Lecture Note on Compact Lie Group

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Contents

Chapter 1

Lie Groups: An Introduction

1.1 Basics

A **Lie group** is a smooth manifold G (without boundary) that is also a group, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by $m(g, h) = gh$, $i(g) = g^{-1}$, are both smooth. A group homomorphism $F : G \to H$ between Lie groups is a Lie group homomorphism if it is also smooth. It is a **Lie group isomorphism** if it is also a diffeomorphism.

Proposition 1.1.1.

(1) *[LeeSM] Theorem 7.5: Every Lie group homomorphism has constant rank.*

(2) *[LeeSM] Corollary 7.6: A Lie group homomorphism is a Lie group isomorphism iff it is bijective.*

(3) *[LeeSM] Theorem 7.7 + 7.9: Let G be a connected Lie group. There exists a simply connected Lie group $\widetilde{G}$, called **universal covering group of** G, that admits a smooth covering map $\pi : \widetilde{G} \to G$ that is also a Lie group homomorphism. It is unique up to Lie group isomorphism.*

Recall that an **embedded submanifold** is a subset with subspace topology and a smooth structure such that the inclusion map is a smooh embedding. An **immersed submanifold** is a subset with a topology with respect to which it is a topological manifold (without boundary), and a smooth structure such that the inclusion map is a smooth immersion. Suppose G is a Lie group. A **Lie subgroup of** G is a subgroup of G endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of G. Note that an embedded subgroup is automatically a Lie subgroup since every embedded submanifold is an immersed submanifold with subspace topology and that subspace topology makes the restrictions of multiplication and inversion still smooth (see [LeeSM] Cor.5.30). The simplest example of an embedded Lie subgroup is the open subgroup by [LeeSM] proposition 5.1, and it is also closed and thus a union of connected components by [LeeSM] Lemma 7.12.

Proposition 1.1.2.

(1) *[LeeSM] proposition 7.14: Suppose G is a Lie group, and $W \subseteq G$ is any neighborhood of the identity.*

  (a) *W generates an open subgroup of G.*

  (b) *If W is connected, it generates a connected open subgroup of G.*

  (c) *If G is connected, then W generates G.*

(2) *[LeeSM] Proposition 7.15: Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

(3) *[LeeSM] Theorem 21.26 (Quotient Theorem for Lie Groups). Suppose G is a Lie group and $K \subseteq G$ is a closed normal subgroup. Then G/K is a Lie group, and the quotient map $\pi : G \to G/K$ is a surjective Lie group homomorphism whose kernel is K.*

(4) *[LeeSM] Proposition 7.17 says that the image of an injective Lie group homomorphism is a Lie subgroup. And a more general result is built from this:*

(5) *[LeeSM] Theorem 21.27 (First Isomorphism Theorem for Lie Groups). If $F : G \to H$ is a Lie group homomorphism, then the kernel of F is a closed normal Lie subgroup of G, the image of F has a unique smooth manifold structure making it into a Lie subgroup of H, and F descends to a Lie group isomorphism $\widetilde{F} : G/\operatorname{Ker} F \to \operatorname{Im} F$. If F is surjective, then $G/\operatorname{Ker} F$ is smoothly isomorphic to H.*

(6) *[LeeSM] Proposition 21.28: Every discrete subgroup of a Lie group is a closed Lie subgroup of dimension zero.*

Example 1.1.3 (Embedded Lie Subgroups)**.** [LeeSM] Example 7.18.

(a) The circle $\mathbb{S}^1$ is an embedded Lie subgroup of $\mathbb{C}^*$ because it is a subgroup and an embedded submanifold.

(b) The set $SL(n, \mathbb{R})$ of $n \times n$ real matrices with determinant equal to 1 is called the special linear group of degree n. Because $SL(n, \mathbb{R})$ is the kernel of the Lie group homomorphism $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$, it is a properly embedded Lie subgroup. Because the determinant function is surjective, it is a smooth submersion by the global rank theorem, so $SL(n, \mathbb{R})$ has dimension $n^2 - 1$.

(c) The subgroup $SL(n, \mathbb{C}) \subseteq GL(n, \mathbb{C})$ consisting of complex matrices of determinant 1 is called the complex special linear group of degree n. It is the kernel of the Lie group homomorphism $\det : GL(n, \mathbb{C}) \to \mathbb{C}^*$. This homomorphism is surjective, so it is a smooth submersion by the global rank theorem. Therefore, $SL(n, \mathbb{C}) = \operatorname{Ker}(\det)$ is a properly embedded Lie subgroup whose codimension is equal to $\dim \mathbb{C}^* = 2$ and whose dimension is therefore $2n^2 - 2$.

◇

Here is an example of a Lie subgroup that is not embedded.

Example 1.1.4 (A Dense Lie Subgroup of the Torus)**.** [LeeSM] Example 7.19.
Let $H \subseteq \mathbb{T}^2$ be the dense submanifold of the torus that is the image of the immersion $\gamma : \mathbb{R} \to \mathbb{T}^2$ defined in [LeeSM] Example 4.20. It is easy to check that γ is an injective Lie group homomorphism, and thus H is an immersed Lie subgroup of $\mathbb{T}^2$ by Proposition 1.1.2 (4). ◇

In general, smooth submanifolds can be closed without being embedded (as is, for example, the figure-eight curve of [LeeSM] Example 5.19) or embedded without being closed (as is the open unit ball in $\mathbb{R}^n$). However, as the next theorem shows, Lie subgroups have the remarkable property that closedness and embeddedness are not independent. This means that every embedded Lie subgroup is properly embedded.

Theorem 1.1.5. *[LeeSM] Theorem 7.21: Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then H is closed in G if and only if it is embedded.*

Compare this with the Cartan's closed subgroup theorem, where the subgroup is not assumed to have a submanifold structure in the first place.

Theorem 1.1.6. *[LeeSM] Theorem 20.12: Suppose G is a Lie group and $H \subseteq G$ is a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup.*

Our main goal now is to develop tools to prove this theorem.

1.2 Lie Algebras

Suppose G is a Lie group. Recall that G acts smoothly and transitively on itself by left translation: $L_g(h) = gh$. A vector field X on G is said to be **left-invariant** if it is invariant under all left translations, in the sense that it is L_g-related to itself for every $g \in G$. More explicitly, this means

$$d\left(L_g\right)_{g'}\left(X_{g'}\right) = X_{gg'}, \quad \text{for all } g, g' \in G$$

Since L_g is a diffeomorphism, this can be abbreviated by writing $\left(L_g\right)_* X = X$ for every $g \in G$.

Because $\left(L_g\right)_*(aX + bY) = a\left(L_g\right)_* X + b\left(L_g\right)_* Y$, the set of all smooth left-invariant vector fields on G, denoted as $\operatorname{Lie}(G)$ is a linear subspace of $\mathfrak{X}(G)$. The Lie bracket $[X, Y]$ of two left-invariant v.f. X, Y is still left-invariant due to [LeeSM] Cor.8.31, so $\operatorname{Lie}(G)$ is a **Lie subalgebra** of the **Lie algebra** $\mathfrak{X}(G)$, i.e., a vector space with skew-symmetric bilinear map that satisfies Jacobi identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$.

Theorem 1.2.1. *Let G be a Lie group. The evaluation map $\varepsilon : \operatorname{Lie}(G) \to T_eG$, given by $\varepsilon(X) = X_e$, is a vector space isomorphism. Thus, $\operatorname{Lie}(G)$ is finite-dimensional, with dimension equal to $\dim G$.*

Proof. It is clear from the definition that ε is linear over $\mathbb{R}$. It is easy to prove that it is injective: if $\varepsilon(X) = X_e = 0$ for some $X \in \operatorname{Lie}(G)$, then left-invariance of X implies that $X_g = d\left(L_g\right)_e\left(X_e\right) = 0$ for every $g \in G$, so $X = 0$.

To show that ε is surjective, let $v \in T_eG$ be arbitrary, and define a (rough) vector field v^{L} on G by

$$\left.v^{\mathrm{L}}\right|_g = d\left(L_g\right)_e(v)$$

If there is a left-invariant vector field on G whose value at the identity is v, clearly it has to be given by this formula.

First we need to check that v^{L} is smooth. By [LeeSM] Proposition 8.14 , it suffices to show that $v^{\mathrm{L}} f$ is smooth whenever $f \in C^\infty(G)$. Choose a smooth curve $\gamma : (-\delta, \delta) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then for all $g \in G$,

$$\begin{aligned} \left(v^{\mathrm{L}} f\right)(g) &= \left.v^{\mathrm{L}}\right|_g f = d\left(L_g\right)_e(v) f = v\left(f \circ L_g\right) = \gamma'(0)\left(f \circ L_g\right) \\ &= \left.\frac{d}{dt}\right|_{t=0}\left(f \circ L_g \circ \gamma\right)(t) \end{aligned}$$

If we define $\varphi : (-\delta, \delta) \times G \to \mathbb{R}$ by $\varphi(t, g) = f \circ L_g \circ \gamma(t) = f(g\gamma(t))$, the computation above shows that $\left(v^{\mathrm{L}} f\right)(g) = \partial\varphi/\partial t(0, g)$. Because φ is a composition of group multiplication, f, and γ, it is smooth. It follows that $\partial\varphi/\partial t(0, g)$ depends smoothly on g, so $v^{\mathrm{L}} f$ is smooth.

Next we show that v^{L} is left-invariant, which is to say that $d\left(L_h\right)_g\left(\left.v^{\mathrm{L}}\right|_g\right) = \left.v^{\mathrm{L}}\right|_{hg}$ for all $g, h \in G$. This follows from the definition of v^{L} and the fact that $L_h \circ L_g = L_{hg}$:

$$d\left(L_h\right)_g\left(\left.v^{\mathrm{L}}\right|_g\right) = d\left(L_h\right)_g \circ d\left(L_g\right)_e(v) = d\left(L_h \circ L_g\right)_e(v) = d\left(L_{hg}\right)_e(v) = \left.v^{\mathrm{L}}\right|_{hg}$$

Thus $v^{\mathrm{L}} \in \operatorname{Lie}(G)$. Since L_e (left translation by the identity) is the identity map of G, it follows that $\varepsilon\left(v^{\mathrm{L}}\right) = \left.v^{\mathrm{L}}\right|_e = v$, so ε is surjective. ☐

Corollary 1.2.2. *Every left-invariant rough vector field on a Lie group is smooth.*

Proof. Let X be a left-invariant rough vector field on a Lie group G, and let $v = X_e$. Now $v^{\mathrm{L}} \in \operatorname{Lie}(G)$ and $\varepsilon\left(\left.v^{\mathrm{L}}\right|_e\right) = X_e$ show that $v^{\mathrm{L}} = X$ by bijectivity of ε. Thus X is smooth. ☐

Corollary 1.2.3. *Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.*

Proof. If G is a Lie group, every basis for $\mathrm{Lie}(G)$ is a left-invariant smooth global frame for G. Explicitly, this basis is $\{b_1^{\mathrm{L}}, \cdots, b_n^{\mathrm{L}}\}$, given a basis $\{b_1, \cdots, b_n\}$ of T_eG. □

Example 1.2.4 (Lie algebras).

(a) Euclidean space $\mathbb{R}^n$: If we consider $\mathbb{R}^n$ as a Lie group under addition, left translation by an element $b \in \mathbb{R}^n$ is given by the affine map $L_b(x) = b + x$, whose differential $d(L_b)$ is represented by the identity matrix in standard coordinates. Thus a vector field $X^i \partial/\partial x^i$ is left-invariant if and only if its coefficients X^i are constants. Because the Lie bracket of two constant-coefficient vector fields is zero, the Lie algebra of $\mathbb{R}^n$ is abelian, and is isomorphic to $\mathbb{R}^n$ itself with the trivial bracket. In brief, $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.

(b) The circle group S^1: Let θ be any angle coordinate on a proper open subset $U \subseteq S^1$, and let $d/d\theta$ denote the corresponding coordinate vector field. Because any other angle coordinate $\tilde{\theta}$ differs from θ by an additive constant in a neighborhood of each point, the transformation law for coordinate vector fields shows that $d/d\theta = d/d\tilde{\theta}$ on their common domain. For this reason, there is a globally defined vector field on S^1 whose coordinate representation is $d/d\theta$ with respect to any angle coordinate. It is a smooth vector field because its component function is constant in any such chart. We denote this global vector field by $d/d\theta$, even though, strictly speaking, it cannot be considered as a coordinate vector field on the entire circle at once. In terms of appropriate angle coordinates, each left translation has a local coordinate representation of the form $\theta \mapsto \theta + c$. Since the differential of this map is the 1×1 identity matrix, it follows that the vector field $d/d\theta$ is left-invariant, and is therefore a basis for the Lie algebra of S^1. This Lie algebra is 1-dimensional and abelian, and therefore $\mathrm{Lie}\left(S^1\right) \cong \mathbb{R}$.

(c) The n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$: choosing an angle function θ^i for the i th circle factor, $i = 1, \ldots, n$, yields local coordinates $\left(\theta^1, \ldots, \theta^n\right)$ for $\mathbb{T}^n$. An analysis similar to that of the previous example shows that the coordinate vector fields $\partial/\partial\theta^1, \ldots, \partial/\partial\theta^n$ are smooth and globally defined on $\mathbb{T}^n$ and form a basis for $\mathrm{Lie}(\mathbb{T}^n)$. Since the Lie brackets of these coordinate vector fields are all zero, $\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n$.

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The Lie groups $\mathbb{R}^n, \mathbb{S}^1$, and $\mathbb{T}^n$ are abelian, and as the discussion above shows, their Lie algebras turn out also to be abelian.

Proposition 1.2.5. *Every abelian Lie group has an abelian Lie algebra (see [LeeSM] Problem 8-25). The converse is true provided that the group is connected ([LeeSM] Problem 20-7).*

Theorem 1.2.6 (Induced Lie Algebra Homomorphisms). *Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Suppose $F : G \to H$ is a Lie group homomorphism. For every $X \in g$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X. With this vector field denoted by F_*X, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.*

Proof. If there is any vector field $Y \in \mathfrak{h}$ that is F-related to X, it must satisfy $Y_e = dF_e(X_e)$, and thus it must be uniquely determined by

$$Y = \left(dF_e\left(X_e\right)\right)^{\mathrm{L}}$$

To show that this Y is F-related to X, we note that the fact that F is a homomorphism implies

$$\begin{aligned} F\left(gg'\right) = F(g)F\left(g'\right) &\Rightarrow F\left(L_g g'\right) = L_{F(g)}F\left(g'\right) \\ &\Rightarrow F \circ L_g = L_{F(g)} \circ F \\ &\Rightarrow dF \circ d\left(L_g\right) = d\left(L_{F(g)}\right) \circ dF \end{aligned}$$

Thus,

$$dF\left(X_g\right)=dF\left(d\left(L_g\right)\left(X_e\right)\right)=d\left(L_{F(g)}\right)\left(dF\left(X_e\right)\right)=d\left(L_{F(g)}\right)\left(Y_e\right)=Y_{F(g)}$$

This says precisely that X and Y are F-related.

For each $X \in \mathfrak{g}$, let $F_* X$ denote the unique vector field in $\mathfrak{h}$ that is F-related to X. It then follows immediately from the naturality of Lie brackets that $F_*[X,Y]=[F_* X, F_* Y]$, so F_* is a Lie algebra homomorphism. □

The map $F_*: \mathfrak{g} \rightarrow \mathfrak{h}$ whose existence is asserted in this theorem is called the **induced Lie algebra homomorphism**. Note that the theorem implies that for any left-invariant vector field $X \in \mathfrak{g}$, $F_* X$ is a well-defined smooth vector field on H, even though F may not be a diffeomorphism.

Proposition 1.2.7 (Properties of Induced Homomorphisms)**.**

(a) The homomorphism $(\mathrm{Id}_G)_*: \operatorname{Lie}(G) \rightarrow \operatorname{Lie}(G)$ *induced by the identity map of* G *is the identity of* $\operatorname{Lie}(G)$.

(b) If $F_1: G \rightarrow H$ *and* $F_2: H \rightarrow K$ *are Lie group homomorphisms, then*

$$(F_2 \circ F_1)_*=(F_2)_* \circ (F_1)_*: \operatorname{Lie}(G) \rightarrow \operatorname{Lie}(K)$$

(c) Isomorphic Lie groups have isomorphic Lie algebras.

If G is a Lie group and $H \subseteq G$ is a Lie subgroup, we might hope that the Lie algebra of H would be a Lie subalgebra of that of G. However, elements of $\operatorname{Lie}(H)$ are vector fields on H, not G, and so, strictly speaking, are not elements of $\operatorname{Lie}(G)$. Nonetheless, the next proposition gives us a way to view $\operatorname{Lie}(H)$ as a subalgebra of $\operatorname{Lie}(G)$.

Theorem 1.2.8 (The Lie Algebra of a Lie Subgroup)**.** *Suppose* $H \subseteq G$ *is a Lie subgroup, and* $\iota: H \hookrightarrow G$ *is the inclusion map. There is a Lie subalgebra* $\mathfrak{h} \subseteq \operatorname{Lie}(G)$ *that is canonically isomorphic to* $\operatorname{Lie}(H)$*, characterized by either of the following descriptions:*

$$\begin{aligned}\mathfrak{h} & =\iota_*(\operatorname{Lie}(H)) \\ & =\left\{X \in \operatorname{Lie}(G): X_e \in T_e H\right\}\end{aligned}$$

1.3 Lie Derivative

Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is the flow of V. For any smooth vector field W on M, define a rough vector field on M, denoted by $\mathcal{L}_V W$ and called the **Lie derivative of** W **with respect to** V, by

$$\begin{aligned}\left(\mathcal{L}_V W\right)_p & =\left.\frac{d}{dt}\right|_{t=0} d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right) \\ & =\lim_{t \rightarrow 0} \frac{d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right)-W_p}{t}\end{aligned}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighborhood of p, and θ_{-t} is the inverse of θ_t, so both $d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$ and W_p are elements of $T_p M$. There are some technical issues regarding the manifold with boundary case (see [LeeSM] p.228). Lemma 9.36 shows that it is indeed a smooth vector field and Theorem 9.38 equates it with $[V, W]$. Corollary 9.39 and Proposition 9.41 and Theorem 9.42 are summarized below.

Proposition 1.3.1. *Suppose* M *is a smooth manifold with or without boundary, and* $V, W, X \in \mathfrak{X}(M)$.

(a) $\mathcal{L}_V W=-\mathcal{L}_W V$

(b) $\mathcal{L}_V[W, X]=\left[\mathcal{L}_V W, X\right]+\left[W, \mathcal{L}_V X\right]$

(c) $\mathcal{L}_{[V, W]} X=\mathcal{L}_V \mathcal{L}_W X-\mathcal{L}_W \mathcal{L}_V X$.

(d) *If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.*

(e) *If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$*

(f) *If $\partial M \neq \varnothing$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any (t_0, p) in the domain of θ,*

$$\left.\frac{d}{dt}\right|_{t=t_0} d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = d\left(\theta_{-t_0}\right)\left(\left(\mathcal{L}_V W\right)_{\theta_{t_0}(p)}\right)$$

(g) *V commutes with W, i.e., $[V, W] \equiv 0 \iff V$ is invariant under the flow of $W \iff W$ is invariant under the flow of V. **Invariance of X under flow η of Y** simply means $d\left(\eta_t\right)_p\left(X_p\right) = X_{\theta_t(p)}$ for all (t, p) in the domain of the flow η.*

Theorem 9.46 is also copied below.

Theorem 1.3.2 (Canonical Form for Commuting Vector Fields). *Let M be a smooth n-manifold, and let $(V_1, \ldots, V_k)$ be a linearly independent k-tuple of smooth commuting vector fields on an open subset $W \subseteq M$. For each $p \in W$, there exists a smooth coordinate chart $\left(U, \left(s^i\right)\right)$ centered at p such that $V_i = \partial/\partial s^i$ for $i = 1, \ldots, k$. If $S \subseteq W$ is an embedded codimension- k submanifold and p is a point of S such that $T_p S$ is complementary to the span of $\left(\left.V_1\right|_p, \ldots, \left.V_k\right|_p\right)$, then the coordinates can also be chosen such that $S \cap U$ is the slice defined by $s^1 = \cdots = s^k = 0$.*

1.4 Exponential Maps

For the theory below, we focus on the $\mathbb{K} = \mathbb{R}$ case. For the $\mathbb{K} = \mathbb{C}$ case, we need a generalization of the usual result of the theory of differential equations to complex setup.

Recall that for $V \in \mathfrak{X}(M)$ an a point $p \in M$, there exists a unique maximal smooth curve $\gamma(t)$ that is an integral curve of V starting at p, i.e., $\gamma(0) = p$ and $\gamma'(t) = V_{\gamma(t)}$ for all t.

Definition 1.4.1. *A **one-parameter subgroup** of a Lie group G is a Lie group homomorphism $\phi : \mathbb{R} \to G$, i.e. ϕ is smooth such that $\phi(s + t) = \phi(s)\phi(t)$ for all $s, t \in \mathbb{R}$.*

Theorem 1.4.2 (Characterization of One-Parameter Subgroups). *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.*

Proof. Note that [LeeSM] Theorem 9.18 shows that $\gamma(t)$ for $X \in \mathrm{Lie}(G)$ starting at e is complete, i.e., defined on $\mathbb{R}$.

To show $\gamma(t)$ is a one-parameter subgroup, we need to show $\gamma(t + s) = \gamma(t)\gamma(s)$ for any $t, s \in \mathbb{R}$.

- Translation lemma says that $\gamma(t + s)$ is again an integral curve of the same vector field generating γ.
- Naturality of integral curves say that smooth map $F : M \to N$ takes integral curves of $X \in \mathfrak{X}(M)$ to integral curves of $Y \in \mathfrak{X}(N)$ when X and Y are F-related. Thus, left-invariance gives X and X $L_{\gamma(t)}$-related and $L_{\gamma(t)}\gamma(s) = \gamma(t)\gamma(s)$ is then an integral curve of X.

By uniqueness of maximal integral curve, these two are the same.

To show the converse, let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup, so $\gamma(0) = e$. Recall the induced Lie algebra homomorphism γ_* that sends a left-invariant vector field to a unique left-invariant vector field that is γ-related to it:

$$\begin{aligned} \gamma_* : \mathrm{Lie}(\mathbb{R}) &\longrightarrow \mathrm{Lie}(G) \\ \frac{d}{dt} &\longmapsto V = \gamma_*\left(\frac{d}{dt}\right) := \left[d\gamma_0\left(\left.\frac{d}{dt}\right|_{t=0}\right)\right]^L \end{aligned}$$

The γ-relatedness of v.f. $\frac{d}{dt}$ and V gives (recall F-relatedness of X and Y means $dF_p(X_p) = Y_{F(p)}$):

$$d\gamma_s \left(\frac{d}{dt}\bigg|_{t=s} \right) = V(\gamma(s)).$$

But the LHS is definition of $\gamma'(s)$, so above shows that γ is an integral curve of V starting at e. $\square$

As a consequence, we get one-to-one correspondences between

- One-parameter subgroups of G.
- Left invariant vector fields on G.
- Tangent vectors at $e \in G$.

So we have three different descriptions of the Lie algebra $\mathfrak{g}$.

Definition 1.4.3. *Given a Lie group G with Lie algebra $\mathfrak{g}$, we define a map $\exp : \mathfrak{g} \to G$, called the* ***exponential map of*** *G, as follows: for any $X \in \mathfrak{g}$, we set*

$$\exp X = \gamma(1)$$

where γ is the one-parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity.

Proposition 1.4.4. *Let G be a Lie group. For any $X \in \mathrm{Lie}(G), \gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X.*

Proof. Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup generated by X, which is the integral curve of X starting at e. For any fixed $s \in \mathbb{R}$, it follows from the rescaling lemma that $\tilde{\gamma}(t) = \gamma(st)$ is the integral curve of sX starting at e, so

$$\exp sX = \tilde{\gamma}(1) = \gamma(s)$$

$\square$

Remark 1.4.5. *Note that the zero vector $0 \in T_eG$ generates the zero vector field on G, whose integral curve through e is the constant curve. So $\exp(0) = e$.*

Example 1.4.6.

(1) For $G = \mathbb{R}^*$, we can identify $T_1G = \mathbb{R}$. For any $x \in T_1G = \mathbb{R}$, the map

$$\phi : \mathbb{R} \to G, \quad t \mapsto e^{tx}$$

is the one-parameter subgroup of G with $\dot{\phi}(0) = x$. It follows $\exp(x) = e^x$.

(2) For $G = S^1$, we can identify $T_1S^1 = i\mathbb{R}$. The one-parameter subgroup corresponding to $ix \in T_1S^1 = i\mathbb{R}$ is

$$\phi : \mathbb{R} \to S^1, \quad t \mapsto e^{itx}.$$

So the exponential map is given by $\exp(ix) = e^{ix}$.

(3) For $G = \mathbb{R}$, we identify $T_0G = \mathbb{R}$. The one-parameter subgroup for $x \in \mathbb{R}$ is

$$\phi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto tx.$$

So the exponential map is $\exp(x) = x$.

$\diamond$

We can also get the one-parameter subgroups and exponential maps of the Lie subgroups of a Lie group.

Proposition 1.4.7. *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in T_eH, i.e., $\{\exp tX : X \in \mathrm{Lie}(G)$ s.t. $X_e \in T_eH\}$. Thus, the exponential map of H is simply $\exp|_{T_eH}$.*

Proposition 1.4.8 (Properties of the Exponential Map)**.** *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

(a) The exponential map is a smooth map from g to G.

(b) For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}, \exp(s+t)X = \exp sX \exp tX$.

(c) For any $X \in \mathfrak{g}, (\exp X)^{-1} = \exp(-X)$.

(d) For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}, (\exp X)^n = \exp(nX)$.

(e) The differential $(d\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself.

(f) The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.

(g) If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi : G \to H$ is a Lie group homomorphism, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\Phi]{} & H \end{array}$$

(h) The flow θ of a left-invariant vector field X is given by $\theta_t = R_{\exp tX}$ (right multiplication by $\exp tX$).

Proof. See [LeeSM] Proposition 20.8. ☐

1.5 Classical groups

Let $\mathbb{K}$ be either $\mathbb{R}$, which gives a real Lie group, or $\mathbb{C}$, which gives a complex Lie group. We discuss the so-called **classical groups**, or various subgroups of the general linear group which are frequently used in linear algebra:

- **General linear group** $\mathrm{GL}(n, \mathbb{K})$, the set of all invertible $n \times n$ matrices with enties in $\mathbb{K}$.
- **Special linear group** $\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}) : \det(A) = 1\}$
- **Orthogonal group** $\mathrm{O}(n, \mathbb{K}) = \{Q \in \mathrm{GL}(n, \mathbb{K}) : Q^TQ = QQ^T = I\}$.
- **Special orthogonal group** $\mathrm{SO}(n, \mathbb{K}) = \{Q \in O(n, \mathbb{K}) : \det(Q) = 1\}$ and more general groups $\mathrm{SO}(p, q; \mathbb{R})$.
- **Symplectic group** $\mathrm{Sp}(n, \mathbb{K}) = \{A : \mathbb{K}^{2n} \to \mathbb{K}^{2n} \mid \omega(Ax, Ay) = \omega(x, y)\}$. Here $\omega(x, y)$ is the skew-symmetric bilinear form $\sum_{i=1}^n x_iy_{i+n} - y_ix_{i+n}$ (which, up to a change of basis, is the unique non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2n}$). Equivalently, one can write $\omega(x, y) = (Jx, y)$, where (,) is the standard symmetric bilinear form on $\mathbb{K}^{2n}$ and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{1.1}$$

  Note that there is some ambiguity with the notation for symplectic group: the group we denoted $\mathrm{Sp}(n, \mathbb{K})$ in some books would be written as $\mathrm{Sp}(2n, \mathbb{K})$.

- **Unitary group** $\mathrm{U}(n) = \{\text{unitary matrices}\} = \{U \in \mathrm{GL}(n, \mathbb{C}) : U^*U = UU^* = I\}$ (note that this is a real Lie group, even though its elements are matrices with complex entries)
- **Special unitary group** $\mathrm{SU}(n) = \{U \in U(n) : \det(U) = 1\}$
- **Group of unitary quaternionic transformations** $U(n, \mathbb{H}) \cong \mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$. Another description of this group, which explains its relation with quaternions, is given in Exercise 2.15 .

We will show that each of the classical groups listed above is a Lie group and will find their Lie algebras and their dimensions.

We start with $\mathrm{GL}(n, \mathbb{K})$: we denote by the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ the vector space $M(n, \mathbb{K})$ with bracket $[A, B] = AB - BA$. We note that $\mathrm{GL}(n, \mathbb{K})$ is an open subset of $\mathfrak{gl}(n, \mathfrak{K})$, so [LeeSM] Proposition 3.9 says that the differential $d\iota_{I_n}$ of the inclusion $\iota : \mathrm{GL}(n, \mathbb{K}) \to \mathfrak{gl}(n, \mathbb{K})$ is an isomorphism between the tangent space $T_{I_n}\mathrm{GL}(n, \mathbb{K})$ and the tangent space of the vector space $\mathfrak{gl}(n, \mathbb{K})$, which is just $\mathfrak{gl}(n, \mathbb{K})$. We have the following results of which the real case is shown at [LeeSM] Proposition 8.41 and Proposition 20.2.

Proposition 1.5.1. *The composition of the natural maps*

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{K})) \xrightarrow{eval} T_{I_n}\mathrm{GL}(n, \mathbb{K}) \xrightarrow{d\iota_{I_n}} \mathfrak{gl}(n, \mathbb{K})$$

gives a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{K}))$ *and* $\mathfrak{gl}(n, \mathbb{K})$.

Proposition 1.5.2. *For any* $A \in \mathfrak{gl}(n, \mathbb{K})$*, let*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2}A^2 + \cdots$$

This series converges to an invertible matrix $e^A \in \mathrm{GL}(n, \mathbb{K})$*, and the one-parameter subgroup of* $\mathrm{GL}(n, \mathbb{K})$ *generated by* $A \in \mathrm{gl}(n, \mathbb{K})$ *is* $\gamma(t) = e^{tA}$*. Therefore, by definition of exponential map, we see* $\exp : A \in \mathfrak{gl}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ *is given by* $\exp(A) = \gamma_A(1) = e^{1A} = e^A$*. The Lie subgroup of it also has* $\exp$ *as the exponential map due to Proposition 1.4.7.*

In a similar way, we define the **logarithmic map** by

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$$

So defined $\log$ is an analytic map defined in a neighborhood of $1 \in \mathfrak{gl}(n, \mathbb{K})$.

How does it help us to study various matrix groups? The key idea is that Proposition 1.4.8 (f) shows that the matrix exponential and logarithmic map diffeomorphically identify some neighborhood of the identity in $\mathrm{GL}(n, \mathbb{K})$ with some neighborhood of 0 in the vector space $\mathfrak{gl}(n, \mathbb{K})$. It turns out that it also does the same for all of the classical groups.

Theorem 1.5.3 (Classical group)**.** *For each classical group* $G \subset \mathrm{GL}(n, \mathbb{K})$*, there exists a vector space* $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ *such that for some some neighborhood* U *of 1 in* $\mathrm{GL}(n, \mathbb{K})$ *and some neighborhood* u *of 0 in* $\mathfrak{gl}(n, \mathbb{K})$ *the following maps are mutually inverse*

$$(U \cap G) \underset{\exp}{\overset{\log}{\rightleftarrows}} (u \cap \mathfrak{g})$$

Before proving this theorem, note that it immediately implies the following important corollary.

Corollary 1.5.4. *Each classical group is a Lie group, with tangent space at identity* $T_1G = \mathfrak{g}$ *and* $\dim G = \dim \mathfrak{g}$*. Groups* $\mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n)$ *are real Lie groups; groups* $\mathrm{GL}(n, \mathbb{K}), \mathrm{SL}(n, \mathbb{K})$*,* $\mathrm{SO}(n, \mathbb{K}), \mathrm{O}(n, \mathbb{K}), \mathrm{Sp}(2n, \mathbb{K})$ *are real Lie groups for* $\mathbb{K} = \mathbb{R}$ *and complex Lie groups for* $\mathbb{K} = \mathbb{C}$.

Proof. By the theorem, for each classical group $G \subset \mathrm{GL}(n, \mathbb{K})$, there exists a neighborhood U of $1 \in \mathrm{GL}(n, \mathbb{K})$ and a neighborhood u of $0 \in \mathfrak{gl}(n, \mathbb{K})$ such that the exponential and logarithm maps are mutually inverse, i.e., $(U \cap G) \xrightarrow{\log} (u \cap \mathfrak{g})$. This implies that $\log : U \cap G \to u \cap \mathfrak{g} \subseteq \mathbb{K}^n$ is a smooth chart, making G locally a submanifold of $\mathrm{GL}(n, \mathbb{K})$ near 1 . For any $g \in G$, consider the neighborhood $g \cdot (U \cap G)$ of g, which is diffeomorphic to $U \cap G$. Thus, the local chart around g is $\log \circ L_g^{-1} : (g \cdot U) \cap G \to u \cap \mathfrak{g}$, where L_g is left multiplication by g. Hence, G is a submanifold of $\mathrm{GL}(n, \mathbb{K})$.

For the tangent space, consider the differential of the exponential map at $0 \in \mathfrak{g}, \exp_* : T_0\mathfrak{g} \to T_1 G$. By Proposition 1.4.8 (e), it is the identity map. This implies $\dim(G) = \dim(\mathfrak{g})$, completing the proof. □

Proof of theorem of classical group. The proof is case by case; it can not be any other way, as "classical groups" are defined by a list rather than by some general definition.

$\mathrm{GL}(n, \mathbb{K})$: Immediate from Proposition 1.4.8; in this case, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$ is the space of all matrices.

$\mathrm{SL}(n, \mathbb{K})$: Suppose $X \in \mathrm{SL}(n, \mathbb{K})$ is close enough to identity. Then $X = \exp(x)$ for some $x \in \mathfrak{gl}(n, \mathbb{K})$. The condition that $X \in \mathrm{SL}(n, \mathbb{K})$ is equivalent to $\det X = 1$, or $\det \exp(x) = 1$. But it is well-known that $\det \exp(x) = \exp(\mathrm{tr}(x))$ (which is easy to see by finding a basis in which x is upper-triangular), so $\exp(x) \in \mathrm{SL}(n, \mathbb{K})$ if and only if $\mathrm{tr}(x) = 0$. Thus, in this case the statement also holds, with $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) \mid \mathrm{tr}\, x = 0\}$.

$\mathrm{O}(n, \mathbb{K}), \mathrm{SO}(n, \mathbb{K})$: The group $\mathrm{O}(n, \mathbb{K})$ is defined by $XX^t = I$. Then X, X^t commute. Writing $X = \exp(x), X^t = \exp(x^t)$ (since exponential map agrees with transposition), we see that x, x^t also commute, and thus $\exp(x) \in \mathrm{O}(n, \mathbb{K})$ implies $\exp(x)\exp(x^t) = \exp(x + x^t) = 1$, so $x + x^t = 0$; conversely, if $x + x^t = 0$, then x, x^t commute, so we can reverse the argument to get $\exp(x) \in \mathrm{O}(n, \mathbb{K})$. Thus, in this case the theorem also holds, with $\mathfrak{g} = \{x \mid x + x^t = 0\}$ the space of skew-symmetric matrices.

What about $\mathrm{SO}(n, \mathbb{K})$? In this case, we should add to the condition $XX^t = 1$ (which gives $x + x^t = 0$) also the condition $\det X = 1$, which gives $\mathrm{tr}(x) = 0$. However, this last condition is unnecessary, because $x + x^t = 0$ implies that all diagonal entries of x are zero. So both $\mathrm{O}(n, \mathbb{K})$ and $\mathrm{SO}(n, \mathbb{K})$ correspond to the same space of matrices $\mathfrak{g} = \{x \mid x + x^t = 0\}$. This might seem confusing until one realizes that $\mathrm{SO}(n, \mathbb{K})$ is exactly the connected component of identity in $\mathrm{O}(n, \mathbb{K})$; thus, neighborhood of 1 in $\mathrm{O}(n, \mathbb{K})$ coincides with the neighborhood of 1 in $\mathrm{SO}(n, \mathbb{K})$.

$\mathrm{U}(n), \mathrm{SU}(n)$: Similar argument shows that for x in a neighborhood of identity in $\mathfrak{gl}(n, \mathbb{C})$ $\exp x \in \mathrm{U}(n) \iff x + x^* = 0$ (where $x^* = \bar{x}^t$) and $\exp x \in \mathrm{SU}(n) \iff x + x^* = 0, \mathrm{tr}(x) = 0$. Note that in this case, $x + x^*$ does not imply that x has zeroes on the diagonal: it only implies that the diagonal entries are purely imaginary. Thus, $\mathrm{tr}\, x = 0$ does not follow automatically from $x + x^* = 0$, so in this case the tangent spaces for $\mathrm{U}(n), \mathrm{SU}(n)$ are different.

$\mathrm{Sp}(n, \mathbb{K})$: Similar argument shows that $\exp(x) \in \mathrm{Sp}(n, \mathbb{K}) \iff x + J^{-1}x^t J = 0$ where J is given by (1.1). Thus, in this case the theorem also holds.

$\mathrm{Sp}(n)$: Same arguments as above show that $\exp(x) \in \mathrm{Sp}(n) \iff x + J^{-1}x^t J = 0, x + x^* = 0$. □

Theorem 1.5.3 gives "local" information about classical Lie groups, i.e. the description of the tangent space at identity. In many cases, it is also important to know "global" information, such as the topology of the group G. In some low-dimensional cases, it is possible to describe the topology of G by establishing a diffeomorphism of G with a known manifold. For example, it is easy to see $\mathrm{SU}(2) = \left\{A \in \mathrm{GL}(2, \mathbb{C}) \mid A\bar{A}^t = 1, \det A = 1\right\} = \left\{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\right\}$ Writing $\alpha = x_1 + ix_2, \beta = x_3 + ix_4, x_i \in \mathbb{R}$, we see that $\mathrm{SU}(2)$ is diffeomorphic to $S^3 = \left\{x_1^2 + \cdots + x_4^2 = 1\right\} \subset \mathbb{R}^4$. It is shown in the exercise below that $\mathrm{SO}(3, \mathbb{R}) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$ and thus is diffeomorphic to the real projective space $\mathbb{RP}^3$. For higher dimensional groups, the standard method of finding their topological invariants such as fundamental groups using [2] Cor.2.12: if G acts transitively on a manifold M, then G is a fiber bundle over M with the fiber G_m-stabilizer of point in M.

Thus we can get information about fundamental groups of G from fundamental groups of M, G_m. Details of this approach for different classical groups are given in the exercises below.

Exercise 1.5.5. *Define a basis in* $\mathfrak{su}(2)$ *by*

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show that the map

$$\begin{aligned} \varphi : \mathrm{SU}(2) &\to \mathrm{GL}(3, \mathbb{R}) \\ g &\mapsto \text{ matrix of } \operatorname{Ad} g \text{ in the basis } i\sigma_1, i\sigma_2, i\sigma_3 \end{aligned}$$

gives a morphism of Lie groups $\mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$.

Exercise 1.5.6. *Let* $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ *be the morphism defined in the previous problem. Compute explicitly the map of tangent spaces* $\varphi_* : \mathfrak{su}(2) \to \mathfrak{so}(3, \mathbb{R})$ *and show that* φ_* *is an isomorphism. Deduce from this that* $\operatorname{Ker} \varphi$ *is a discrete normal subgroup in* $\mathrm{SU}(2)$, *and that* $\operatorname{Im} \varphi$ *is an open subgroup in* $\mathrm{SO}(3, \mathbb{R})$.

Exercise 1.5.7. *Prove that the map* φ *used in two previous exercises establishes an isomorphism* $\mathrm{SU}(2)/\mathbb{Z}_2 \to \mathrm{SO}(3, \mathbb{R})$ *and thus, since* $\mathrm{SU}(2) \simeq S^3, \mathrm{SO}(3, \mathbb{R}) \simeq \mathbb{RP}^3$.

Exercise 1.5.8. *Using [2] Example 2.24, show that for* $n \geqslant 1$, *we have* $\pi_0(\mathrm{SU}(n+1)) = \pi_0(\mathrm{SU}(n)), \pi_0(\mathrm{U}(n+1)) = \pi_0(\mathrm{U}(n))$ *and deduce from it that groups* $\mathrm{U}(n), \mathrm{SU}(n)$ *are connected for all* n. *Similarly, show that for* $n \geqslant 2$, *we have* $\pi_1(\mathrm{SU}(n+1)) = \pi_1(\mathrm{SU}(n)), \pi_1(\mathrm{U}(n+1)) = \pi_1(\mathrm{U}(n))$ *and deduce from it that for* $n \geqslant 2, \mathrm{SU}(n)$ *is simply-connected and* $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$.

Exercise 1.5.9. *Using [2] Example 2.24, show that for* $n \geqslant 2$, *we have* $\pi_0(\mathrm{SO}(n+1, \mathbb{R})) = \pi_0(\mathrm{SO}(n, \mathbb{R}))$ *and deduce from it that groups* $\mathrm{SO}(n)$ *are connected for all* $n \geqslant 2$. *Similarly, show that for* $n \geqslant 3$, $\pi_1(\mathrm{SO}(n+1, \mathbb{R})) = \pi_1(\mathrm{SO}(n, \mathbb{R}))$ *and deduce from it that for* $n \geqslant 3, \pi_1(\mathrm{SO}(n, \mathbb{R})) = \mathbb{Z}_2$

The following tables summarize the results of Theorem 1.5.3 and computation of the fundamental groups of classical Lie groups given in the exercises. For non-connected groups, $\pi_1(G)$ stands for the fundamental group of the connected component of identity.

G	$\mathrm{O}(n, \mathbb{R})$	$\mathrm{SO}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$
$\mathfrak{g}$	$x + x^t = 0$	$x + x^t = 0$	$x + x^* = 0$	$x + x^* = 0, \operatorname{tr} x = 0$	$x + J^{-1}x^tJ = x + x^* = 0$
$\dim G$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 (n \geqslant 3)$	$\mathbb{Z}_2 (n \geqslant 3)$	$\mathbb{Z}$	$\{1\}$	$\{1\}$

Table 1.1: Compact classical groups. Here π_0 is the set of connected components, π_1 is the fundamental group (for disconnected groups, π_1 is the fundamental group of the connected component of identity), and J is given by (1.1)

G	$\mathrm{GL}(n, \mathbb{R})$	$\mathrm{SL}(n, \mathbb{R})$	$\mathrm{Sp}(n, \mathbb{R})$
$\mathfrak{g}$	$\mathfrak{gl}(n, \mathbb{R})$	$\operatorname{tr} x = 0$	$x + J^{-1}x^tJ = 0$
$\dim G$	n^2	$n^2 - 1$	$n(2n+1)$
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}_2 (n \geqslant 3)$	$\mathbb{Z}_2 (n \geqslant 3)$	$\mathbb{Z}$

Table 1.2: Noncompact real classical groups.

For complex classical groups, the Lie algebra and dimension are given by the same formula as for real groups. However, the topology of complex Lie groups is different and is given in the table below. We do not give a proof of these results, referring the reader to more advanced books such as [32].

G	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 1.3: Complex classical groups.

Note that some of the classical groups are not simply-connected. As was shown in Proposition 1.1.1 (3), in this case the universal cover has a canonical structure of a Lie group. Of special importance is the universal cover of $\mathrm{SO}(n,\mathbb{R})$ which is called the spin group and is denoted $\mathrm{Spin}(n)$; since $\pi_1(\mathrm{SO}(n,\mathbb{R}))=\mathbb{Z}_2$, this is a twofold cover, so $\mathrm{Spin}(n)$ is a compact Lie group.

1.6 Exercises

Exercise 1.6.1. *Let G be a Lie group and $H-$ a closed Lie subgroup.*
(1) Let $\bar{H}$ be the closure of H in G. Show that $\bar{H}$ is a subgroup in G.
(2) Show that each coset $Hx, x\in\bar{H}$, is open and dense in $\bar{H}$.
(3) Show that $\bar{H}=H$, that is, every Lie subgroup is closed.

Exercise 1.6.2. *(1) Show that every discrete normal subgroup of a connected lie group is central (hint: consider the map $G\to N: g\mapsto ghg^{-1}$ where h is a fixed element in N). (2) By applying part (1) to kernel of the map $\widetilde{G}\to G$, show that for any connected Lie group G, the fundamental group $\pi_1(G)$ is commutative.*

Exercise 1.6.3. *Let $f: G_1\to G_2$ be a morphism of connected Lie groups such that $f_*: T_1G_1\to T_1G_2$ is an isomorphism (such a morphism is sometimes called local isomorphism). Show that f is a covering map, and $\operatorname{Ker} f$ is a discrete central subgroup.*

Exercise 1.6.4. *Define a bilinear form on $\mathfrak{su}(2)$ by $(a,b)=\frac{1}{2}\operatorname{tr}\left(a\bar{b}^t\right)$. Show that this form is symmetric, positive definite, and invariant under the adjoint action of $\mathrm{SU}(2)$.*

Exercise 1.6.5. *Using Gram-Schmidt orthogonalization process, show that $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R})$ is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal. Deduce from this that $\mathrm{GL}(n,\mathbb{R})$ is homotopic (as a topological space) to $\mathrm{O}(n,\mathbb{R})$.*

Exercise 1.6.6. *Let L_n be the set of all Lagrangian subspaces in $\mathbb{R}^{2n}$ with the standard symplectic form ω defined in Section 2.7. (A subspace V is Lagrangian if $\dim V=n$ and $\omega(x,y)=0$ for any $x,y\in V$.)*

Show that the group $\mathrm{Sp}(n,\mathbb{R})$ acts transitively on L_n and use it to define on L_n a structure of a smooth manifold and find its dimension.

Exercise 1.6.7. *Let $\mathbb{H}=\{a+bi+cj+dk\mid a,b,c,d\in\mathbb{R}\}$ be the algebra of quaternions, defined by $ij=k=-ji$, $jk=i=-kj, ki=j=-ik, i^2=j^2=k^2=-1$, and let $\mathbb{H}^n=\{(h_1,\ldots,h_n)\mid h_i\in\mathbb{H}\}$. In particular, the subalgebra generated by $1,i$ coincides with the field $\mathbb{C}$ of complex numbers.*

Note that $\mathbb{H}^n$ has a structure of both left and right module over $\mathbb{H}$ defined by

$$h\left(h_1,\ldots,h_n\right)=\left(hh_1,\ldots,hh_n\right),\quad\left(h_1,\ldots,h_n\right)h=\left(h_1h,\ldots,h_nh\right)$$

(1) Let $\operatorname{End}_{\mathbb{H}}\left(\mathbb{H}^n\right)$ be the algebra of endomorphisms of $\mathbb{H}^n$ considered as right $\mathbb{H}$-module:

$$\operatorname{End}_{\mathbb{H}}\left(\mathbb{H}^n\right)=\left\{A:\mathbb{H}^n\to\mathbb{H}^n\mid A\left(\mathbf{h}+\mathbf{h}'\right)=A(\mathbf{h})+A\left(\mathbf{h}'\right),A(\mathbf{h}h)=A(\mathbf{h})h\right\}$$

Show that $\operatorname{End}_{\mathbb{H}}\left(\mathbb{H}^n\right)$ is naturally identified with the algebra of $n\times n$ matrices with quaternion entries.

(2) Define an $\mathbb{H}$-valued form (,) on $\mathbb{H}^n$ by

$$\left(\mathbf{h},\mathbf{h}'\right)=\sum_i\overline{h_i}h_i'$$

where $\overline{a+bi+cj+dk} = a-bi-cj-dk$. *(Note that* $\overline{uv} = \overline{v}\overline{u}$*.) Let* $\mathrm{U}(n,\mathbb{H})$ *be the group of "unitary quaternionic transformations":*

$$\mathrm{U}(n,\mathbb{H}) = \left\{A \in \mathrm{End}_{\mathbb{H}}\left(\mathbb{H}^n\right) \mid \left(A\mathbf{h}, A\mathbf{h}'\right) = \left(\mathbf{h},\mathbf{h}'\right)\right\}$$

Show that this is indeed a group and that a matrix A *is in* $\mathrm{U}(n,\mathbb{H})$ *iff* $A^*A = 1$*, where* $\left(A^*\right)_{ij} = \overline{A_{ji}}$*.*

(3) Define a map $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ *by*

$$(z_1,\ldots,z_{2n}) \mapsto (z_1 + jz_{n+1},\ldots,z_n + jz_{2n})$$

Show that it is an isomorphism of complex vector spaces (if we consider $\mathbb{H}^n$ *as a complex vector space by* $z\left(h_1,\ldots h_n\right) = \left(h_1 z,\ldots,h_n z\right)$*) and that this isomorphism identifies*

$$\mathrm{End}_{\mathbb{H}}\left(\mathbb{H}^n\right) = \left\{A \in \mathrm{End}_{\mathbb{C}}\left(\mathbb{C}^{2n}\right) \mid \bar{A} = J^{-1}AJ\right\}$$

where J *is defined by (1.1). (Hint: use* $jz = \bar{z}j$ *for any* $z \in \mathbb{C}$ *to show that* $\mathbf{h} \mapsto \mathbf{h}j$ *is identified with* $\mathbf{z} \mapsto J\overline{\mathbf{z}}$*.)*

(4) Show that under identification $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ *defined above, the quaternionic form (,) is identified with*

$$\left(\mathbf{z},\mathbf{z}'\right) - j\left\langle\mathbf{z},\mathbf{z}'\right\rangle$$

where $\left(\mathbf{z},\mathbf{z}'\right) = \sum \overline{z_i} z_i'$ *is the standard Hermitian form in* $\mathbb{C}^{2n}$ *and* $\left\langle\mathbf{z},\mathbf{z}'\right\rangle = \sum_{i=1}^{n}\left(z_{i+n}z_i' - z_i z_{i+n}'\right)$ *is the standard bilinear skew-symmetric form in* $\mathbb{C}^{2n}$*. Deduce from this that the* group $\mathrm{U}(n,\mathbb{H})$ *is identified with* $\mathrm{Sp}(n) = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(2n)$*.*

Exercise 1.6.8. *(1) Show that* $\mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq S^3$*.*
(2) Using the previous exercise, show that we have a natural transitive action of $\mathrm{Sp}(n)$ *on the sphere* S^{4n-1} *and a stabilizer of a point is isomorphic to* $\mathrm{Sp}(n-1)$*.*
(3) Deduce that $\pi_1(\mathrm{Sp}(n+1)) = \pi_1(\mathrm{Sp}(n)), \pi_0(\mathrm{Sp}(n+1)) = \pi_0(\mathrm{Sp}(n))$*.*

Chapter 2

Cartan Closed Subgroup Theorem

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