A \in \mathcal{M}$, then $X \backslash A \in \mathcal{M}$,

(iii) if $A_1, A_2, \cdots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

(2) If $\mu(A) = 0$, then $A \in \mathcal{M}$.

(3) If $A_1, A_2, \cdots \in \mathcal{M}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

(4) If $A_1, A_2, \cdots \in \mathcal{M}$, then

(i) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ provided $A_1 \subset A_2 \subset \ldots$,

(ii) $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ provided $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1) < \infty$.

It is also good to remember that the first statement of (4) holds without the measurability assumption if μ is **regular**, that is, for every $A \subset X$ there is a μ measurable set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

Recall that the family of **Borel sets** in X is the smallest σ-algebra containing the open (or equivalently closed) subsets of X. We shall often consider measures with some of the following properties.

Definition 1.1.5. Let μ be a measure on X.

(1) μ is **locally finite** if for every $x \in X$ there is $r > 0$ such that

$$\mu(B(x, r)) < \infty.$$

(2) μ is a **Borel measure** if all Borel sets are μ measurable, i.e., $\mathrm{Bor}(X) \subseteq \mathcal{M}$.

(3) μ is **Borel regular** if it is a Borel measure and if for every $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

(4) μ is a **Radon measure** if it is a Borel measure and

(i) $\mu(K) < \infty$ for compact sets $K \subset X$,

(ii) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact }\}$ for open sets $V \subset X$,

(iii) $\mu(A) = \inf\{\mu(V) : A \subset V, V \text{ is open }\}$ for $A \subset X$.

We shall give a few simples examples. Many others will be encountered later on.

Example 1.1.6.
(1) The **Lebesgue measure** $\mathcal{L}^n$ on $\mathbb{R}^n$ is a Radon measure.

(2) The **Dirac measure** δ_a at a point $a \in X$ is defined by $\delta_a(A) = 1$, if $a \in A, \delta_a(A) = 0$, if $a \notin A$ (that is, $\delta_a(A) = \chi_A(a)$). It is a Radon measure on any metric space X.

(3) The **counting measure** n on X is defined by letting $n(A)$ be the number of elements in A, possibly ∞. It is Borel regular on any metric space X, but it is a Radon measure only if every compact subset of X is finite, that is, X is discrete.

In general, Radon measures are always Borel regular as a rather immediate consequence of the definition. The converse is not true as the above example (3) shows. Clearly in $\mathbb{R}^n$ the local finiteness means that compact sets have finite measure.

A measure μ on X is called a **metric outer measure** if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $A, B \subseteq X$ are **positively separated**, i.e., $d(A, B) > 0$.

Theorem 1.1.7 (Carathéodory's criterion)**.** Suppose μ is a measure in X, then it is a Borel measures if and only if it is a metric outer measure.

Proof. Theorem 1.5 of [1] (but note that measure here is called outer measure in [1]) shows that every metric outer measure is a Borel measure.

Conversely, suppose μ is a Borel measure. Let $A, B \subseteq X$ with $m := d(A, B) > 0$. Then define

$$U = \bigcup_{x \in A} B\left(x, \frac{m}{2}\right).$$

Clearly, $A \subseteq U$, $B \cap U = \varnothing$, and U is open and thus $U \in \operatorname{Bor}(X) \subseteq \mathcal{M}_\mu$. Therefore, by measurability of U, we have

$$\mu(A \cup B) = \mu((A \cup B) \cap U) + \mu((A \cup B) \cap U^c) = \mu(A) + \mu(B).$$

□

Given a measure μ and a subset A of X we can form a new measure by restricting μ to A.

Definition 1.1.8. The **restriction of a measure** μ to a set $A \subset X, \mu \llcorner A$, is defined by

$$(\mu \llcorner A)(B) = \mu(A \cap B) \quad \text{for } B \subset X.$$

It is clear that $\mu L A$ is a measure. Many of the relations between μ and $\mu \llcorner A$ are easy to derive. For example,

Theorem 1.1.9.

(1) Every μ measurable set is also $\mu \llcorner A$ measurable.

(2) If μ is Borel regular and A is μ measurable with $\mu(A) < \infty$, then $\mu \llcorner A$ is Borel regular.

Proof. The first statement is readily checked from the definitions. Note that A can be quite arbitrary there. We prove the second part.

Let B be a Borel set with $A \subset B$ and $\mu(A) = \mu(B)$. Then $\mu(B \backslash A) = 0$. Given $C \subset X$ let D be a Borel set with $B \cap C \subset D$ and $\mu(B \cap C) = \mu(D)$. Then $C \subset D \cup (X \backslash B) = E$, say, and

$$\begin{aligned}(\mu \llcorner A)(E) &\leq \mu(B \cap E) = \mu(B \cap D) \leq \mu(D)\\ &= \mu(B \cap C) = \mu(A \cap C) = (\mu \llcorner A)(C).\end{aligned}$$

Thus $(\mu \llcorner A)(E) = (\mu \llcorner A)(C)$, and so $\mu \llcorner A$ is Borel regular. □

The following approximation theorem will be extremely useful, see [2] Theorem 2.2.2 for instance.

Theorem 1.1.10. Let μ be a Borel regular measure on X, A a μ measurable set, and $\varepsilon > 0$.

(1) If $\mu(A) < \infty$, there is a closed set $C \subset A$ such that $\mu(A \backslash C) < \varepsilon$.

(2) If there are open sets $V_1, V_2, \ldots$ such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < \infty$ for all i, then there is an open set V such that $A \subset V$ and $\mu(V \backslash A) < \varepsilon$.

Note. The result holds for any Borel measure provided A is a Borel set. In $\mathbb{R}^n$ it follows immediately that the set C in (1) can be taken to be compact. This holds of course in any σ**-compact space** X, where every closed set is a countable union of compact sets.

Corollary 1.1.11. A measure μ on $\mathbb{R}^n$ is a Radon measure if and only if it is locally finite and Borel regular.

The proof is left as an exercise.

In what follows we shall mainly work with Borel regular measures or Radon measures for convenience. But often they could quite easily be replaced by Borel measures or locally finite Borel measures, for example with the help of Exercise 1.1.2.

We shall often encounter measures μ which are carried by a proper subset F of X, that is, $\mu(X \backslash F) = 0$. It is not hard to see that in the case where μ is a Borel measure and X is separable, there exists a unique smallest closed set with this property.

Definition 1.1.12. If μ is a Borel measure on a separable metric space X, the **support** of μ, spt μ, is the smallest closed set F such that $\mu(X \backslash F) = 0$. In other words,

$$\begin{aligned}\operatorname{spt} \mu &= X \backslash \bigcup \{V : V \text{ open}, \mu(V) = 0\}\\ &= X \backslash \{x : \exists r > 0 \text{ such that } \mu(B(x, r)) = 0\}.\end{aligned}$$

Example 1.1.13.
(1) Let f be a non-negative continuous function on $\mathbb{R}^n$. Define a measure μ_f by

$$\mu_f(A) = \int_A f d\mathcal{L}^n$$

for $\mathcal{L}^n$ measurable sets A. Then the support of μ_f agrees with that of f :

$$\operatorname{spt} \mu_f = \operatorname{spt} f = \operatorname{Cl}\{x : f(x) \neq 0\},$$

where Cl refers to closure.

(2) Let $Q = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers, and

$$\mu = \sum_{i=1}^{\infty} 2^{-i} \delta_{q_i},$$

where δ_{q_i} is the Dirac measure at q_i. Then μ is a finite Radon measure on $\mathbb{R}$ with spt $\mu = \mathbb{R}$. Nevertheless, μ is carried by the countable set Q in the sense that $\mu(\mathbb{R} \backslash Q) = 0$.

1.2 Integration

The integral

$$\int_A f d\mu = \int_A f(x) d\mu x$$

with respect to a measure μ over a set A of a function f is defined in the usual way, as well as the μ measurability and integrability of f. When the domain of the integration A is the whole space X, we often omit it using the notation

$$\int f d\mu = \int_X f d\mu$$

In $\mathbb{R}^n$ we abbreviate the Lebesgue integral

$$\int_A f(x) dx = \int_A f(x) d\mathcal{L}^n x.$$

The integral $\int f d\mu$ is defined for any non-negative μ measurable function on X. Even when $f : X \to [0, \infty]$ is not μ measurable we can define the lower and upper integrals by

$$\int_* f d\mu = \sup_\varphi \int \varphi d\mu \text{ and } \int^* f d\mu = \inf_\psi \int \psi d\mu,$$

where φ and ψ run through the μ measurable functions $X \to [0, \infty]$ such that $\varphi \leq f \leq \psi$.

The μ integrability of $f : X \to \overline{\mathbb{R}}$ (in the last two chapters of $f : X \to \mathbb{C}$) means that f is μ measurable and $\int |f| d\mu < \infty$. As usual, for $1 \leq p < \infty$ the space of μ measurable functions $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) with $\int |f|^p d\mu < \infty$ is denoted by $L^p(\mu)$, and $L^\infty(\mu)$ is the space of functions which are essentially bounded with respect to μ.

A function $f : A \to \overline{\mathbb{R}}$ is a **Borel function** if A is a Borel set and the sets $\{x \in A : f(x) < c\}$ are Borel sets for all $c \in \mathbb{R}$. A mapping $f : X \to Y$ between metric spaces X and Y is a Borel mapping if $f^{-1}(U)$ is a Borel set for every open set $U \subset Y$.

We shall mention here only a few of the well-known properties of the integral. The following form of Fubini's theorem will be frequently used.

Theorem 1.2.1. Suppose that X and Y are separable metric spaces, and μ and ν are locally finite Borel measures on X and Y, respectively. If f is a non-negative Borel function on $X \times Y$, then

$$\iint f(x, y) d\mu x d\nu y = \iint f(x, y) d\nu y d\mu x$$

In particular, when f is the characteristic function of a Borel set A,

$$\int \mu(\{x : (x, y) \in A\}) d\nu y = \int \nu(\{y : (x, y) \in A\}) d\mu x$$

There are many more general forms of Fubini's theorem, see [2] §2.6. To formulate an extension, define the product measure $\mu \times \nu$ by

$$(\mu \times \nu)(C) = \inf \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i)$$

where the infimum is taken over all sequences $A_1, A_2, \ldots$ of μ measurable sets and $B_1, B_2, \ldots$ of ν measurable sets such that

$$C \subset \bigcup_{i=1}^{\infty} A_i \times B_i$$

Here $0 \cdot \infty = \infty \cdot 0 = 0$. It is easy to see that $\mu \times \nu$ is a measure over $X \times Y$. Moreover, if both μ and ν are either Borel, Borel regular, or Radon measures, $\mu \times \nu$ has the same property. The statement of Theorem 1.14 is valid for all $\mu \times \nu$ measurable functions f which are non-negative or $\mu \times \nu$ integrable (i.e. $\int |f| d(\mu \times \nu) < \infty$), and the iterated integrals agree with the $\mu \times \nu$ integral:

$$\int f d(\mu \times \nu) = \iint f(x, y) d\mu x d\nu y$$

The assumption that X and Y are separable, which of course implies that $X \times Y$ is separable, guarantees that the Borel sets and functions are $\mu \times \nu$ measurable.

As an application of Fubini's theorem we record the following useful formula.

Theorem 1.2.2. Let μ be a Borel measure and f a non-negative Borel function on a separable metric space X. Then

$$\int f d\mu = \int_0^\infty \mu(\{x \in X : f(x) \geq t\}) dt.$$

Proof. Let $A = \{(x, t) : f(x) \geq t\}$. Then

$$\begin{aligned}
&\int_0^\infty \mu(\{x \in X : f(x) \geq t\}) dt = \int_0^\infty \mu(\{x : (x, t) \in A\}) dt \\
&= \int \mathcal{L}^1(\{t \in [0, \infty) : (x, t) \in A\}) d\mu x = \int \mathcal{L}^1([0, f(x)]) d\mu x \\
&= \int f(x) d\mu x
\end{aligned}$$

□

Another way to look at the Radon measures and integrals with respect to them is to consider them as linear functionals on $C_0(X)$, the space of compactly supported continuous real-valued functions on X. That is, if μ is a Radon measure on X, we can associate to it the linear functional

$$L : C_0(X) \to \mathbb{R}, \quad Lf = \int f d\mu.$$

This is obviously **positive** in the sense that

$$Lf \geq 0 \text{ for } f \geq 0.$$

In the case where X is locally compact the converse also holds, see e.g. **[?]** 2.14.

Theorem 1.2.3 (Riesz representation theorem)**.** Let X be a locally compact metric space and $L : C_0(X) \to \mathbb{R}$ a positive linear functional. Then there is a unique Radon measure μ such that

$$Lf = \int f d\mu \quad \text{for } f \in C_0(X)$$

1.3 Image measures

We can map measures from one metric space X to another, Y.

Definition 1.3.1. The image of a measure μ under a mapping $f : X \to Y$ is defined by

$$f_\sharp \mu(A) = \mu\left(f^{-1} A\right) \quad \text{for } A \subset Y.$$

It is apparent that $f_\sharp\mu$ is a measure on Y. It is also immediate that A is $f_\sharp\mu$ measurable whenever $f^{-1}(A)$ is μ measurable. Hence if μ is a Borel measure and f a Borel function, $f_\sharp\mu$ is a Borel measure. The following simple criterion on the Radonness of $f_\sharp\mu$ will suffice for us. For more general results, see e.g. [2] 2.2.17.

Theorem 1.3.2. Let X and Y be separable metric spaces. If $f : X \to Y$ is continuous and μ is a Radon measure on X with compact support, then $f_\sharp\mu$ is a Radon measure. Moreover, spt $f_\sharp\mu = f(\operatorname{spt}\mu)$.

Proof. Replacing X by the subspace spt μ we may assume X is compact. Statement (i) of Definition 1.1.5(4) is trivial, as μ, and hence also $f_\sharp\mu$, are finite measures. We leave (ii) as an exercise and prove only (iii). □

Let $A \subset Y$ and $\varepsilon > 0$. Since μ is a Radon measure there is an open set $U \subset X$ such that $f^{-1}A \subset U$ and $\mu(U) \le \mu\left(f^{-1}A\right) + \varepsilon$. Set $V = Y \backslash f(X \backslash U)$. Then V is open, as X is compact, $A \subset V$ and

$$\begin{aligned} f_\sharp\mu(V) &= \mu\left(f^{-1}(Y \backslash f(X \backslash U))\right) \\ &= \mu\left(X \backslash f^{-1}(f(X \backslash U))\right) \le \mu(U) \\ &\le \mu\left(f^{-1}A\right) + \varepsilon = f_\sharp\mu(A) + \varepsilon. \end{aligned}$$

This yields (iii). We leave the last statement on supports also as an exercise.

The following theorem can be proven via a rather straightforward approximation by simple functions. It can also be easily deduced from Theorem 1.2.2.

Theorem 1.3.3. Suppose $f : X \to Y$ is a Borel mapping, μ is a Borel measure on X, and g is a non-negative Borel function on Y. Then

$$\int g df_\sharp\mu = \int (g \circ f) d\mu$$

When Y is locally compact, all this could also be done in the reverse order: letting

$$Lg = \int (g \circ f) d\mu \quad \text{for } g \in C_0(Y)$$

we obtain a linear functional on $C_0(Y)$ which by the Riesz representation theorem 1.16 corresponds to a Radon measure $f_\sharp\mu$.

It is clear that pulling back measures is not nearly as natural as pushing them forward: the formula $\mu(A) = \nu(fA)$ does not usually define a Borel measure even for very nice measures ν if f fails to be injective. Still it is often possible to find such pull-backs abstractly. The following proof can be found in Schwartz's *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures* §I.5.

Theorem 1.3.4. Let X and Y be compact metric spaces and $f : X \to Y$ a continuous surjection. For any Radon measure ν on Y there exists a Radon measure μ on X such that $f_\sharp\mu = \nu$.

1.4 Weak convergence

Next we consider a convergence of measures.

Definition 1.4.1. Let $\mu, \mu_1, \mu_2, \ldots$ be Radon measures on a metric space X. We say that the sequence (μ_i) converges weakly to μ,

$$\mu_i \xrightarrow{\mathbf{w}} \mu,$$

if

$$\lim_{i\to\infty} \int \varphi d\mu_i = \int \varphi d\mu \quad \text{for all } \varphi \in C_0(X)$$

Example 1.4.2.
(1) In $\mathbb{R}, \delta_i \xrightarrow{\mathbf{w}} 0$ as $i \to \infty$.

(2) Let

$$\mu_k = \frac{1}{k} \sum_{i=1}^{k} \delta_{i/k}.$$

Then $\mu_k \xrightarrow{\mathbf{w}} \mathcal{L}^1 \llcorner [0,1]$.

The weak convergence is useful because a very general compactness theorem holds. We prove it only for $\mathbb{R}^n$.

Theorem 1.4.3. If $\mu_1, \mu_2, \ldots$ are Radon measures on $\mathbb{R}^n$ with

$$\sup \{\mu_i(K) : i = 1, 2, \ldots\} < \infty$$

for all compact sets $K \subset \mathbb{R}^n$, then there is a weakly convergent subsequence of (μ_i).

Proof. The space $C_0(\mathbb{R}^n)$ is separable under the norm

$$\|\varphi\| = \max \{|\varphi(x)| : x \in \mathbb{R}^n\},$$

whence it has a countable dense subset D. For example, choosing functions $\varphi_i \in C_0(\mathbb{R}^n), i = 1, 2, \ldots$, with $\varphi_i = 1$ on $B(i)$, one can by the Weierstrass approximation theorem take for D the set of all products $\varphi_i P$ where $i = 1, 2, \ldots$ and P runs through polynomials with rational coefficients. For each $\varphi \in D$ the bounded sequence $\left(\int \varphi d\mu_i\right)$ of real numbers has a convergent sub-sequence. Using the diagonal method we can thus extract a sub-sequence (μ_{i_k}) such that the limit

$$L\varphi = \lim_{k \to \infty} \int \varphi d\mu_{i_k}$$

exists and is finite for all $\varphi \in D$. The denseness of D then implies that this actually holds for all $\varphi \in C_0(\mathbb{R}^n)$, and the Riesz representation theorem 1.16 gives the limit measure. □

As Example 1.4.2 shows $\mu_i \xrightarrow{w} \mu$ need not imply that $\mu_i(A) \to \mu(A)$ even when $A = \mathbb{R}^n$. However, the following semicontinuity properties hold.

Theorem 1.4.4. Let $\mu_1, \mu_2, \ldots$ be Radon measures on a locally compact metric space. If $\mu_i \xrightarrow{w} \mu, K \subset X$ is compact and $G \subset X$ is open, then

$$\mu(K) \geq \limsup_{i \to \infty} \mu_i(K),$$
$$\mu(G) \leq \liminf_{i \to \infty} \mu_i(G).$$

Proof.
(1) Let $\varepsilon > 0$. By property (4) (iii) of Definition 1.1.5 there is an open set V such that $K \subset V$ and $\mu(V) \leq \mu(K) + \varepsilon$. By Urysohn's lemma, see e.g. Rudin $[1, 2.12]$, there is $\varphi \in C_0(X)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K and $\operatorname{spt} \varphi \subset V$. Thus

$$\begin{aligned} \mu(K) &\geq \mu(V) - \varepsilon \geq \int \varphi d\mu - \varepsilon \\ &= \lim_{i \to \infty} \int \varphi d\mu_i - \varepsilon \geq \limsup_{i \to \infty} \mu_i(K) - \varepsilon, \end{aligned}$$

and (1) follows.

(2) is proven similarly through approximation of G with compact sets from inside. □

1.5 Approximate identities

We shall now show that arbitrary Radon measures in $\mathbb{R}^n$ can be approximated weakly by smooth functions, that is, by measures of the form $A \mapsto \int_A g d\mathcal{L}^n$ where $g \in C^\infty(\mathbb{R}^n)$, the space of infinitely differentiable real-valued functions on $\mathbb{R}^n$. First we define convolutions.

Definition 1.5.1. Let f and g be real-valued functions on $\mathbb{R}^n$ and μ a Radon measure on $\mathbb{R}^n$. The **convolutions** $f * g$ of f and g, and $f * \mu$ of f and μ, are defined by

$$f * g(x) = \int f(x-y) g(y) dy,$$
$$f * \mu(x) = \int f(x-y) d\mu y,$$

provided the integral exists.

We now consider an **approximate identity** $\{\psi_\varepsilon\}_{\varepsilon>0}$. By this we mean that each ψ_ε is a non-negative continuous function on $\mathbb{R}^n$ such that

$$\operatorname{spt} \psi_\varepsilon \subset B(\varepsilon) \text{ and } \int \psi_\varepsilon d\mathcal{L}^n = 1.$$

Any continuous function $\psi : \mathbb{R}^n \to [0, \infty)$ with spt $\psi \subset B(1)$ and $\int \psi d\mathcal{L}^n = 1$ obviously gives such an approximate identity by

$$\psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon).$$

In particular we may take

$$\begin{aligned} &\psi_\varepsilon(x) = c(\varepsilon) e^{-1/\left(\varepsilon^2 - |x|^2\right)} && \text{for } |x| < \varepsilon, \\ &\psi_\varepsilon(x) = 0 && \text{for } |x| \geq \varepsilon, \end{aligned}$$

where $c(\varepsilon)$ is determined by $\int \psi_\varepsilon d\mathcal{L}^n = 1$, to get an approximate identity consisting of C^∞ functions. It is shown in many textbooks that for any such approximate identity consisting of C^∞ functions the functions $\psi_\varepsilon * f$, where $f \in L^p(\mathbb{R}^n)$, are also C^∞ and they converge to f in L^p. We now study $\psi_\varepsilon * \mu$ in the same spirit.

Theorem 1.5.2. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity and μ a Radon measure on $\mathbb{R}^n$. Then the functions $\psi_\varepsilon * \mu$ are infinitely differentiable and they converge weakly to μ as $\varepsilon \downarrow 0$, that is,

$$\lim_{\varepsilon \downarrow 0} \int \varphi \left(\psi_\varepsilon * \mu\right) d\mathcal{L}^n = \int \varphi d\mu$$

for all $\varphi \in C_0(\mathbb{R}^n)$. If $\mu(\mathbb{R}^n) < \infty$, this holds for all uniformly continuous bounded functions $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Proof. By studying the difference quotients and using induction one can verify in a straightforward manner that for all $i_j \in \{1, \ldots, n\}, j = 1, \ldots, k$,

$$\partial_{i_1} \ldots \partial_{i_k} \left(\psi_\varepsilon * \mu\right) = \left(\partial_{i_1} \ldots \partial_{i_k} \psi_\varepsilon\right) * \mu$$

where ∂_i means the partial derivative with respect to the i-th coordinate. It follows that $\psi_\varepsilon * \mu$ has partial derivatives of all orders.

To prove the second statement we use Fubini's theorem, change of variable and the facts that spt $\psi_\varepsilon \subset B(\varepsilon)$

and $\int \psi_\varepsilon d\mathcal{L}^n = 1$ to compute

$$\begin{aligned}
&\int \varphi\left(\psi_\varepsilon * \mu\right) d\mathcal{L}^n - \int \varphi d\mu \\
&= \int \varphi(x) \int \psi_\varepsilon(x-y) d\mu y dx - \int \varphi(y) \int \psi_\varepsilon(x) dx d\mu y \\
&= \int \left[\int \varphi(x)\psi_\varepsilon(x-y) dx - \int \varphi(y)\psi_\varepsilon(x) dx\right] d\mu y \\
&= \iint_{B(\varepsilon)} [\varphi(x+y) - \varphi(y)]\psi_\varepsilon(x) dx d\mu y.
\end{aligned}$$

Since φ is uniformly continuous with compact support and $\int \psi_\varepsilon d\mathcal{L}^n = 1$, this goes to zero as $\varepsilon \downarrow 0$. The last statement follows also by the above proof. □

We finish this chapter with some remarks on lower semicontinuous functions. We shall need this concept only for non-negative functions. One way to define them is to say that a non-negative function g on $\mathbb{R}^{\boldsymbol{n}}$ is lower semicontinuous if there are non-negative functions $\varphi_i \in C_0\left(\mathbb{R}^n\right)$, $i = 1, 2, \ldots$, such that $\varphi_1 \leq \varphi_2 \leq \ldots$ and $g = \lim_{i \to \infty} \varphi_i$. An equivalent definition is that the sets $\{x : g(x) > c\}$ are open for all $c \in \mathbb{R}$. Examples are characteristic functions of open sets and $x \mapsto |x|^p, p \in \mathbb{R}$ (with value ∞ at 0 if $p < 0$).

Chapter 2

Hausdorff Measure and Dimension

To get some motivations, see [7]. Mendelbrot also has many writings on the connection between fractals and the nature.

We will introduce Hausdorff measures and dimension for measuring the metric size of quite general sets. They will be one of the basic means for studying geometric properties of sets and expressing results that these studies lead to. Hausdorff measures also provide a fruitful source for getting examples to which several later results on general measures apply. The basic definitions and first results on Hausdorff measures and dimension are due to Carathéodory and Hausdorff. We shall start with a more general construction, called Carathéodory's construction.

2.1 Carathéodory's construction

Let X be a metric space, $\mathcal{F}$ a family of subsets of X and ζ a non-negative function on $\mathcal{F}$. We make the following two assumptions.

(1) For every $\delta>0$ there are $E_1, E_2, \cdots \in \mathcal{F}$ such that $X=\bigcup_{i=1}^{\infty} E_i$ and $d\left(E_i\right) \leq \delta$.

(2) For every $\delta>0$ there is $E \in \mathcal{F}$ such that $\zeta(E) \leq \delta$ and $d(E) \leq \delta$. For $0<\delta \leq \infty$ and $A \subset X$ we define

$$\psi_\delta(A)=\inf \left\{\sum_{i=1}^{\infty} \zeta\left(E_i\right): A \subset \bigcup_{i=1}^{\infty} E_i, d\left(E_i\right) \leq \delta, E_i \in \mathcal{F}\right\}$$

Assumption (1) was only introduced to guarantee that such coverings always exist. The role of (2) is to have $\psi_\delta(\varnothing)=0$ (we fix δ for ψ_δ but δ in (2) can be arbitrarily small). It also allows us to use coverings $\{E_i\}_{i\in I}$ with I finite or countable without changing the value of $\psi_\delta(A)$.

It is easy to see that ψ_δ is monotonic and subadditive so that it is a measure. Usually it is highly non-additive and not a Borel measure. See Exercise below.

Exercise 2.1.1. Let U be an open ball in $\mathbb{R}^n, n \geq 2$, with $d(U)=\delta$. Show that for $0 \leq s \leq 1, \mathcal{H}_\delta^s(U)=\mathcal{H}_\delta^s(\bar{U})=\overline{\mathcal{H}}_\delta^s(\partial U)$.

Evidently,

$$\psi_\delta(A) \leq \psi_{\varepsilon}(A) \quad \text{whenever } 0<\varepsilon<\delta \leq \infty.$$

Hence we can define $\psi=\psi(\mathcal{F}, \zeta)$ by

$$\psi(A)=\lim_{\delta \downarrow 0} \psi_\delta(A)=\sup_{\delta>0} \psi_\delta(A) \quad \text{for } A \subset X.$$

The measure-theoretic behaviour of ψ is much better than that of ψ_δ.

Theorem 2.1.2. (1) ψ is a Borel measure. (2) If the members of $\mathcal{F}$ are Borel sets, ψ is Borel regular.

Proof.
(1) The proof that ψ is a measure is straightforward and left to the reader. To show that ψ is a Borel measure, we verify the condition of Theorem 1.7. Let $A, B \subset X$ with $d(A,B) > 0$. Choose δ with $0 < \delta < d(A,B)/2$. If the sets $E_1, E_2, \ldots \in \mathcal{F}$ cover $A \cup B$ and satisfy $d(E_i) \leq \delta$, then none of them can meet both A and B. Hence

$$\begin{aligned}\sum_i \zeta(E_i) &\geq \sum_{A\cap E_i \neq \varnothing} \zeta(E_i) + \sum_{B\cap E_i \neq \varnothing} \zeta(E_i) \\ &\geq \psi_\delta(A) + \psi_\delta(B).\end{aligned}$$

Taking the infimum over all such coverings we have $\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B)$. But the opposite inequality holds also as ψ_δ is a measure, and so $\psi_\delta(A\cup B) = \psi_\delta(A) + \psi_\delta(B)$. Letting $\delta \downarrow 0$, we obtain $\psi(A\cup B) = \psi(A) + \psi(B)$ as required. (2) If $A \subset X$, choose for every $i = 1, 2, \ldots$ sets $E_{i,1}, E_{i,2}, \cdots \in \mathcal{F}$ such that

$$\begin{gathered}A \subset \bigcup_j E_{i,j}, d(E_{i,j}) \leq 1/i \text{ and} \\ \sum_j \zeta(E_{i,j}) \leq \psi_{1/i}(A) + 1/i.\end{gathered}$$

Then $B = \bigcap_i \bigcup_j E_{i,j}$ is a Borel set such that $A \subset B$ and $\psi(A) = \psi(B)$. Thus ψ is Borel regular. □

2.2 Hausdorff measures

Let X be separable, $0 \leq s < \infty$, and choose

$$\begin{aligned}&\mathcal{F} = \{E : E \subset X\}, \\ &\zeta(E) = \zeta_s(E) = d(E)^s\end{aligned}$$

with the interpretations $0^0 = 1$ and $d(\varnothing)^s = 0$. The resulting measure ψ is called the **s-dimensional Hausdorff measure** and denoted by $\mathcal{H}^s$. So

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A)$$

where

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_i d(E_i)^s : A \subset \bigcup_i E_i, d(E_i) \leq \delta\right\}$$

The integral dimensional Hausdorff measures play a special role. Let us start from $s = 0$. It is easy to see that $\mathcal{H}^0$ is the counting measure:

$$\mathcal{H}^0(A) = \operatorname{card} A = \text{ the number of points in } A.$$

Next, for $s = 1, \mathcal{H}^1$ also has a concrete interpretation as a generalized length measure. In particular, for a rectifiable curve Γ in $\mathbb{R}^n, \mathcal{H}^1(\Gamma)$ can be shown to equal the length of Γ. (If the length is defined in some other reasonable way; of course, $\mathcal{H}^1(\Gamma)$ can also be taken as the definition of the length of Γ.) For unrectifiable curves $\Gamma, \mathcal{H}^1(\Gamma) = \infty$. More generally, if m is an integer, $1 \leq m < n$, and M is a sufficiently regular m-dimensional surface in $\mathbb{R}^n$ (for example, C^1 submanifold), then the restriction $\mathcal{H}^m LM$ gives a constant multiple of the surface measure on M. This follows for example from the area formula, see [2] 3.2.3. For $s = n$ in $\mathbb{R}^n$,

$$\mathcal{H}^n = 2^n \alpha(n)^{-1} \mathcal{L}^n, \tag{1}$$

whence

$$\mathcal{H}^n(B(x,r)) = (2r)^n \quad \text{for } x \in \mathbb{R}^n, 0 < r < \infty. \tag{2}$$

Often one normalizes Hausdorff measures (as in [2]) so that $\mathcal{H}^n$ will equal $\mathcal{L}^n$, but since we shall not usually be interested in the exact values of Hausdorff measures, we use the simpler definition. The proof of the equality (1) is rather complicated and based on the so-called isodiametric inequality

$$\mathcal{L}^n(A) \leq 2^{-n}\alpha(n)d(A)^n \quad \text{for } A \subset \mathbb{R}^n$$

see [2] 2.10.33. But to see that $\mathcal{H}^n = c\mathcal{L}^n$ with some positive and finite constant is much easier. All we have to do is to verify that both $\mathcal{H}^n$ and $\mathcal{L}^n$ are **uniformly distributed measures** (i.e., Borel regular measures μ on a metric space X such that $0 < \mu(B(x,r)) = \mu(B(y,r)) < \infty$ for $x, y \in X, 0 < r < \infty$) and use the following theorem (That $\mathcal{H}^n$ is Borel regular will be noted in Corollary 2.2.3.) We shall use the formulas (1) and (2) many times, but almost always the weaker information that they hold with some unspecified constants would suffice.

Theorem 2.2.1. Let μ and ν be uniformly distributed Borel regular measures on a separable metric space X. Then there is a constant c such that $\mu = c\nu$.

Proof. Let g and h be the functions giving the μ and ν measures of the balls of radius r :

$$g(r) = \mu(B(x,r)), h(r) = \nu(B(x,r)) \text{ for } x \in X, 0 < r < \infty.$$

Let U be a non-empty bounded open subset of X. Clearly the limit $\lim_{r\downarrow 0}(\nu(U \cap B(x,r))/h(r))$ exists and equals 1 for $x \in U$. Hence by Fatou's lemma and Fubini's theorem

$$\begin{aligned}
\mu(U) &= \int_U \lim_{r\downarrow 0} h(r)^{-1}\nu(U \cap B(x,r))d\mu x \\
&\leq \liminf_{r\downarrow 0} h(r)^{-1} \int \nu(U \cap B(x,r))d\mu x \\
&= \liminf_{r\downarrow 0} h(r)^{-1} \int_U \mu(B(y,r))d\nu y \\
&= \left(\liminf_{r\downarrow 0} g(r)/h(r)\right)\nu(U)
\end{aligned}$$

Interchanging μ and ν we obtain similarly

$$\nu(U) \leq \left(\liminf_{r\downarrow 0} \frac{h(r)}{g(r)}\right)\mu(U).$$

It follows that the limit $c = \lim_{r\downarrow 0}(g(r)/h(r))$ exists and $\mu(U) = c\nu(U)$ for every open set U. That $\mu = c\nu$ then follows by Theorem 1.1.10(2) and the Borel regularity of μ and ν. □

For any $s > n, \mathcal{H}^s$ in $\mathbb{R}^n$ is uninteresting since $\mathcal{H}^s(\mathbb{R}^n) = 0$ (see Theorem 2.2.5).

Hausdorff measures behave nicely under translations and dilations in $\mathbb{R}^n$: for $A \subset \mathbb{R}^n, a \in \mathbb{R}^n, 0 < t < \infty$,

$$\begin{aligned}
&\mathcal{H}^s(A+a) = \mathcal{H}^s(A) \quad \text{where } A + a = \{x + a : x \in A\}, \\
&\mathcal{H}^s(tA) = t^s\mathcal{H}^s(A) \quad \text{where } tA = \{tx : x \in A\}.
\end{aligned}$$

These are readily verified from the definition. In particular,

$$\mathcal{H}^s(B(x,r)) = c(s,n)r^s \quad \text{for } x \in \mathbb{R}^n, 0 < r < \infty.$$

But, as follows from Theorem 2.2.5, $c(s, n)$ is positive and finite only when $s = n$; for $s > n, c(s, n) = 0$, for $s < n, c(s, n) = \infty$. Thus only $\mathcal{H}^n$ is uniformly distributed in $\mathbb{R}^n$. To prove that $0 < c(n, n) < \infty$, one can use any of the standard proofs for the fact that the unit ball (or cube) has positive and finite Lebesgue measure.

We shall now derive some simple properties of Hausdorff measures in a general separable metric space X.

Theorem 2.2.2. Let $0 \leq s < n$ and $\zeta(E) = d(E)^s$ for $E \subset X$. If

(1) $\mathcal{F} = \{F \subset X : F \text{ is closed }\}$ or

(2) $\mathcal{F} = \{U \subset X : U \text{ is open }\}$ or

(3) $X = \mathbb{R}^n$ and $\mathcal{F} = \{K \subset \mathbb{R}^n : K \text{ is convex }\}$, then $\psi(\mathcal{F}, \zeta) = \mathcal{H}^s$.

The first and last statement follow from the fact that the closure and convex hull of a set E have the same diameter as E. The second statement holds since for any $\varepsilon > 0, \{x : d(x, E) < \varepsilon\}$ is open and has diameter at most $d(E) + 2\varepsilon$. We leave the details as an exercise. Recalling Theorem 2.1.2(2) we have

Corollary 2.2.3. $\mathcal{H}^s$ is Borel regular.

Notice that usually $\mathcal{H}^s$ is not a Radon measure since it need not be locally finite. For example, if $s < n$ every non-empty open set in $\mathbb{R}^n$ has non- σ-finite $\mathcal{H}^s$ measure. But taking any $\mathcal{H}^s$ measurable set A in $\mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$, the restriction $\mathcal{H}^s \llcorner A$ is a Radon measure by Theorem 1.1.9(2) and Corollary 1.1.11.

Often one is only interested in knowing which sets have $\mathcal{H}^s$ measure zero. For this it is enough to use any of the approximating measures $\mathcal{H}^s_\delta$, for example $\mathcal{H}^s_\infty$; in fact we don't really need any measure at all.

Lemma 2.2.4. Let $A \subset X, 0 \leq s < \infty$ and $0 < \delta \leq \infty$. Then the following conditions are equivalent:

(1) $\mathcal{H}^s(A) = 0$.

(2) $\mathcal{H}^s_\delta(A) = 0$.

(3) $\forall \varepsilon > 0 \exists E_1, E_2, \ldots \subset X$ such that

$$A \subset \bigcup_i E_i \text{ and } \sum_i d\left(E_i\right)^s < \varepsilon.$$

The proof is left as an exercise.

We shall now compare measures $\mathcal{H}^s$ with each other.

Theorem 2.2.5. For $0 \leq s < t < \infty$ and $A \subset X$,

(1) $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$,

(2) $\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = \infty$.

Proof. To prove (1), let $A \subset \bigcup_i E_i$ with $d\left(E_i\right) \leq \delta$ and $\sum_i d\left(E_i\right)^s \leq \mathcal{H}^s_\delta(A) + 1$. Then

$$\mathcal{H}^t_\delta(A) \leq \sum_i d\left(E_i\right)^t \leq \delta^{t-s} \sum_i d\left(E_i\right)^s \leq \delta^{t-s}\left(\mathcal{H}^s_\delta(A) + 1\right),$$

which gives (1) as $\delta \downarrow 0$.

(2) is really only a restatement of (1). But we have emphasized this simple theorem by doublestating it, because it leads to one of the fundamental concept the Hausdorff dimension. ☐

2.3 Hausdorff dimension

According to Theorem 2.2.5, we may define

Definition 2.3.1. The **Hausdorff dimension** of a set $A \subset X$ is

$$\begin{aligned} \dim A &= \sup \{s : \mathcal{H}^s(A) > 0\} = \sup \{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf \left\{t : \mathcal{H}^t(A) < \infty\right\} = \inf \left\{t : \mathcal{H}^t(A) = 0\right\} \end{aligned}$$

(Sometimes some of these sets may be empty, but we leave the obvious interpretations to the reader.)

Clearly the Hausdorff dimension has the natural properties of monotonicity and stability with respect to countable unions:

$$\begin{array}{ll} \dim A \leq \dim B & \text{for } A \subset B \subset X \\ \dim \bigcup_{i=1}^{\infty} A_i = \sup_i \dim A_i & \text{for } A_i \subset X, i = 1, 2, \ldots \end{array}$$

To state the definition in other words, $\operatorname{din} A$ is the unique number (it may be ∞ in some metric spaces) for which

$$\begin{array}{ll} s < \dim A & \text{implies } \mathcal{H}^s(A) = \infty, \\ t > \dim A & \text{implies } \mathcal{H}^t(A) = 0. \end{array}$$

At the borderline case $s = \dim A$ we cannot have any general nontrivial information about the value $\mathcal{H}^s(A)$; all three cases $\mathcal{H}^s(A) = 0$, $0 < \mathcal{H}^s(A) < \infty, \mathcal{H}^s(A) = \infty$ are possible. But if for some given A we can find s such that $0 < \mathcal{H}^s(A) < \infty$, then s must equal $\dim A$. Since $\mathbb{R}^n$ has infinite but σ-finite $\mathcal{H}^n$ measure, it follows that

$$\dim \mathbb{R}^n = n.$$

Hence $0 \leq \dim A \leq n$ for all $A \subset \mathbb{R}^n$. We shall soon see that for all $s \in [0, n], \dim A = s$ for some subset A of $\mathbb{R}^n$.

To find the Hausdorff dimension or to estimate the Hausdorff measures of a given set, it is always possible and often advantageous to use coverings with some simpler sets like balls or, in $\mathbb{R}^n$, dyadic cubes. This is easy to see and we shall return to it in the next chapter.

Recalling Lemma 2.2.4(3) we observe that we do not really need Hausdorff measures to define Hausdorff dimension.

Remark 2.3.2. Although the Hausdorff dimension measures the metric size of any subset of our metric space, the values of the Hausdorff measures often do not give much extra information. This is so since there may be no value s for which the set has positive and finite $\mathcal{H}^s$ measure. But often replacing $\zeta_s(E) = d(E)^s$ by some other function of the diameter, one can find measures measuring the given set in a more delicate manner.

Let $h : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $h(0) = 0$. We take again

$$\mathcal{F} = \{E : E \subset X\} \quad \text{and} \quad \zeta(E) = h(d(E))$$

(with $d(\varnothing) = 0$). Then the corresponding measure $\psi(\mathcal{F}, \zeta) = \Lambda_h$ is called the Hausdorff h measure. Of course, $\Lambda_h = \mathcal{H}^s$ when $h(t) = t^s$.

There are many cases where some other h than t^s is more useful and natural. Among the most important are sets related to Brownian motion in $\mathbb{R}^n$. For example, the trajectories of the Brownian motion in $\mathbb{R}^n$ have positive and σ-finite Λ_h measure almost surely with (for small t)

$$\begin{array}{ll} h(t) = t^2 \log \log t^{-1} & \text{in the case } n \geq 3, \text{ and} \\ h(t) = t^2 \log t^{-1} \log \log \log t^{-1} & \text{in the case } n = 2. \end{array}$$

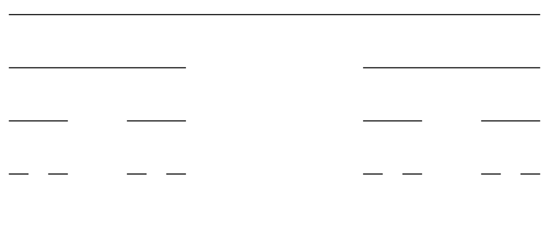
We have now introduced measures for measuring the size of very general sets. It is time to look at some examples with which Hausdorff measures are convenient and useful. We begin with the most classical.

2.4 Cantor sets

2.4.1 Cantor sets in $\mathbb{R}^1$

Let $0 < \lambda < 1/2$. Denote $I_{0,1} = [0,1]$, and let $I_{1,1}$ and $I_{1,2}$ be the intervals $[0,\lambda]$ and $[1-\lambda,1]$, respectively. We continue this process of selecting two subintervals of each already given interval. If we have defined intervals $I_{k-1,1},\ldots,I_{k-1,2^{k-1}}$, we define $I_{k,1},\ldots,I_{k,2^k}$ by deleting from the middle of each $I_{k-1,j}$ an interval of length $(1-2\lambda)d\left(I_{k-1,j}\right) = (1-2\lambda)\lambda^{k-1}$. All the intervals $I_{k,j}$ thus obtained have length λ^k. We define a kind of limit set of this construction by

$$C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}$$



Then $C(\lambda)$ is an uncountable compact set without interior points and with zero Lebesgue measure. The most commonly used case is the Cantor middle-third set $C(1/3)$, see the figure.

We shall now study the Hausdorff measures and dimension of $C(\lambda)$. As usual, it is much simpler to find upper bounds than lower bounds for the Hausdorff measures. This is due to the definition: a judiciously chosen covering will give an upper estimate, but a lower estimate requires finding an infimum over arbitrary coverings. For every $k = 1,2,\ldots, C(\lambda) \subset \bigcup_j I_{k,j}$, and so

$$\mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq \sum_{j=1}^{2^k} d\left(I_{k,j}\right)^s = 2^k \lambda^{ks} = \left(2\lambda^s\right)^k.$$

In order for this upper bound to be useful, it should stay bounded as $k \to \infty$. The smallest value of s for which this happens is given by $2\lambda^s = 1$, that is,

$$s = \log 2/\log(1/\lambda).$$

For this choice we have

$$\mathcal{H}^s(C(\lambda)) = \lim_{k\to\infty} \mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq 1.$$

Thus $\dim C(\lambda) \leq s$. Next we shall show

$$\mathcal{H}^s(C(\lambda)) \geq 1/4 \tag{2.1}$$

which will give

$$\dim C(\lambda) = \log 2/\log(1/\lambda).$$

To prove (2.1), it suffices to show that

$$\sum_j d\left(I_j\right)^s \geq 1/4 \tag{2.2}$$

whenever open intervals $I_1, I_2, \ldots$ cover $C(\lambda)$. Since $C(\lambda)$ is compact, finitely many I_j 's cover $C(\lambda)$ so that we may assume that there were only $I_1,\ldots,I_n$ to begin with. Since $C(\lambda)$ has no interior points, we can, making I_j slightly larger if necessary, assume that the end-points of each I_j are outside $C(\lambda)$. Then there is $\delta > 0$ such that the distance from all these end-points to $C(\lambda)$ is at least δ. Choosing k so large that

$\delta > \lambda^k = d(I_{k,i})$, it follows that every interval $I_{k,i}$ is contained in some I_j. We shall now show that for any open interval I and any fixed ℓ,

$$\sum_{I_{\ell,i} \subset I} d(I_{\ell,i})^s \leq 4d(I)^s \tag{2.3}$$

This gives (2.2), since

$$4\sum_j d(I_j)^s \geq \sum_j \sum_{I_{k,i} \subset I_j} d(I_{k,i})^s \geq \sum_{i=1}^{2^k} d(I_{k,i})^s = 1$$

To verify (2.3), suppose there are some intervals $I_{\ell,i}$ inside I and let n be the smallest integer for which I contains some $I_{n,i}$. Then $n \leq \ell$. Let $I_{n,j_1}, \ldots, I_{n,j_p}$ be all the n-th generation intervals which meet I. Then $p \leq 4$, since otherwise I would contain some $I_{n-1,i}$. Thus

$$4d(I)^s \geq \sum_{m=1}^{p} d(I_{n,j_m})^s = \sum_{m=1}^{p} \sum_{I_{\ell,i} \subset I_{n,j_m}} d(I_{\ell,i})^s \geq \sum_{I_{\ell,i} \subset I} d(I_{\ell,i})^s .$$

Actually it is not hard to show that (2.2) can be improved to $\sum d(I_j)^s \geq 1$, which gives the precise value $\mathcal{H}^s(C(\lambda)) = 1$, see [1] Theorem 1.14. However, the above argument can be generalized to many situations where the exact value of the measure is practically impossible to compute.

Note that $\dim C(\lambda)$ measures the sizes of the Cantor sets $C(\lambda)$ in a natural way: when λ increases, the sizes of the deleted holes decrease and the sets $C(\lambda)$ become larger, and also $\dim C(\lambda)$ increases. Notice also that when λ runs from 0 to $1/2, \dim C(\lambda)$ takes all the values between 0 and 1.

2.4.2 Generalized Cantor sets in $\mathbb{R}^1$

Instead of keeping constant the ratios of the lengths of the intervals in every two successive stages of the construction, we can vary it in the following way. Let $T = (\lambda_i)$ be a sequence of numbers in the open interval $(0, 1/2)$. We construct a set $C(T)$ otherwise as above, but take the intervals $I_{k,j}$ to have length $\lambda_k d(I_{k-1,i})$. Then for every k we get 2^k intervals $I_{k,j}$ of length

$$s_k = \lambda_1 \cdots \lambda_k.$$

Let $h : [0, \infty) \to [0, \infty)$ be a continuous increasing function such that

$$h(s_k) = 2^{-k}. \tag{2.4}$$

Then by the above argument

$$1/4 \leq \Lambda_h(C(T)) \leq 1.$$

Conversely, we can start from any continuous increasing function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and $h(2r) < 2h(r)$ for $0 < r < \infty$, and inductively select $\lambda_1, \lambda_2, \ldots$ such that (2.4) is valid. Thus for any such h there is a compact set $C_h \subset \mathbb{R}^1$ such that $0 < \Lambda_h(C_h) < \infty$. Choosing $h(r) = r^s \log(1/r)$ for small values of r, where $0 < s \leq 1$, we have $\dim C_h = s$ and $\mathcal{H}^s(C_h) = 0$. On the other hand, choosing $h(r) = r^s / \log(1/r)$ for small r, where $0 \leq s < 1, C_h$ has non- σ-finite $\mathcal{H}^s$ measure and dimension s. In particular, the extreme cases $s = 1$ and $s = 0$ give a set of the dimension 1 with zero Lebesgue measure and an uncountable set of dimension zero.

2.4.3 Cantor sets in $\mathbb{R}^n$

We can use the same ideas as above to construct Cantor-type sets in $\mathbb{R}^n$ having a given Hausdorff dimension s. We can start from a ball, cube, rectangle etc. and at each stage of the construction select similar geometric figures inside the previous ones. One can then often use the following proposition.

Suppose for $k = 1, 2, \ldots$ we have compact sets $E_{i_1,\ldots,i_k}, i_j = 1, \ldots, m_j$, such that

$$
\begin{aligned}
& E_{i_1,\ldots,i_k,i_{k+1}} \subset E_{i_1,\ldots,i_k}, \\
& d_k = \max_{i_1\ldots i_k} d\left(E_{i_1,\ldots,i_k}\right) \rightarrow 0 \text{ as } k \rightarrow \infty \\
& \sum_{j=1}^{m_{k+1}} d\left(E_{i_1,\ldots,i_k,j}\right)^s = d\left(E_{i_1,\ldots,i_k}\right)^s \\
& \sum_{B \cap E_{i_1,\ldots,i_k} \neq \varnothing} d\left(E_{i_1,\ldots,i_k}\right)^s \leq c d(B)^s
\end{aligned}
$$

for any ball B with $d(B) \geq d_k$, where c is a positive constant. Then

$$
0 < \mathcal{H}^s\left(\bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k} E_{i_1,\ldots,i_k}\right) < \infty .
$$

We leave the proof as an exercise. Notice that the above conditions are satisfied for example in the following situation: select all the sets $E_{i_1,\ldots,i_k}$ to be balls of radius r_k. Choose the balls $E_{i_1,\ldots,i_k,j}$ fairly uniformly distributed inside $E_{i_1,\ldots,\ell_k}$ and so that $m_{k+1} r_{k+1}^s = r_k^s$. If r_k tends to zero very rapidly (or equivalently, m_k grows very rapidly), the diameter $2r_{k+1}$ of $d\left(E_{i_1,\ldots,i_k,i}\right)$ is much smaller for large k than the distance from $E_{i_1,\ldots,i_k,i}$ to the nearest neighbour $E_{i_1,\ldots,i_k,j}$; this distance is of magnitude $r_k^{1-s/n} r_{k+1}^{s/n}$. Hence sets with large Hausdorff dimension (even equal to n) can look extremely porous at arbitrarily small scales, cf. Figure 2.1.

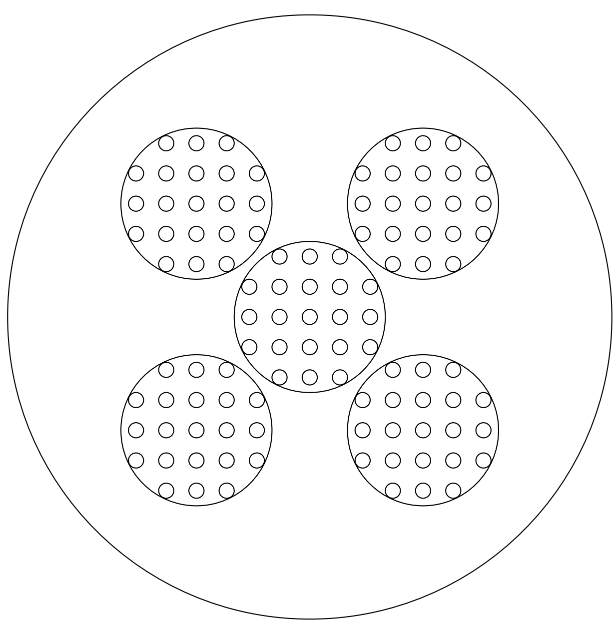


Figure 2.1: A very porous Cantor set.

2.5 Self-similar sets

Roughly speaking, a subset of $\mathbb{R}^n$ is selfsimilar if it can be split into parts which are geometrically similar to the whole set. The Cantor sets $C(\lambda)$ in subsection 2.4.1 are simple examples. If the parts $C(\lambda) \cap [0, \lambda]$ and $C(\lambda) \cap [1-\lambda, 1]$ are magnified in ratio $1/\lambda$ we get (a translate of) the original Cantor set. We shall briefly describe parts of the more general elegant theory of Hutchinson. For more details see [1]. The self-similarity of $C(\lambda)$ above can be expressed by the formula

$$C(\lambda) = S_1(C(\lambda)) \cup S_2(C(\lambda))$$

where the similarity maps $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ are defined by $S_1(x) = \lambda x$, $S_2(x) = \lambda x + 1 - \lambda$. Another standard example is von Koch's "snowflake" curve, see Figure 4.3. In the construction one replaces at each stage a segment of length d by four segments of length $d/3$ as in the figure. The von Koch curve K is a limit of the polygonal curves thus obtained. It is a non-rectifiable curve having tangents at none of its points. It can also be presented in terms of similarity maps S_i in the form

$$K = S_1K \cup S_2K \cup S_3K \cup S_4K.$$

Here $S_1, \ldots, S_4$ are the orientation-preserving similarities of ratios $1/3$ of the plane which map the first initial segment onto the four next ones.

We now state the basic ideas of Hutchinson's general theory. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a **similitude** if there is $r, 0 < r < 1$, such that

$$|S(x) - S(y)| = r|x - y| \quad \text{for } x, y \in \mathbb{R}^n.$$

Similitudes are exactly those maps S which can be written as

$$S(x) = rg(x) + z, \quad x \in \mathbb{R}^n,$$

for some $g \in O(n), z \in \mathbb{R}^n$ and $0 < r < 1$. Suppose $\mathcal{S} = \{S_1, \ldots, S_N\}$, $N \geq 2$, is a finite sequence of similitudes with contraction ratios $r_1, \ldots, r_N$. We say that a non-empty compact set K is **invariant** under $\mathcal{S}$ if

$$K = \bigcup_{i=1}^{N} S_iK.$$

Then for any such $\mathcal{S}$ there exists a unique invariant compact set. A quick way to prove this is to use the fact that the family of all non-empty compact subsets of $\mathbb{R}^n$ is a complete metric space with the Hausdorff metric ρ,

$$\rho(E, F) = \max\{d(x, F), d(y, E) : x \in E, y \in F\},$$

see e.g. [2] 2.10.21. The map $\widetilde{S} : E \mapsto \bigcup_{i=1}^{N} S_iE$ is readily seen to be a contraction in the Hausdorff metric, whence it has a unique fixed point. By definition, this is the invariant set we wanted.

In addition, it follows by the simple general properties of contractions in complete metric spaces that however we choose an initial compact set $F \subset \mathbb{R}^n$, the iterations

$$\tilde{S}^m(F) = \tilde{S} \circ \cdots \circ \widetilde{S}(F) = \bigcup_{i_1=1}^{N} \cdots \bigcup_{i_m=1}^{N} S_{i_1} \circ \cdots \circ S_{i_m}(F)$$

will converge to K. Moreover, for any m the set K satisfies

$$K = \bigcup_{i_1=1}^{N} \cdots \bigcup_{i_m=1}^{N} S_{i_1} \circ \cdots \circ S_{i_m}(K).$$

Since

$$d\left(S_{i_1}\circ\cdots\circ S_{i_m}(K)\right)\leq\left(\max_{1\leq i\leq N}r_i\right)^m d(K)\to 0,\quad \text{as } m\to\infty,$$

an invariant set can be expressed as a union of arbitrarily small sets geometrically similar to itself. We define an invariant set under $\mathcal{S}$ to be **self-similar** if with $s=\dim K$,

$$\mathcal{H}^s\left(S_i(K)\cap S_j(K)\right)=0\quad \text{for } i\neq j.$$

This definition is rather awkward to use, but the following somewhat stronger separation condition, called the **open set condition**, is very convenient: There is a non-empty open set O such that

$$\bigcup_{i=1}^N S_i(O)\subset O \text{ and } S_i(O)\cap S_j(O)=\varnothing\quad \text{for } i\neq j.$$

This is satisfied if the different parts $S_i(K)$ are disjoint as for the classical Cantor sets. Then we can use as O the ε-neighbourhood $\{x : d(x,K)<\varepsilon\}$ for sufficiently small ε. The open set condition also holds in many other interesting cases. For example, in the case of the von Koch curve we can take for O the open triangle which is the interior of the convex hull of the polygonal line consisting of the first four line segments, see Figure 2.2.

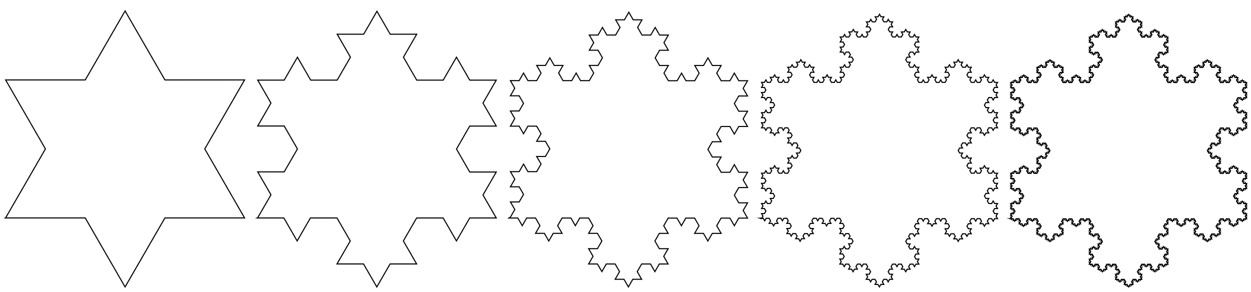


Figure 2.2: von Koch curve (as part of von Koch snowflake)

Under the open set condition the dimension of K is explicitly determined by the contraction ratios $r_1,\ldots,r_N$ of the similitudes S_i in $\mathcal{S}$:

Theorem 2.5.1. If $\mathcal{S}$ satisfies the open set condition, then the invariant set K is self-similar and $0<\mathcal{H}^s(K)<\infty$, whence $s=\dim K$, where s is the unique number for which

$$\sum_{i=1}^N r_i^s=1.$$

Moreover, there are positive and finite numbers a and b such that

$$ar^s\leq\mathcal{H}^s(K\cap B(x,r))\leq br^s\quad \text{for } x\in K, 0<r\leq 1.$$

For a proof see [1]. If in the above $r_1=\cdots=r_N=r$ we have $\dim K=\log N/\log(1/r)$ in accordance with what we previously proved about the Cantor sets $C(\lambda)$. For the von Koch curve K this gives $\dim K=\log 4/\log 3$.

Chapter 3

Other Dimensions and Measures

The main part of this chapter will deal with Minkowski and packing dimensions and packing measures and their relations to Hausdorff measures. We begin with two slight modifications of Hausdorff measures.

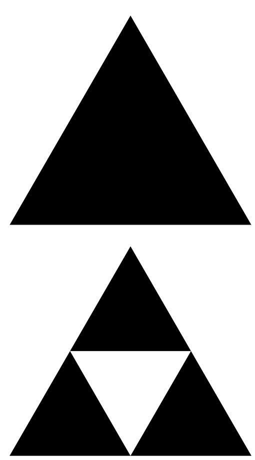
3.1 Spherical measures

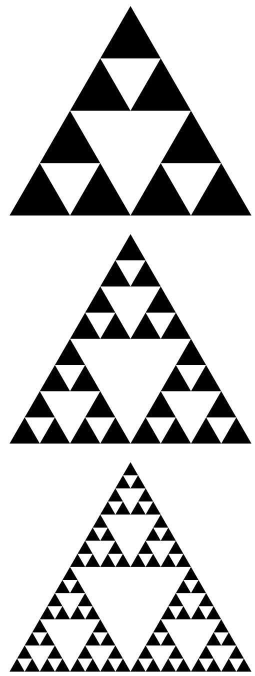
Let $0 \leq t < \infty$. If we apply Carathéodory's construction taking $\mathcal{F}$ to be the family of all closed (or open) balls in a separable metric space X and $\zeta(B) = d(B)^t$, the resulting measure $\psi(\mathcal{F}, \zeta)$ is called t dimensional **spherical measure**. We denote it by $\mathcal{S}^t$. In $\mathbb{R}^n$, for $n \geq 2$ and $0 < t < n$, it differs from the t-dimensional Hausdorff measure, but they are related by the inequalities

$$\mathcal{H}^t(A) \leq \mathcal{S}^t(A) \leq 2^t \mathcal{H}^t(A).$$

The left hand inequality follows immediately from the definitions and the right one from the fact that any bounded set $E \subset X$ is contained in a ball of diameter $2d(E)$. Hence for example for finding the Hausdorff dimension of a given set, we can use spherical measures and coverings with balls in place of Hausdorff measures and more general coverings.

We give an example of a compact set S in $\mathbb{R}^2$ for which $\mathcal{H}^t(S) < \mathcal{S}^t(S)$. The self-similar set indicated by figure below, a Sierpinski gasket, suffices. Besicovitch studied in detail for $t = 1$ a modified example giving the biggest possible ratio $\mathcal{S}^1(A)/\mathcal{H}^1(A) = 2/\sqrt{3}$.





3.2 Net measures

The **net measures** are denoted by $\mathcal{N}^t$ and they are obtained from the Carathéodory construction in $\mathbb{R}^n$ by taking again $\zeta(E)=d(E)^t$ and as $\mathcal{F}$ the family of half-open dyadic cubes in $\mathbb{R}^n$, that is, cubes of the form

$$\left\{x \in \mathbb{R}^n : k_i 2^{-m} \leq x_i<\left(k_i+1\right) 2^{-m} \text{ for } i=1, \ldots, n\right\}$$

where k_i and m are arbitrary integers. The net measures are often easier to handle than Hausdorff measures, because every family $\mathcal{A}$ of such dyadic cubes with $\sup_{Q \in \mathcal{A}} d(Q)<\infty$ has a disjoint subfamily with the same union; select those cubes in $\mathcal{A}$ which are not contained in any other. As for spherical measures one gets

$$\mathcal{H}^t(A) \leq \mathcal{N}^t(A) \leq 4^t n^{t/2} \mathcal{H}^t(A).$$

For applications of net measures to Hausdorff measures, see e.g. [1] Chapter 5.

The Hausdorff dimension is a natural parameter to measure the metric size of any given set in a metric space. However, it is not the only one. There are other parameters whose use is well justified from the point of view both of the geometric contents of the very definitions and of the applications.

3.3 Minkowski dimensions

The Hausdorff dimension is defined by looking at the coverings of a set by small sets E_i and inspecting the sums $\sum d\left(E_i\right)^s$. As noted before the sets E_i could be arbitrary or they could be balls or, in $\mathbb{R}^n$, dyadic cubes. One of the most immediate modifications from this leads to coverings with balls, for example, of the same size. Although the following makes sense in any metric space, we restrict attention to $\mathbb{R}^n$.

Let A be a non-empty bounded subset of $\mathbb{R}^n$. For $0<\varepsilon<\infty$, let $N(A, \varepsilon)$ be the smallest number of ε-balls needed to cover A :

$$N(A, \varepsilon)=\min \left\{k: A \subset \bigcup_{i=1}^k B\left(x_i, \varepsilon\right) \quad \text{for some } x_i \in \mathbb{R}^n\right\}.$$

The **upper** and **lower Minkowski dimensions** of A are defined by

$$\overline{\dim}_M A = \inf\left\{s : \limsup_{\varepsilon \downarrow 0} N(A,\varepsilon)\varepsilon^s = 0\right\}$$

and

$$\underline{\dim}_M A = \inf\left\{s : \liminf_{\varepsilon \downarrow 0} N(A,\varepsilon)\varepsilon^s = 0\right\}.$$

It is obvious that

$$\begin{aligned}\overline{\dim}_M A &= \inf\left\{s : \limsup_{\varepsilon \downarrow 0} N(A,\varepsilon)\varepsilon^s < \infty\right\}\\ &= \sup\left\{s : \limsup_{\varepsilon \downarrow 0} N(A,\varepsilon)\varepsilon^s = \infty\right\}\\ &= \sup\left\{s : \limsup_{\varepsilon \downarrow 0} N(A,\varepsilon)\varepsilon^s > 0\right\},\end{aligned}$$

and similarly for $\underline{\dim}_M A$. It follows immediately from the definitions that

$$\dim A \leq \dim_M A \leq \overline{\dim}_M A \leq n,$$

and these inequalities can be strict. For the left inequality one can get an example even from countable compact sets. For instance,

$$\underline{\dim}_M(\{0\} \cup \{1/i : i = 1,2,\ldots\}) = 1/2.$$

We leave the proof as an exercise.

We now briefly indicate how to construct a compact set $E \subset \mathbb{R}^1$ with $\dim_M E < \dim_M E$. Let $0 < s < t < 1$. As in 4.10, start constructing a Cantor set $C(\lambda)$ of Hausdorff dimension less than s, i.e. $s > \log 2/\log(1/\lambda)$. Thus we have two subintervals $I_{1,1}$ and $I_{1,2}$ whose lengths d_1 satisfy $2d_1^s \leq 1$. In each $I_{1,j}$ perform now the construction of $C(\mu)$ of dimension greater than t sufficiently many times so that you will have altogether 2^{k_2} subintervals $I_{2,1},\ldots,I_{2,2^{k_2}}$ of $[0,1]$ whose lengths d_2 satisfy $2^{k_2}d_2^t \geq 1$. After that continue again with the construction of $C(\lambda)$ and so on. The resulting Cantor set of this process will have the lower Minkowski dimension at most s and the upper at least t. We omit the details.

To obtain a compact set $E \subset \mathbb{R}^1$ with $0 < s = \dim E < \underline{\dim}_M E = t < 1$, perform a Cantor construction where inside the intervals I already selected one chooses many intervals I_j of different sizes such that $\sum_j d\left(I_j\right)^s = d(I)^s$ but $N\left(\bigcup_j I_j, \varepsilon\right)\varepsilon^t \geq 1$ for all $0 < \varepsilon \leq d(I)$. A combination and modification of these ideas shows that for any $0 \leq s \leq t \leq u \leq 1$ there is a compact set $E \subset \mathbb{R}^1$ with $\dim E = s, \underline{\dim}_M E = t$ and $\overline{\dim}_M E = u$.

There are some obvious equivalent definitions of Minkowski dimensions. For example,

$$\overline{\dim}_M A = \limsup_{\varepsilon \downarrow 0} \frac{\log N(A,\varepsilon)}{\log(1/\varepsilon)},$$
$$\underline{\dim}_M A = \liminf_{\varepsilon \downarrow 0} \frac{\log N(A,\varepsilon)}{\log(1/\varepsilon)}.$$

The proofs are left as exercises. The corresponding formulas can be given also in terms of the packing numbers $P(A,\varepsilon)$ instead of the covering numbers $N(A,\varepsilon)$. Let $P(A,\varepsilon)$ be the greatest number of disjoint ε-balls with centres in A :

$$\begin{aligned}P(A,\varepsilon) = \max\{k :\ &\text{there are disjoint balls } B\left(x_i,\varepsilon\right),\\ &i = 1,\ldots,k, \text{ with } x_i \in A\}.\end{aligned}$$

Then

$$N(A, 2\varepsilon) \leq P(A, \varepsilon) \leq N(A, \varepsilon/2). \tag{3.1}$$

To verify the first inequality, let $k = P(A, \varepsilon)$ and choose disjoint balls $B(x_i, \varepsilon), x_i \in A, i = 1, \ldots, k$. If there exists $x \in A \backslash \bigcup_{i=1}^{k} B(x_i, 2\varepsilon)$, the balls $B(x_1, \varepsilon), \ldots, B(x_k, \varepsilon), B(x, \varepsilon)$ would be disjoint giving $k+1 \leq P(A, \varepsilon) = k$. Hence the balls $B(x_i, 2\varepsilon)$ cover A, and so $N(A, 2\varepsilon) \leq k = P(A, \varepsilon)$.

For the second inequality let $N = N(A, \varepsilon/2)$ and $k = P(A, \varepsilon)$, and choose $x_1, \ldots, x_N \in \mathbb{R}^n, y_1, \ldots, y_k \in A$ such that $A \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon/2)$ and the balls $B(y_j, \varepsilon), j = 1, \ldots, k$, are disjoint. Then each y_j belongs to some $B(x_i, \varepsilon/2)$ and no $B(x_i, \varepsilon/2)$ contains more than one point y_j, the balls $B(y_j, \varepsilon)$ being disjoint. Thus $k \leq N$, which gives $P(A, \varepsilon) \leq N(A, \varepsilon/2)$.

The inequalities (3.1) give immediately the formulas for the Minkowski dimensions in terms of $P(A, \varepsilon)$. For example,

$$\overline{\dim}_M A = \limsup_{\varepsilon \downarrow 0} \frac{\log P(A, \varepsilon)}{\log(1/\varepsilon)}.$$

The Minkowski dimensions can also easily be seen to be determined with dyadic cubes: let $\tilde{N}_m(A)$ be the number of dyadic cubes ("boxes") of side-length 2^{-m} which meet A. Then

$$\overline{\dim}_M A = \limsup_{m \to \infty} \frac{\log \tilde{N}_m(A)}{m \log 2}$$

3.4 Packing dimensions and measures

Earlier we observed that even a compact countable set can have positive Minkowski dimension. This is a reflection of the fact that the Minkowski dimensions are lacking one of the fundamental properties of the Hausdorff dimension:

$$\dim \left(\bigcup_{i=1}^{\infty} A_i \right) = \sup \{ \dim A_i : i = 1, 2, \ldots \}.$$

We can easily modify the Minkowski dimensions to arrive at dimensions which have this property. We call them **upper** and **lower packing dimensions** and they can be defined for any subset A of $\mathbb{R}^n$ by

$$\overline{\dim}_p A = \inf \left\{ \sup_i \overline{\dim}_M A_i : A = \bigcup_{i=1}^{\infty} A_i, A_i \text{ is bounded} \right\}$$
$$\underline{\dim}_p A = \inf \left\{ \sup_i \underline{\dim}_M A_i : A = \bigcup_{i=1}^{\infty} A_i, A_i \text{ is bounded} \right\}$$

Clearly,

$$\dim A \leq \underline{\dim}_p A \leq \underline{\dim}_M A$$

and

$$\underline{\dim}_p A \leq \overline{\dim}_p A \leq \overline{\dim}_M A.$$

All these inequalities can be strict. But now $\overline{\dim}_p A = 0$ for all countable sets.

The upper packing dimension can also be defined in terms of the packing measures, which we now introduce. Because of this the upper packing dimension is often called just packing dimension.

Let $0 \leq s < \infty$. For $A \subset \mathbb{R}^n$ and $0 < \delta < \infty$, put

$$P_\delta^s(A) = \sup \sum_i d(B_i)^s$$

where the supremum is taken over all disjoint families (packings) of closed balls $\{B_1, B_2, \ldots\}$ such that $d(B_i) \leq \delta$ and the centres of the B_i 's are in A. Then $P_\delta^s(A)$ is non-decreasing with respect to δ and we set

$$P^s(A) = \lim_{\delta \downarrow 0} P_\delta^s(A) = \inf_{\delta > 0} P_\delta^s(A).$$

Obviously P^s is monotonic and $P^s(\varnothing) = 0$, but unfortunately it is not countably subadditive. To get a measure out of it we use a standard procedure and define

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P^s(A_i) : A = \bigcup_{i=1}^{\infty} A_i \right\}.$$

Then $\mathcal{P}^s$ is a Borel regular measure on $\mathbb{R}^n$. That $\mathcal{P}^s$ is a Borel measure can be verified as in the case of Carathéodory's construction. To see that it is Borel regular, notice first that always $P_\delta^s(\bar{A}) = P_\delta^s(A)$, whence $P^s(\bar{A}) = P^s(A)$. Hence

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P^s(F_i) : A \subset \bigcup_{i=1}^{\infty} F_i, F_i \text{ is closed} \right\},$$

from which the Borel regularity follows as in Theorem 2.1.2. This last formula also gives for Borel sets $B \subset \mathbb{R}^n$,

$$\mathcal{P}^s(B) = \inf \left\{ \sum_{i=1}^{\infty} P^s(B_i) : B = \bigcup_{i=1}^{\infty} B_i, B_i \text{ s are disjoint Borel sets} \right\}$$

Observe also that trivially $\mathcal{P}^s(A) \leq P^s(A)$. It is evident that

$$\mathcal{P}^t(A) = 0 \text{ whenever } \mathcal{P}^s(A) < \infty \text{ and } 0 \leq s < t.$$

Hence the packing measures determine a dimension in the same way as Hausdorff measures. We now show that it is the upper packing dimension defined earlier via the upper Minkowski dimension.

Theorem 3.4.1. For any $A \subset \mathbb{R}^n$,

$$\begin{aligned} \overline{\dim}_p A &= \inf\{s : \mathcal{P}^s(A) = 0\} = \inf\{s : \mathcal{P}^s(A) < \infty\} \\ &= \sup\{s : \mathcal{P}^s(A) > 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\} \end{aligned}$$

Proof. The last four terms are easily seen to equal, and we shall only show

$$\overline{\dim}_p A = d \equiv \inf\{s : \mathcal{P}^s(A) = 0\}.$$

Clearly, for bounded sets $B \subset \mathbb{R}^n$,

$$P(B, \varepsilon/2)\varepsilon^s \leq P_\varepsilon^s(B)$$

which leads to $\overline{\dim}_p A \leq d$. To prove the opposite inequality, let $0 < t < s < d$ and $A_i \subset \mathbb{R}^n$ be bounded with $A = \bigcup_{i=1}^{\infty} A_i$. It is enough to show that $\overline{\dim}_M A_i \geq t$ for some i. Since $\mathcal{P}^s(A) > 0$, there is i such that $P^s(A_i) > 0$. Let $0 < \alpha < P^s(A_i)$. Then for $\delta > 0$, $P_\delta^s(A_i) > \alpha$ and there exist disjoint closed balls $B_1, B_2, \ldots$ with centres in A_i such that $d(B_j) \leq \delta$ and

$$\sum_j d(B_j)^s \geq \alpha$$

Assuming $\delta \leq 1$, let for $m = 0, 1, 2, \ldots, k_m$ be the number of the balls B_j for which $2^{-m-1} < d(B_j) \leq 2^{-m}$. Then

$$\sum_{m=0}^{\infty} k_m 2^{-ms} \geq \sum_j d(B_j)^s \geq \alpha.$$

This yields for some integer $N \geq 0$,

$$2^{Nt}\left(1-2^{t-s}\right)\alpha \leq k_N$$

since otherwise

$$\sum_{m=0}^{\infty} k_m 2^{-ms} < \sum_{m=0}^{\infty} 2^{mt}\left(1-2^{t-s}\right)2^{-ms}\alpha = \alpha.$$

Since $d\left(B_j\right) \leq \delta$ for all j, we have $2^{-N-1} \leq \delta$. Therefore

$$P\left(A_i, 2^{-N-1}\right) \geq k_N \geq 2^{Nt}\left(1-2^{t-s}\right)\alpha,$$

which gives

$$\sup_{0<\varepsilon\leq\delta} P\left(A_i,\varepsilon\right)\varepsilon^t \geq P\left(A_i, 2^{-N-1}\right)2^{-Nt-t} \geq 2^{-t}\left(1-2^{t-s}\right)\alpha.$$

Letting $\delta \downarrow 0$, we obtain

$$\limsup_{\varepsilon\downarrow 0} P\left(A_i,\varepsilon\right)\varepsilon^t > 0$$

and so $\overline{\dim}_M A_i \geq t$ as required. □

Next we state without proof (see [4] Theorem 5.12) a comparison between packing measures and Hausdorff measures.

Theorem 3.4.2. For all $A \subset \mathbb{R}^n, \mathcal{H}^s(A) \leq \mathcal{P}^s(A)$.

Chapter 4

Energies and Capacities

4.1 Energies

Oftentime, we need to study the geometric properties of general Radon measures μ on $\mathbb{R}^n$. Conditions we often impose on them guarantee that not too much measure is concentrated on small regions. This can be expressed for example by the growth condition with some positive numbers s and c,

$$\mu(B(x,r)) \leq cr^s \quad \text{for } x \in \mathbb{R}^n, 0 < r < \infty, \tag{4.1}$$

or by the finiteness of t-**energy** $I_t(\mu)$,

$$I_t(\mu) = \iint |x-y|^{-t} d\mu x d\mu y < \infty. \tag{4.2}$$

We shall see that the conditions (4.1) and (4.2) are very closely related to each other and also to the Hausdorff measures and dimension.

To get some feeling what (4.1) and (4.2) mean, consider $\mu = \mathcal{L}^1 \llcorner [0,1]$. Then (8.1) holds if and only if $s \leq 1$, and (4.2) holds if and only if $t < 1$. This is of course very easy to see. It takes a little more work to show that for any non-zero Radon measure μ on $[0,1]$,(4.1) can hold only if $s \leq 1$, and (4.2) can hold only if $t < 1$. Thus in both cases the range of the possible parameters s and t is bounded from above by 1, which is also the dimension of $[0,1]$. This is no coincidence, and we come to that in greater generality soon.

Let us compare first the conditions (4.1) and (4.2). Using Theorem 1.2.2,

$$\begin{aligned} \int |x-y|^{-t} d\mu y &= \int_0^\infty \mu\left(\left\{y : |x-y|^{-t} \geq u\right\}\right) du \\ &= \int_0^\infty \mu\left(B\left(x, u^{-1/t}\right)\right) du = t \int_0^\infty r^{-t-1} \mu(B(x,r)) dr \end{aligned}$$

by a change of variable. If $\mu\left(\mathbb{R}^n\right) < \infty$ and if for some $s > t, \mu(B(x,r)) \leq cr^s$ for $x \in \mathbb{R}^n, r > 0$, then we immediately see that $I_t(\mu) < \infty$. On the other hand if $I_s(\mu) < \infty$, (4.1) need not quite hold, but it holds for a suitable restriction of μ. Namely, assuming $0 < \mu\left(\mathbb{R}^n\right) < \infty$, the Borel set

$$A = \left\{x : \int |x-y|^{-s} d\mu y \leq M\right\}$$

has positive μ measure for some M. If $\nu = \mu \llcorner A$, then

$$r^{-s} \nu(B(x,r)) \leq \int_{B(x,r)} |x-y|^{-s} d\nu y \leq M \text{ for } x \in A, r > 0.$$

To see that ν really satisfies (4.1), let $x \in \mathbb{R}^n$ and $r > 0$. If $B(x,r) \cap A = \varnothing, \nu(B(x,r)) = 0$. If there is $z \in B(x,r) \cap A$, we have by the above

$$r^{-s}\nu(B(x,r)) \leq 2^s(2r)^{-s}\nu(B(z,2r)) \leq 2^s M.$$

This discussion shows that the two least upper bounds in the next definition agree. For $A \subset \mathbb{R}^n$, let

$$\mathcal{M}(A) = \{\mu : \mu \text{ is a Radon measure with compact support, spt } \mu \subset A \text{ and } 0 < \mu(\mathbb{R}^n) < \infty\}.$$

Definition 4.1.1. The **capacitary dimension** of a set $A \subset \mathbb{R}^n$ is

$$\begin{aligned} \dim_c A &= \sup\{s : \exists \mu \in \mathcal{M}(A) \text{ with } \mu(B(x,r)) \leq r^s \text{ for } x \in \mathbb{R}^n, r > 0\} \\ &= \sup\{t : \exists \mu \in \mathcal{M}(A) \text{ with } I_t(\mu) < \infty\} \end{aligned}$$

Here the supremum is interpreted as 0 if there are no such parameters s or t. For the first this occurs only if $A = \varnothing$. For the second there is no non-zero Radon measure μ on A with $I_t(\mu) < \infty$ for some $t > 0$ if A is finite or countable. There are also uncountable compact sets with this property; in fact, we shall soon see that they are exactly those having Hausdorff dimension zero.

4.2 Capacities and Hausdorff measures

We can also arrive at the capacitary dimension through set functions called Riesz capacities.

Definition 4.2.1. Let $s > 0$. The (Riesz) s-capacity of a set $A \subset \mathbb{R}^n$ is defined by

$$C_s(A) = \sup\left\{I_s(\mu)^{-1} : \mu \in \mathcal{M}(A) \text{ with } \mu(\mathbb{R}^n) = 1\right\}$$

with the interpretation $C_s(\varnothing) = 0$.

The following result is merely a restatement of the definitions.

Theorem 4.2.2. For $s > 0$ and $A \subset \mathbb{R}^n$,

$$\dim_c A = \sup\{s : C_s(A) > 0\} = \inf\{s : C_s(A) = 0\}.$$

Remark 4.2.3. By a trivial approximation we could drop the requirement that the measures μ have compact support in the definitions of $\dim_c A$ and $C_s(A)$.

Before going on let us make one more trivial observation. In contrast to Hausdorff measures the finiteness of capacities says very little: it is easy to see that any bounded set in $\mathbb{R}^n$ has finite s-capacity for all $s > 0$.

Theorem 4.2.4. Let $A \subset \mathbb{R}^n$.

(1) If $s > 0$ and $\mathcal{H}^s(A) < \infty$, then $C_s(A) = 0$.

(2) $\dim_c A \leq \dim A$.

Proof. (1) Suppose $C_s(A) > 0$. Then there is $\mu \in \mathcal{M}(A)$ with $\mu(A) = 1$ and $I_s(\mu) < \infty$. Thus $\int |x-y|^{-s} d\mu y < \infty$ for μ almost all $x \in \mathbb{R}^n$, whence for such x,

$$\lim_{r \downarrow 0} \int_{B(x,r)} |x-y|^{-s} d\mu y = 0.$$

Consequently, given $\varepsilon > 0$ there are $B \subset A$ and $\delta > 0$ such that $\mu(B) > 1/2$ and

$$\mu(B(x,r)) \leq r^s \int_{B(x,r)} |x-y|^{-s} d\mu y \leq \varepsilon r^s \quad \text{for } x \in B \text{ and } 0 < r \leq \delta.$$

Choose sets $E_1, E_2, \ldots$ such that

$$B \subset \bigcup_i E_i, \quad B \cap E_i \neq \varnothing, \quad d(E_i) \leq \delta \quad \text{and}$$
$$\sum_i d(E_i)^s \leq \mathcal{H}^s(A) + 1.$$

Picking $x_i \in B \cap E_i$ and setting $r_i = d(E_i)$, we have

$$1/2 < \mu(B) \leq \sum_i \mu(B(x_i, r_i)) \leq \varepsilon \sum_i r_i^s \leq \varepsilon(\mathcal{H}^s(A) + 1).$$

Letting $\varepsilon \downarrow 0$ we conclude $\mathcal{H}^s(A) = \infty$, which proves (1). (2) follows immediately from (1). □

We state without proof the Frostman's lemma (see [4] Theorem 8.8).

Theorem 4.2.5. Let B be a Borel set in $\mathbb{R}^n$. Then $\mathcal{H}^s(B) > 0$ if and only if there exists $\mu \in \mathcal{M}(B)$ such that $\mu(B(x,r)) \leq r^s$ for $x \in \mathbb{R}^n$ and $r > 0$. Moreover, we can find μ so that $\mu(B) \geq c\mathcal{H}^s_\infty(B)$ where $c > 0$ depends only on n.

Using Frostman's lemma we can now give more complete information about the relations of Hausdorff measures and capacities of Borel sets. The following theorem is often very useful for the estimation of Hausdorff dimension from below.

Theorem 4.2.6. Let A be a Borel set in $\mathbb{R}^n$.

(1) If $s > 0$ and $\mathcal{H}^s(A) < \infty$, then $C_s(A) = 0$.

(2) If $s > 0$ and $C_s(A) = 0$, then $\mathcal{H}^t(A) = 0$ for $t > s$.

(3) $\dim_c A = \dim A$.

Proof. (1) was already stated in Theorem 4.2.4 (1). If $\mathcal{H}^t(A) > 0$, Frostman's lemma gives $\mu \in \mathcal{M}(A)$ for which $\mu(B(x,r)) \leq r^t$. Then for $0 < s < t, I_s(\mu) < \infty$ by the discussion at the beginning of this chapter. Hence $C_s(A) > 0$ and (2) follows. (3) is an immediate consequence of (1) and (2). □

Chapter 5

Analytic Capacity

In this chapter we will introduce the notion of analytic capacity and we will study some of its basic properties, its connection with the Painlevé problem, and its relationship with Hausdorff measures. We will also review some results on the Cauchy transform and Vitushkin's localization operator that are useful for the study of analytic capacity.

A classical problem in complex analysis is the following: which compact sets $E \subset \mathbb{C}$ are removable for bounded analytic functions in the following sense?

Q:
If U is an open set in $\mathbb{C}$ containing E and $f : U \backslash E \to \mathbb{C}$ is a bounded analytic function, then f has an analytic extension to U.

This problem has been studied for almost a century, but a geometric characterization of such removable sets is still lacking. We shall prove some partial results and discuss some other results and conjectures. For many different function classes a complete solution has been given in terms of Hausdorff measures or capacities.

5.1 Basic definitions and properties

Ahlfors introduced a set function γ, called **analytic capacity**, whose null-sets are exactly the removable sets of **Q** above. It is defined for compact sets $E \subset \mathbb{C}$ by

$$\gamma(E) = \sup \left\{ |f'(\infty)| : f \text{ is analytic in } \mathbb{C}\backslash E \text{ with } \|f\|_\infty \leq 1 \right\}. \tag{5.1}$$

Here

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{C}\backslash E\}$$

and

$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$$

with

$$f(\infty) = \lim_{z \to \infty} f(z)$$

(Usually $f'(\infty) \neq \lim_{z\to\infty} f'(z)$.)

One says that a function $f : \mathbb{C}\backslash E \longrightarrow \mathbb{C}$ is **admissible** for E if it is analytic and bounded by 1 in modulus in $\mathbb{C}\backslash E$. So the supremum that defines $\gamma(E)$ is taken over all admissible functions for E.

Let us consider a couple of easy examples: the capacity of a point is zero, because the only functions that are analytic and bounded in the complementary are constants. On the other hand the capacity of a ball $\bar{B}(0,r)$ is positive, as can be seen by taking the function $f(z)=r/z$. Consider the Laurent expansion of f near ∞ :

$$f(z)=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots.$$

Clearly, $f'(\infty)=a_1$, and also

$$f'(\infty)=\frac{1}{2\pi i}\int_\Gamma f(z)dz, \tag{5.2}$$

where Γ is any rectifiable curve surrounding E with a suitable orientation. Let us remark that, in general, $f'(\infty)\neq\lim_{z\to\infty}f'(z)$. Instead, $f'(\infty)=g'(0)$ for $g(z)=f\left(\frac{1}{z}\right)$. If $A\subset\mathbb{C}$ is an arbitrary set, then we define

$$\gamma(A)=\sup_{E\subset A,E\text{ compact}}\gamma(E).$$

The outer boundary of a compact set E is the boundary of the unbounded component of $\mathbb{C}\backslash E$. It is denoted by $\partial_o E$. Obviously, $\partial_o E\subset\partial E$.

Proposition 5.1.1. The following properties hold:

(a) If $E\subset F$, then $\gamma(E)\leq\gamma(F)$.

(b) For all $z,\lambda\in\mathbb{C}$,
$$\gamma(z+\lambda E)=|\lambda|\gamma(E).$$

(c) For every compact set $E\subset\mathbb{C}$,
$$\gamma(E)=\gamma\left(\partial_o E\right).$$

Proof. The statements (a) and (c) are straightforward consequences of the definition, while (b) follows by an easy change of variables. □

Proposition 5.1.2. Let $E\subset\mathbb{C}$ be compact. The supremum in (5.1) is attained, and any admissible function f which attains it satisfies $f(\infty)=0$ if $\gamma(E)>0$.

As a consequence, of this proposition, one obtains an equivalent definition for $\gamma(E)$ if in the supremum (1.1) one asks $f(\infty)=0$ too.

Proof. To see that the supremum is attained one just has to notice that the family of admissible functions for E is a normal family, and so one can consider a sequence of admissible functions $\{f_k\}_k$ such that $f_k'(\infty)\to\gamma(E)$, and take f to be the limit of a convergent subsequence.

To check that $f(\infty)=0$ if f is admissible and attains the supremum when $\gamma(E)>0$, consider the function

$$g(z)=\frac{f(z)-f(\infty)}{1-\overline{f(\infty)}f(z)}.$$

It is clear $|g(z)|\leq 1$ for all $z\in\mathbb{C}\backslash E, g(\infty)=0$, and moreover,

$$g'(\infty)=\lim_{z\to\infty}\frac{z(f(z)-f(\infty))}{1-\overline{f(\infty)}f(z)}=\frac{f'(\infty)}{1-|f(\infty)|^2},$$

and thus, $|g'(\infty)|>|f'(\infty)|$ if $\gamma(E)=f'(\infty)\neq 0$. □

Proposition 5.1.3. Let $E\subset\mathbb{C}$ be a compact connected set different from a single point, and let f be a conformal map of the unbounded connected component of $\mathbb{C}_\infty\backslash E$ to the unit disk satisfying $f(\infty)=0$. Then $\gamma(E)=|f'(\infty)|$.

Proof. Since f is admissible, it is obvious that $\gamma(E) \geq |f'(\infty)|$. On the other hand, if g is also admissible and $g(\infty) = 0$, then $g \circ f^{-1} : B(0,1) \to B(0,1)$ is analytic and fixes the origin. Thus, by the Schwarz lemma, $\left|g \circ f^{-1}(z)\right| \leq |z|$ for all z in the unit disk. Consequently, $|g(z)| \leq |f(z)|$ for all $z \in \mathbb{C} \backslash E$, which implies that $|g'(\infty)| \leq |f'(\infty)|$ □

Proposition 5.1.4. The analytic capacity of a disk is its radius. The analytic capacity of a segment of length ℓ equals $\ell/4$.

Proof. The first statement follows by taking into account that, for a disk $B(z_0, r)$, the function

$$f(z) = \frac{r}{z - z_0}$$

maps conformally $\mathbb{C}_\infty \backslash \bar{B}(z_0, r)$ to the unit disk and $f'(\infty) = r$. For the second one, notice that the function

$$f(z) = \left(z + \frac{1}{z}\right)\frac{\ell}{4}$$

maps conformally the unit disk to $\mathbb{C}_\infty \backslash [-\ell/2, \ell/2]$ and satisfies $f(0) = \infty$. Therefore,

$$\gamma([-\ell/2, \ell/2]) = \lim_{z \to \infty} \left|z f^{-1}(z)\right| = \lim_{z \to 0} |f(z) z| = \frac{\ell}{4} \tag{5.3}$$

□

Recall that the $1/4$ Koebe theorem asserts that if $f : B(0,1) \to \mathbb{C}$ is analytic, univalent, and $f(0) = 0, f'(0) = 1$, then $B(0, 1/4) \subset f(B(0,1))$. In the next proposition we show an important application of this result to analytic capacity.

Proposition 5.1.5. If $E \subset \mathbb{C}$ is compact and connected, then

$$\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E).$$

Proof. The second inequality follows from the preceding proposition and the fact that E is contained in a closed disk with radius $\operatorname{diam}(E)$.

For the first one, let $U \subset \mathbb{C}_\infty$ be the unbounded component of $\mathbb{C}_\infty \backslash E$ and consider the conformal mapping $f : U \to B(0,1)$, with $f(\infty) = 0$. Take $z_1, z_2 \in E$ such that $|z_1 - z_2| = \operatorname{diam}(E)$ and consider the function

$$g(z) = \frac{\gamma(E)}{f^{-1}(z) - z_1}.$$

This is a univalent map in the unit disk because f^{-1} is so. Further, $g(0) = 0$ and arguing as in (1.3), $\left|z f^{-1}(z)\right| \to \gamma(E)$ as $z \to 0$, and thus

$$|g'(0)| = \lim_{z \to 0} \left|\frac{\gamma(E)}{z\left(f^{-1}(z) - z_1\right)}\right| = 1.$$

As z_2 is not in the range of $f^{-1}, \gamma(E)/\left(z_2 - z_1\right)$ is not either in the range of g. Therefore, by Koebe's $1/4$ theorem,

$$\frac{\gamma(E)}{|z_2 - z_1|} \geq \frac{1}{4},$$

and the proposition follows. □

Corollary 5.1.6. If $\gamma(E) = 0$, then E is totally disconnected.

As shown above, for a continuum $E \subset \mathbb{C}$, the analytic capacity can be computed easily if one knows the conformal mapping between the unbounded component of $\mathbb{C}_\infty \backslash E$ and the unit disk, and moreover one has the estimate $\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E)$. For disconnected sets the situation is much more difficult. Obtaining precise identities in simple cases is already difficult and estimates analogous to the preceding one are missing.

Proposition 5.1.7 (Outer regularity of γ)**.** Let $\{E_n\}_{n\geq 0}$ be a sequence of compact sets in $\mathbb{C}$ such that $E_{n+1} \subset E_n$ for each n. Then

$$\gamma\left(\bigcap_{n\geq 0} E_n\right) = \lim_{n\to\infty} \gamma\left(E_n\right).$$

Proof. Let us write $E = \bigcap_{n\geq 0} E_n$. Since $\{\gamma(E_n)\}_n$ is a non-increasing sequence, it is clear that the limit on the right-hand side above exists. Also, taking into account that $E \subset E_n$ for all n, we get $\gamma(E) \leq \lim_{n\to\infty} \gamma\left(E_n\right)$.

For the converse inequality, for each n, take a function f_n admissible for E_n such that $f_n'(\infty) = \gamma\left(E_n\right)$. Since $\{f_n\}_n$ is a normal family on $\mathbb{C}\backslash E$, there exists some subsequence $\{f_{n_k}\}_k$ which is uniformly convergent on compact subsets of $\mathbb{C}\backslash E$ to some function f. It turns out that f is admissible for E and moreover, using (5.2),

$$f'(\infty) = \lim_{k\to\infty} f_{n_k}'(\infty) = \lim_{k\to\infty} \gamma\left(E_{n_k}\right) = \lim_{n\to\infty} \gamma\left(E_n\right).$$

Therefore, $\gamma(E) \geq |f'(\infty)| = \lim_{n\to\infty} \gamma\left(E_n\right)$. □

Corollary 5.1.8. If $E \subset \mathbb{C}$ is compact, then

$$\gamma(E) = \inf\{\gamma(U) : U \text{ open}, U \supset E\}.$$

Proposition 5.1.9. For every compact set E, there is an admissible function f for E such that $f'(\infty) = \gamma(E)$. Such a function is unique in the unbounded component of $\mathbb{C}_\infty \backslash E$ if $\gamma(E) > 0$. The function f in the proposition is called the **Ahlfors function** of E.

Proof. The existence of f has already been shown in Proposition 1.2. So let us turn to the uniqueness. Suppose that there are two admissible functions f_1, f_2, such that $f_1'(\infty) = f_2'(\infty) = \gamma(E)$ and $f_1(\infty) = f_2(\infty) = 0$. Then $f = \frac{f_1+f_2}{2}$ is also admissible and $f'(\infty) = \gamma(E), f(\infty) = 0$. Let $g = \frac{f_2-f_1}{2}$, so that $f_1 = f - g$ and $f_2 = f + g$. From the inequalities

$$|f \pm g|^2 = |f|^2 + |g|^2 \pm 2\operatorname{Re}(f\bar{g}) \leq 1,$$

we infer that $|f|^2 + |g|^2 \leq 1$. Using also that $1 + |f| \leq 2$, we deduce

$$\frac{|g|^2}{2} \leq \frac{1-|f|^2}{2} = \frac{(1-|f|)(1+|f|)}{2} \leq 1 - |f|.$$

Thus,

$$|f| + \frac{|g|^2}{2} \leq 1$$

If $f_1 \neq f_2$ near ∞, then $g \neq 0$ in the unbounded component of $\mathbb{C}\backslash E$, and we may consider the Laurent series of $\frac{g^2}{2}$ for z near ∞ :

$$\frac{g(z)^2}{2} = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + \cdots, \quad a_n \neq 0.$$

Since $g(\infty)=0$, we have $n \geq 2$. For $\varepsilon>0$, take the function

$$\widetilde{f}(z)=f(z)+\varepsilon \overline{a_n} z^{n-1} \frac{g(z)^2}{2}.$$

If ε is small enough so that $\left|\varepsilon \overline{a_n} z^{n-1}\right| \leq 1$ in some bounded neighborhood of E, then we deduce that

$$|\tilde{f}(z)| \leq|f(z)|+\left|\varepsilon \overline{a_n} z^{n-1} \frac{g(z)^2}{2}\right| \leq|f(z)|+\frac{|g(z)|^2}{2} \leq 1$$

in that neighborhood, and thus in the whole unbounded component of $\mathbb{C} \backslash E$ by the maximum principle. On the other hand,

$$\tilde{f}^{\prime}(\infty)=f^{\prime}(\infty)+\varepsilon\left|a_n\right|^2>\gamma(E),$$

which is a contradiction. □

Let us remark that a slight modification in the arguments for Proposition 5.1.7 shows that the Ahlfors functions of $E_n, n \geq 0$, converge on compact subsets to the Ahlfors function of $\bigcap_{n \geq 0} E_n$.

5.2 Removable sets and the Painlevé problem

A compact set $E \subset \mathbb{C}$ is said to be **removable** for bounded analytic functions (or just, removable) if for every open set Ω containing E, each bounded function analytic on $\Omega \backslash E$ has an analytic extension to Ω. For example, by a well-known theorem of Riemann, if E is a finite collection of points, then it is removable. Also, using Baire's category theorem, one can extend this result to the case where E is countable and compact. On the other hand, a disk $\bar{B}\left(z_0, r\right)$ is not removable: just consider the function $f(z)=1 /\left(z-z_0\right)$, which is analytic and bounded in $\mathbb{C} \backslash \bar{B}\left(z_0, r\right)$ and cannot be extended analytically to the whole $\bar{B}\left(z_0, r\right)$.

Painlevé's problem consists in characterizing removable sets for bounded analytic functions in a metric/geometric way. Because of the next result, essentially due to Ahlfors, this turns out to be equivalent to describing compact sets with zero analytic capacity in metric/geometric terms.

Proposition 5.2.1. Let $E \subset \mathbb{C}$ be compact. The following are equivalent: (i) E is removable for bounded analytic functions. (ii) There exists an open set $\Omega \supset E$ such that every bounded function analytic in $\Omega \backslash E$ has an analytic extension to Ω. (iii) Every function analytic and bounded in $\mathbb{C} \backslash E$ is constant. (iv) $\gamma(E)=0$.

Proof. That (i) $\Rightarrow$ (ii) is trivial. The implication (ii) $\Rightarrow$ (iii) is also immediate: given $f: \mathbb{C} \backslash E \rightarrow \mathbb{C}$ analytic and bounded, considering its restriction to $\Omega \backslash E$, it turns out that f can be extended to an entire bounded function, and thus it must be constant by Liouville's theorem.

(iii) $\Rightarrow$ (iv) is a direct consequence of the definition of γ. To prove the converse implication, let $E \subset \mathbb{C}$ be compact with $\gamma(E)=0$. Suppose that there exists a non-constant bounded analytic function $f: \mathbb{C} \backslash E \rightarrow \mathbb{C}$, so that $f(\infty) \neq f\left(z_0\right)$ for some $z_0 \in \mathbb{C} \backslash E$. Consider the function

$$g(z)=\frac{f(z)-f\left(z_0\right)}{z-z_0} \quad \text { for } z \neq z_0,$$

and $g(z)=f^{\prime}\left(z_0\right)$. It is easy to check that g is bounded in $\mathbb{C} \backslash E, g(\infty)=0$ and $g^{\prime}(\infty)=f(\infty)-f\left(z_0\right) \neq 0$, and thus $\gamma(E)>0$, which is a contradiction.

To show that (iii) $\Rightarrow$ (ii), just take $\Omega=\mathbb{C}$. Finally, let us see that (iii) $\Rightarrow$ (i). To this end, consider an arbitrary open set $\Omega \supset E$ and take $f: \Omega \backslash E \rightarrow \mathbb{C}$ analytic and bounded. We have to show that f can be extended analytically to the whole of Ω. We may assume that Ω is connected (otherwise, we consider each component separately). Then $\Omega \backslash E$ is connected because E is totally disconnected (by Corollary 5.1.6, since (iii) implies that $\gamma(E)=0$). Take smooth curves $\Gamma_1, \Gamma_2 \subset \Omega$ surrounding E, with Γ_2 very close to E, and consider a point z inside Γ_1 and outside Γ_2. Then we have

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}dw-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(w)}{w-z}dw=f_1(z)+f_2(z).$$

It is easy to check that $f_1(z)$ and $f_2(z)$ do not depend on the precise curves Γ_1, Γ_2, as long as z is inside Γ_1 and outside Γ_2. Then f_1 is analytic in Ω and f_2 in $\mathbb{C}\backslash E$. Moreover, since f_1 is bounded near ∂E, it turns out that f_2 is also bounded near ∂E, and so in the whole $\mathbb{C}\backslash E$, by the maximum principle. Since f_2 vanishes at ∞, (iii) implies that $f_2=0$, and thus $f=f_1$ is analytic in Ω. □

Let us insist on the fact that saying that a compact set is removable is the same as saying that it has zero analytic capacity. In a sense, the reader should think that analytic capacity measures the size of a set as a non-removable singularity for bounded analytic functions.

A stronger version of the implication (iv) $\Rightarrow$ (i) will be proved in Proposition 1.18 below. Moreover, that proof avoids the technical problem of the construction of the curves Γ_1 and Γ_2 in the preceding proposition.

5.3 The Cauchy transform and Vitushkin's localization operator

The **Cauchy transform** of a (possibly complex) finite measure ν on $\mathbb{C}$ is defined by

$$\mathcal{C}\nu(z)=\int\frac{1}{\xi-z}d\nu(\xi). \tag{5.4}$$

The integral is absolutely convergent for a.e. $z\in\mathbb{C}$, with respect to Lebesgue measure. This follows easily from Fubini's theorem, taking into account that $\int_K \frac{1}{|z|}d\mathcal{L}^2(z)<\infty$ on compact sets K.

The Cauchy transform appears naturally in complex analysis. For instance, by Cauchy's integral formula, if f is a function analytic in a simply connected

open set $\Omega\subset\mathbb{C}, \Gamma\subset\Omega$ is a closed rectifiable Jordan curve, and z is a point which belongs to the bounded component of $\mathbb{C}\backslash\Gamma$ (and so $z\in\Omega$), then $f(z)=\mathcal{C}\nu(z)$, where ν is the complex measure

$$\nu=\frac{1}{2\pi i}f(z)dz_\Gamma.$$

Proposition 5.3.1. If ν is a complex measure, then $\mathcal{C}\nu$ is locally integrable in $\mathbb{C}$ (with respect to Lebesgue measure). Moreover, the integral that defines the Cauchy transform in (5.4) is absolutely convergent for a.e. $z\in\mathbb{C}$, with respect to Lebesgue measure. Further, $\mathcal{C}\nu$ is analytic in $\mathbb{C}\backslash\operatorname{supp}\nu, \mathcal{C}\nu(\infty)=0$ and $(\mathcal{C}\nu)'(\infty)=-\nu(\mathbb{C})$.

The easy proof is left for the reader. A straightforward consequence of the proposition is that, if a set E supports a measure ν such that $\nu(\mathbb{C})\neq 0$ and $\mathcal{C}\nu$ is bounded in $\mathbb{C}\backslash E$, then $\gamma(E)\neq 0$ and so E is not removable. One should view the Cauchy transform as a tool for constructing analytic functions. Usually, the difficulties arise when one tries to check that the constructed functions are bounded in modulus.

In the remainder of this section we assume that the reader is familiar with the very basics of the theory of distributions. By using distributions, the properties of the Cauchy transform and the localization operator of Vitushkin that we will see below become much more natural. In fact, it is possible to talk about Vitushkin's localization operator without appealing to distributions, but then the results look less transparent. The reader will find all the necessary results, and much more, in [9] Chapter 6, for example.

The definition of Cauchy transform also makes sense if ν is a compactly supported (complex) distribution. In this case we set

$$\mathcal{C}\nu = -\frac{1}{z} * \nu.$$

A key fact that explains why the Cauchy transform is so important in complex analysis is the following:

Theorem 5.3.2. The kernel $\frac{1}{\pi z}$ is the fundamental solution of the $\bar{\partial}$ operator. That is,

$$\bar{\partial}\frac{1}{\pi z} = \delta_0,$$

where δ_0 is the Dirac delta at the origin. As a consequence, if ν is a compactly supported distribution on $\mathbb{C}$,

$$\bar{\partial}(\mathcal{C}\nu) = -\pi\nu \tag{5.5}$$

Also, if $f \in L^1_{\text{loc}}(\mathbb{C})$ (or, more generally, $f \in \mathcal{D}'$) is analytic in a neighborhood of ∞ and $f(\infty) = 0$, then

$$\mathcal{C}(\bar{\partial}f) = -\pi f$$

All the identities in the preceding theorem must be understood in the sense of distributions. For the proof, see Conway [19, p. 195], for instance. Notice that, as a consequence of the last statement in the theorem, if a distribution ν satisfies $\bar{\partial}\nu = 0$ (and, in particular, it is analytic in a neighborhood of ∞) and vanishes at ∞, then $\nu = 0$. Further, any function $f \in L^1_{loc}(\mathbb{C})$ or distribution $f \in \mathcal{D}'$ which is analytic in a neighborhood of ∞ and vanishes at ∞ is the Cauchy transform of a unique compactly supported distribution, namely $\frac{1}{\pi}\bar{\partial}f$.

Proposition 5.3.3. If ν is a compactly supported distribution, then $\mathcal{C}\nu$ is analytic in $\mathbb{C}\setminus\operatorname{supp}(\nu)$, and moreover,

$$\mathcal{C}\nu(\infty) = 0 \quad \text{and} \quad (\mathcal{C}\nu)'(\infty) = -\langle \nu, 1\rangle = -\nu(\mathbb{C}).$$

Here, $\langle\cdot,\cdot\rangle$ stands for the pairing between distributions of compact support and the corresponding test functions (i.e. C^∞ functions).

Proof. By the preceding theorem, $\bar{\partial}(\mathcal{C}\nu) = -\pi\nu$, and so $\mathcal{C}\nu$ is analytic out of $\operatorname{supp}(\nu)$. To show that $\mathcal{C}\nu(\infty) = 0$, take $r > 0$ such that $\operatorname{supp}(\nu) \subset B(0,r)$, and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a C^∞ radial function such that $0 \leq \varphi \leq 1$, which vanishes on $B(0, r/2)$, and equals 1 on $\mathbb{C}\setminus B(0,r)$. Write $k_r(z) = \varphi(z)\frac{1}{z}$, for $z \in \mathbb{C}$. It is easy to check that

$$\nu * \frac{1}{z} = \nu * k_r \quad \text{in } \mathbb{C}\setminus\bar{B}(0,2r),$$

in the sense of distributions. Moreover, since $k_r(z)$ is a C^∞ radial function,

$$\nu * k_r(z) = \langle \nu, \tau_z k_r\rangle,$$

where $\tau_z k_r(w) = k_r(w-z)$. It is easy to check that $\tau_z k_r \to 0$ as $z \to \infty$ in C^∞ with the topology of test functions, and so

$$\langle \nu, \tau_z k_r\rangle \to 0 \quad \text{as } z \to \infty.$$

That is, $\mathcal{C}\nu(\infty) = -\nu * \frac{1}{z}(\infty) = -\nu * k_r(\infty) = 0$.

To prove that $(\mathcal{C}\nu)'(\infty) = \langle \nu, -1\rangle$, consider a radial C^∞ approximation of the identity $\{\psi_\varepsilon\}_{\varepsilon>0}$, so that $\operatorname{supp}\psi_\varepsilon \subset B(0,\varepsilon)$ and $\int \psi_\varepsilon d\mathcal{L}^2 = 1$ for all $\varepsilon > 0$. Then we have

$$\psi_\varepsilon * \mathcal{C}\nu = \psi_\varepsilon * \left(-\frac{1}{z} * \nu\right) = -\frac{1}{z} * (\psi_\varepsilon * \nu) = \mathcal{C}(\psi_\varepsilon * \nu)$$

Thus we deduce that $\mathcal{C}(\psi_\varepsilon * \nu)$ is analytic in $\mathbb{C}\backslash\overline{\mathcal{U}_\varepsilon(\operatorname{supp}(\nu))}$, converges locally uniformly to $\mathcal{C}\nu$ on compact subsets of $\mathbb{C}\backslash\overline{\mathcal{U}_\varepsilon(\operatorname{supp}(\nu))}$, and vanishes at ∞. From equation (5.2) one infers easily that

$$\lim_{\varepsilon\to 0}\left(\mathcal{C}(\psi_\varepsilon * \nu)\right)'(\infty) = (\mathcal{C}\nu)'(\infty)$$

On the other hand, by Proposition 5.3.1, since $\psi_\varepsilon * \nu$ is a C^∞ function (and thus a measure),

$$\left(\mathcal{C}(\psi_\varepsilon * \nu)\right)'(\infty) = -\int 1 d(\psi_\varepsilon * \nu) = -\langle(\psi_\varepsilon * \nu), 1\rangle = -\langle \nu, 1 * \psi_\varepsilon\rangle = -\langle \nu, 1\rangle,$$

and thus the last claim in the proposition follows. □

Given $f \in L^1_{\text{loc}}(\mathbb{C})$ and $\varphi \in C^\infty$ compactly supported, we define

$$V_\varphi f := \varphi f + \frac{1}{\pi}\mathcal{C}(f\bar{\partial}\varphi) \tag{5.6}$$

For a fixed function φ, one calls V_φ **Vitushkin's localization operator** (associated with φ). The same definition (5.6) makes sense if f is a distribution from $\mathcal{D}'$.

Proposition 5.3.4. Let $f \in L^1_{loc}(\mathbb{C})$ (or more generally, $f \in \mathcal{D}'$) and $\varphi \in C^\infty$ be compactly supported. Then we have

$$V_\varphi f = \frac{-1}{\pi}\mathcal{C}(\varphi\bar{\partial} f)$$

in the sense of distributions. In the identity above, $\bar{\partial} f$ should be understood in the sense of distributions.

Proof. By (5.6) and (5.5),

$$\bar{\partial}(V_\varphi f) = f\bar{\partial}\varphi + \varphi\bar{\partial} f + \frac{1}{\pi}\bar{\partial}\mathcal{C}(f\bar{\partial}\varphi) = \varphi\bar{\partial} f = \bar{\partial}\left(\frac{-1}{\pi}\mathcal{C}(\varphi\bar{\partial} f)\right).$$

Moreover, since $\varphi f, f\bar{\partial}\varphi$ and $\varphi\bar{\partial} f$ are all compactly supported, it follows that both $V_\varphi f$ and $\frac{-1}{\pi}\mathcal{C}(\varphi\bar{\partial} f)$ are analytic in a neighborhood of ∞ and vanish at ∞. By the remark after Theorem 5.3.2, it turns out that both distributions are equal. □

Observe that if $f = \mathcal{C}\nu$, where ν is a compactly supported complex measure or distribution, then

$$V_\varphi(\mathcal{C}\nu) = \mathcal{C}(\varphi\nu) \tag{5.7}$$

This fundamental identity justifies why V_φ is called (Vitushkin's) localization operator: $\mathcal{C}\nu$ is a analytic in $\mathbb{C}\setminus\operatorname{supp}(\nu)$, and so $\operatorname{supp}(\nu)$ can be understood as the set of singularities of $\mathcal{C}\nu$. By (1.7), it turns out that $V_\varphi(\mathcal{C}\nu)$ is analytic in the larger set $\mathbb{C}\setminus\operatorname{supp}(\varphi\nu)$. So the singularities are now localized to $\operatorname{supp}(\nu)\cap\operatorname{supp}(\varphi)$.

In the next proposition we show that the operator V_φ enjoys other nice properties, besides the one about localization of singularities.

Proposition 5.3.5. Let $\varphi \in C^\infty$ be supported in a ball B_r of radius r, with $\|\varphi\|_\infty \leq c_4$ and $\|\nabla\varphi\|_\infty \leq c_4/r$. For any function $f \in L^1_{\text{loc}}(\mathbb{C})$, the following properties hold:

(i) $\|V_\varphi f\|_\infty \leq c_5\|f\chi_{B_r}\|_\infty$, (ii) $\|V_\varphi f\|_\infty \leq c_6\omega_f(r)$, where ω_f stands for the modulus of continuity of f, (iii) $V_\varphi f$ is holomorphic outside $\operatorname{supp}(\bar{\partial} f)\cap\operatorname{supp}(\varphi)$, (iv) if f is bounded in B_r, then $V_\varphi f$ is continuous where f is.

The constants c_5 and c_6 depend only on c_4.

Proof. By (5.6),

$$\|V_\varphi f\|_\infty \le \|\varphi f\|_\infty + \frac{1}{\pi}\|\mathcal{C}(f\bar\partial\varphi)\|_\infty.$$

Clearly, $\|\varphi f\|_\infty \le c_4 \|\chi_{B_r} f\|_\infty$. Also, for any $z \in \mathbb{C}$,

$$\begin{aligned}|\mathcal{C}(f\bar\partial\varphi)(z)| &\le \int_{B_r} \frac{1}{|w-z|}|f(w)||\bar\partial\varphi(w)|d\mathcal{L}^2(w) \\ &\le \|f\chi_{B_r}\|_\infty \|\nabla\varphi\|_\infty \int_{B_r}\frac{1}{|w-z|}d\mathcal{L}^2(w) \le c\|f\chi_{B_r}\|_\infty\end{aligned}$$

and thus (i) follows. By Proposition 5.3.4, V_φ vanishes on constants. Then replacing f by $f - f(z_0)$ in (i), where z_0 is the center of B_r, we get (ii).

The third statement is a direct consequence of the identity $V_\varphi f = \frac{-1}{\pi}\mathcal{C}(\varphi\bar\partial f)$. The fourth one follows from (5.6), taking into account that the integral

$$\int \frac{f(w)}{w-z} d\mathcal{L}^2(w)$$

depends continuously on z when f is bounded on B_r. □

Proposition 5.3.6. Let $\Omega \subset \mathbb{C}$ be open and $E \subset \mathbb{C}$ compact with $\gamma(E) = 0$. Then every function analytic and bounded in $\Omega\backslash E$ can be extended analytically to the whole set Ω.

Notice that it is not assumed that $E \subset \Omega$. In particular, it may happen that $E \cap \partial\Omega \neq \varnothing$. This is the main difference with Proposition 5.2.1 above.

Proof. We may assume Ω to be bounded. Consider a grid of squares $\{Q_i\}_{i\in I}$ in $\mathbb{C}$ with side length $\ell(Q_i) = \ell$ for every $i \in I$. Let $\{\varphi_i\}_{i\in I}$ be a partition of unity of C^∞ functions subordinated to the squares $\{2Q_i\}_{i\in I}$, so that $\operatorname{supp}(\varphi_i) \subset 2Q_i$ for each $i \in I$ and $\sum_{i\in I}\varphi_i \equiv 1$ on $\mathbb{C}$.

Extend f by zero to $\mathbb{C}\backslash(\Omega\backslash E)$. Since f vanishes out of a bounded set, $f = \frac{-1}{\pi}\mathcal{C}(\bar\partial f)$ and $V_\varphi f$ is identically zero except for finitely many indices $i \in I$. So we have

$$f = \frac{-1}{\pi}\sum_{i\in I}\mathcal{C}\left(\varphi_i\bar\partial f\right) = \sum_{i\in I} V_{\varphi_i} f$$

Notice that, since $\operatorname{supp}(\bar\partial f) \subset E \cup \partial\Omega$, then for each $i \in I$,

$$\operatorname{supp}\left(\bar\partial V_{\varphi_i} f\right) \subset 2Q_i \cap (E \cup \partial\Omega).$$

As a consequence, if $2Q_i \cap \partial\Omega = \varnothing$, then $V_{\varphi_i} f$ is analytic out of $2Q_i \cap E$. Since $V_{\varphi_i} f$ vanishes at ∞, $\|V_\varphi f\|_\infty < \infty$ (by (i) in the preceding proposition), and

$$\gamma(2Q_i \cap E) \le \gamma(E) = 0,$$

we infer that

$$V_{\varphi_i} f \equiv 0 \quad \text{if } 2Q_i \cap \partial\Omega = \varnothing.$$

Therefore,

$$f = \sum_{i\in I: 2Q_i\cap\partial\Omega\neq\varnothing} V_{\varphi_i} f.$$

Thus, f is analytic in $\Omega\backslash\overline{\mathcal{U}_{4\ell}(\partial\Omega)}$. Since ℓ is arbitrarily small, it turns out that f is analytic in the whole of Ω. □

5.4 Analytic capacity, Riesz capacity and Hausdorff measures

We shall now give two simple relations between analytic capacity and Hausdorff measures. Recall that $\dim E > 1$ implies $C_1(E) > 0$ for Borel sets by Theorem 4.2.6.

Theorem 5.4.1. If $E \subset \mathbb{C}$ is compact and $C_1(E) > 0$, then $\gamma(E) > 0$.

Proof. Since $C_1(E) > 0$ there is a Radon measure μ with $\operatorname{spt}\mu \subset E$, $0 < \mu(E) < \infty$, and

$$\int \frac{d\mu\zeta}{|\zeta - z|} \leq 1 \quad \text{for } z \in \mathbb{C}$$

(exercise: take a suitable restriction of a measure ν with $I_1(\nu) < \infty$.) Setting

$$f(z) = \int \frac{d\mu\zeta}{\zeta - z}, \quad z \in \mathbb{C}\backslash E$$

a direct computation shows that f has complex derivative in $\mathbb{C}\backslash E$, whence it is analytic, $f(\infty) = 0$, and

$$f'(\infty) = \lim_{z\to\infty} \int \frac{1}{\zeta/z - 1} d\mu\zeta = -\mu(E) \neq 0.$$

Thus $\gamma(E) > 0$. □

In the other direction we have a theorem of Painlevé from the last century.

Theorem 5.4.2. If $E \subset \mathbb{C}$ is compact and $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.

Proof. Let $z \in \mathbb{C}\backslash E$ and let $0 < \varepsilon < d(z, E)/2$. We can cover the compact set E with discs $B_j, j = 1, \ldots, k$, such that $E \cap B_j \neq \varnothing$ and $\sum_{j=1}^k d(B_j) < \varepsilon$. Let $f : \mathbb{C}\backslash E \to \mathbb{C}$ be analytic with $\|f\|_\infty \leq 1$ and $f(\infty) = 0$. Choosing R such that $E \cup \{z\} \subset B(R)$ and letting $\Gamma = \partial\left(\bigcup_{j=1}^k B_j\right)$, we have by the Cauchy integral formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{S(R)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Since $f(\infty) = 0$, the first integral tends to zero as R tends to infinity. Thus

$$|f(z)| \leq \frac{1}{2\pi} \int_\Gamma \frac{|f(\zeta)|}{|\zeta - z|} d\mathcal{H}^1\zeta \leq \frac{1}{\pi d(z, E)} \sum_{j=1}^k \mathcal{H}^1(\partial B_j) \leq \frac{\varepsilon}{d(z, E)}.$$

Letting $\varepsilon \downarrow 0$, we obtain $f(z) = 0$. Hence $\gamma(E) = 0$. □

Our main interest here is to know which compact sets have zero analytic capacity. The two simple theorems above show that we only have to worry about sets of Hausdorff dimension one. We shall mainly pay attention to sets of finite $\mathcal{H}^1$ measure and later on give some comments on those having infinite, or more essentially, non- σ-finite, $\mathcal{H}^1$ measure.

We shall see that there are many sets of positive $\mathcal{H}^1$ measure and zero analytic capacity. But before this we briefly look at an important class of sets for which $\mathcal{H}^1$ and γ are simultaneously zero. We give the proof for the following deep theorem only in a very special case.

Theorem 5.4.3. Let $\Gamma \subset \mathbb{C}$ be a rectifiable curve (i.e., $\Gamma = f(I)$ for a Lipschitz map f) and E a compact subset of Γ. Then $\gamma(E) = 0$ if and only if $\mathcal{H}^1(E) = 0$.

proof for $\Gamma \subset \mathbb{R}$. Let $E \subset \mathbb{R} \subset \mathbb{C}$ with $\mathcal{L}^1(E) > 0$. Set

$$g(z) = \int_E \frac{1}{x-z} d\mathcal{L}^1 x, \quad z \in \mathbb{C} \backslash E.$$

By a direct computation the values of g are contained in the strip $S = \{x + \mathrm{i}y : |y| < \pi\}$. Let $h : S \to U(1)$ be a conformal map. Then $f = h \circ g$ is a bounded non-constant analytic function in $\mathbb{C} \backslash E$, whence $\gamma(E) > 0$. □

Remark 5.4.4. The above proof gives an estimate $\gamma(E) \geq c\mathcal{L}^1(E)$ for $E \subset \mathbb{R}$. One can show more precisely that $\gamma(E) = \mathcal{L}^1(E)/4$ for compact subsets E of $\mathbb{R}$.

An immediate consequence of above theorem is that if E is a compact subset of $\mathbb{C}$ with $\mathcal{H}^1(E) < \infty$ and $\gamma(E) = 0$, then E is purely 1-unrectifiable. Indeed, otherwise $\mathcal{H}^1(E \cap \Gamma) > 0$ for some rectifiable curve Γ and so $\gamma(E) > 0$. A reasonable conjecture seems to be that the converse also holds (see [4] p.275 "Analytic capacity and rectifiability" for a partial result of this conjecture).

Conjecture. Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then $\gamma(E) = 0$ if and only if E is purely 1-unrectifiable.

Remark 5.4.5. For more on rectifiability, see [4] chapter 15 "Rectifiable sets and approximate tangent planes." For more on the relationship between analytic capacity and rectifiability, see [5].

5.5 Semiadditivity of analytic capacity

See [10] Chapter 6 for more on semiadditivity of analytic capacity, where the following result is obtained. Proposition 1.19 shows how Vitushkin's localization operator can be used to prove the semiadditivity of analytic capacity in two very particular cases (when the compact sets are of particular geometric shapes).

Theorem 5.5.1. Let $E \subset \mathbb{C}$ be compact. Let $E_i, i \geq 1$, be Borel sets such that $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\gamma(E) \leq c \sum_{i=1}^{\infty} \gamma(E_i)$$

where c is an absolute constant.

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