Split Spoils: Solution to Stolen Necklace Problem Via Borsuk-Ulam Theorem

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1 Introduction: a puzzle

Mathematics is never dull and dry. It comes naturally from our daily lives and speaks its own language with polished delicacy after centuries of development and perfection. There are many illuminating and enlightening puzzles around us which turn out to be elegantly solvable through mathematical means. This article is intended to demonstrate how seemingly unrelated subjects can be magically put together and to impress you presumably as a math-oriented high school student the sheer beauty of math. Let us have some math-flavored appetizers first: you probably have encountered the following three problems during your math journey.

1. How to arrange the order of a line of people getting water so that the total waiting time of all people is minimized?
2. How to walk through the seven bridges of Königsberg without revisiting them?
3. How to draw a closed curve of a given length l to encircle the largest area on a plane? A circle? But can you prove that?

Each of them is quite classic and relates to a different area of mathematics: rearrangement inequality, Euler path (due to Euler in 1735), and isoperimetric inequality (due to Hurwitz in 1901). Pause and think about them if you haven't seen a solution yet. You may find yourself using dozens of paper to draw paths and closed curves. While the three previous problems are in the field of basic algebra, graph theory, and Fourier analysis, the following puzzle will reveal you something a bit advanced, something called topology, but sharing a similar feature of deceptive easiness.

The Stolen Necklace Problem (raw version):
Try to imagine this situation. In midnight one day, two thieves sneak into a luxury spot to rob jewelry. They found a necklace with 8 sapphires and 10 emeralds (see Figure 1). Since they both like the jewelry on the necklace, they decided to share it equally. They want to cut the necklace into several pieces so that both of them would get 4 sapphires and 5 emeralds in total. However, they wonder if there is a way to have as fewest cuts as possible to do that fair division. Can you come up with such a cutting scheme? How many cuts in total and where to cut?

Figure 1: A necklace with 8 blue sapphires and 10 green emeralds

Notice that 8 and 10 are both even numbers in our problem for two thieves to possibly equally divide the beads. Of course, you may find such scenario not sufficiently pragmatical. For example, most necklaces are designed to be perfectly symmetric rather than messing around; thieves can have merely a single cut and take out the whole cord to leave the beads alone, being able to be easily half-divided.

Then, why is this problem is potentially of great interests? Think about this. If this necklace is replaced by many hills of some continuous mountains with valuable minerals coveted by two neighboring countries. Suppose they want to divide the mineral resources equally, what should they do? The distribution of minerals are certainly far from perfectly symmetric, and "cuts" in this

Figure 2: A necklace with perfect symmetry: the white and black beads are evenly distributed. In fact, in this case, a single cut in the middle suffices to divide the necklace. Note that this necklace is a closed one, which, to make it open, needs an additional cut than our raw version of stolen necklace problem.

case become a series of territorial demarcations. The "cuts" are so expensive to safeguarded that the two countries want to have a plan to minimize the number of "cuts." You can of course name many other scenarios and variants. Division of big data can be costly due to computer memory, and electrical engineers consider the "cuts" on the circuits.

Therefore, we may need to rephrase our problem to be a model with mathematical variables and to incorporate some assumptions.

The Stolen Necklace Problem (refined version):
Suppose two thieves have a necklace with k sapphires and l emeralds, both divisible by 2 (k and l being even integers). Minimize the number of places to cut the necklace so that each thieve have $\frac{k}{2}$ sapphires and $\frac{l}{2}$ emeralds.

We shall proceed solving it using different techniques in math, those you will later use too. We abstract, classify, simplify, and translate problems, where translation is a transformation of math problems by their equivalences. This last technique is the core of this article.

2 Necklace division theorem

The goal of this section is to present the following theorem. At the end of the article, we want to generalize it to cases where there are more thieves and beads (more complexity and dimensionality).

Theorem 2.1 (Necklace division theorem)**.** *For the (refined) stolen necklace problem, two cuts suffice.*

Since this is a low-dimensional case, you may again try to guess and draw a series of two-cut schemes on a necklace with sapphires and emeralds to get some guesses. Since we already rule out possibilities of schemes with three or even more cuts, it should be more doable.

Perhaps annoyed by how performance of one cut depends on the other, which in turns depends on the first one, you may find the two cuts to be analogous to two variables controlling the amount of beads assigned to each thief. If you happen to be familiar with some number theory, you may try to express this relationship in indeterminate equation. Relating something you have learned is a standard trick to do math, and the inspiration you can get from your imaginative mind is endless. In fact, mathematics solves real-life problems by first extract and abstract the fundamental relationships between these variables to simplify the description of the problem. We already see that by refining the raw version necklace problem, with strategies we shall continue to use.

Another basic observation is that there are three ways to assign the three pieces of necklace divided from two cuts. Let us symbolically denote the three pieces as a, b, and c and the whole necklace as $a|b|c$ with $|$ representing the cut. Three assignments are then:

- (ab, c): thief 1: a,b; thief 2: c
- (a, bc): thief 1: a; thief 2: b,c
- (ac, b): thief 1: a,c; thief 2: b

Notice that the assignment (a, bc) is regarded indifferently as (bc, a) because switching allocations should not affect the amount of beads they get for the division to be fair. In the first situation, the first cut $|$ in $a|b|c$ is not in effect for assignment, and the same for the second cut in the second situation. Thus, in these two situations, one cut divides the necklace into two parts, which is an equal division if and only if the amounts of two types of beads on the right of the midpoint is the same as those on the left. A completely symmetric necklace is a special case of this (see Figure 2 for example). You can then immediately tell the cutting scheme by first checking this characterization of the necklace, so you may further simplify the necklace problem by focusing on the third situation: the middle piece belongs to one thief and the remaining belongs to the other.

Now, to solve this stolen necklace problem completely, we shall translate it to another problem by first entering a whole new world–topology, where you will have a taste of math's powerful intercommunication.

3 Borsuk-Ulam theorem

3.1 A new world: topology

Topology studies properties of spaces that are invariant under continuous deformation. Its nickname is "rubber-sheet geometry" because the objects can be stretched and contracted like rubber, but cannot be broken. For example, a square can be deformed into a circle without breaking it, but a figure 8 cannot. Hence a square is topologically equivalent to a circle, but different from a figure 8.

Here are some examples of typical questions in topology: How many holes are there in an object? How can you define the holes in a torus or sphere? What is the boundary of an object? Is a space connected? Does every continuous function from the space to itself have a fixed point?

When comparing topology and analysis (a more advanced version of calculus, which I suppose to be introduced in high school math), one can tell the difference of how they organize information and describe relationship between mathematical objects. For example, topology tends to use relative position between two points, while analysis tends to use the absolute distance between the two point. Topologist may describe a separation between two point as this: there are two disjoint sets containing the two points. In contrast, analyst may calculate the distance of two points (x_1, y_1) and (x_2, y_2) on a plane by using the formula $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ to say how far they are apart from each other.

To prove the following theorem in its two dimensional case, which is a classic result in algebraic topology, however, you don't need to know the exact definitions of some mathematical objects. Mathematical rigor is sacrificed to give way to a more vivid illustration of the essence of mathematical beauty. What you need to do is simply equipping yourself with full imagination to fill the gaps made by any informality.

3.2 Borsuk-Ulam theorem

We need some terminologies there:

- Topologists like using S^2 to represent the **unit sphere**, which is the set $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ because it is a member of a whole families of balls in different dimensions, S^n, $n = 1, 2, 3, \ldots$.
- **Antipodal points** refers to a pair of points (x, y, z) and $(-x, -y, -z)$ because one is the opposite of the other with respect to the origin.
- **Continuity** of a map in higher dimension: you may already know from your calculus class how continuity of a map g from a some domain in $\mathbb{R}$ going to some range in $\mathbb{R}$ is defined. The following diagram illustrates a function g and G a continuous-discontinuous pair in one-dimension on the first row and also such a pair h and H in two-dimension on the second row.

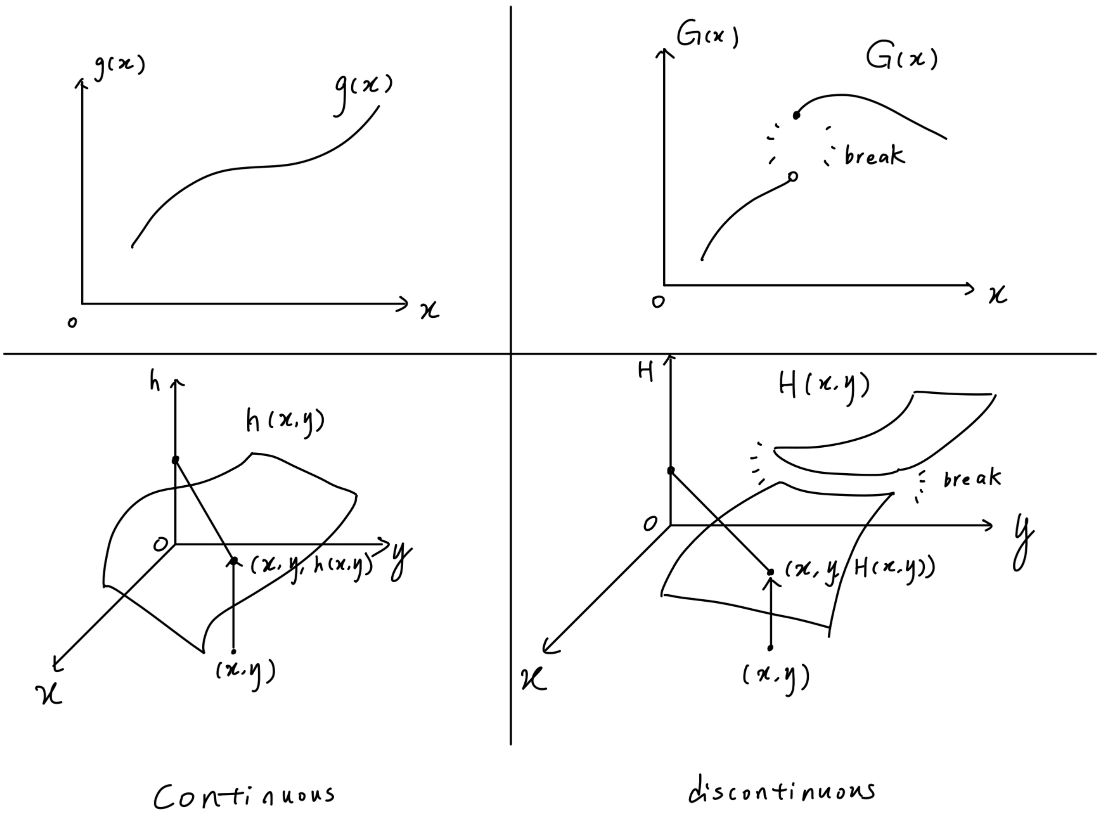

Figure 3: Graph of g,G,h,H.

> An intuitive understanding of discontinuity is "holes" and "breaking up" of the graph of function, while such intuition demands more complexity in higher dimension. On the other hand, continuity means that if you move from a point P in the domain of the function f to a point P' very near to P, you should not see a big abrupt change of $f(P)$ and $f(P')$. For if there is such a change there will be a "gap" interrupting the continuity.

You may understand the gist of the following theorem now.

Theorem 3.1 (Borsuk-Ulam Theorem for S^2)**.** *Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point (x, y, z) of S^2 such that $f(x, y, z) = f(-x, -y, -z)$.*

3.3 Proof of theorem 3.1

The theorem says that if we map points on the unit sphere continuously to the plane, there will be an antipodal pair on the sphere such that they are mapped to the same point of the plane. Let's define a function

$$g(x, y, z) := f(x, y, z) - f(-x, -y, -z)$$

Therefore, we need to find a point (x_0, y_0, z_0) on the sphere such that $g(x_0, y_0, z_0) = (0, 0)$ as this gives $f(x_0, y_0, z_0) - f(-x_0, -y_0, -z_0) = (0, 0)$, or $f(x_0, y_0, z_0) = f(-x_0, -y_0, -z_0)$, allowing us the find the desired antipodal pair. In fact, f is an even function for point (x_0, y_0, z_0),but g is an odd function for any point (x, y, z) on sphere because

$$g(-x, -y, -z) = f(-x, -y, -z) - f(x, y, z) = -g(x, y, z)$$

Suppose $g(x, y, z) = (s, t)$ where (s, t) is a point on the plane. The equation in the line above then indicates that if we draw the points $g(x, y, z)$ and $g(-x, -y, -z)$ on the plane, we will see (s, t) and $(-s, -t)$, which are a pair of antipodal points too.

Let us pick the equator of the sphere (i.e. the circle with latitude $0°$). If the point (x, y, z) walks through the equator, what's its trail on the plane by mapping through g? You may answer this by noticing two facts: (1) we just

observed that g maps antipodal points on sphere to antipodal points on the plane; (2) The equator is a big circle where you can find infinitely many pairs of antipodal points.
In fact, the trail is a closed curve around the origin such that for any line passing through the origin that divide this closed curve into two halves, one half is exactly identical to the other half after rotating it with 180°. Why a closed curve? That's because of the continuity. The image of the equator under g cannot be something broken as what we have noted before. Of course, if the closed curve already passes through the origin by some special g, and in this case, the theorem is proved.

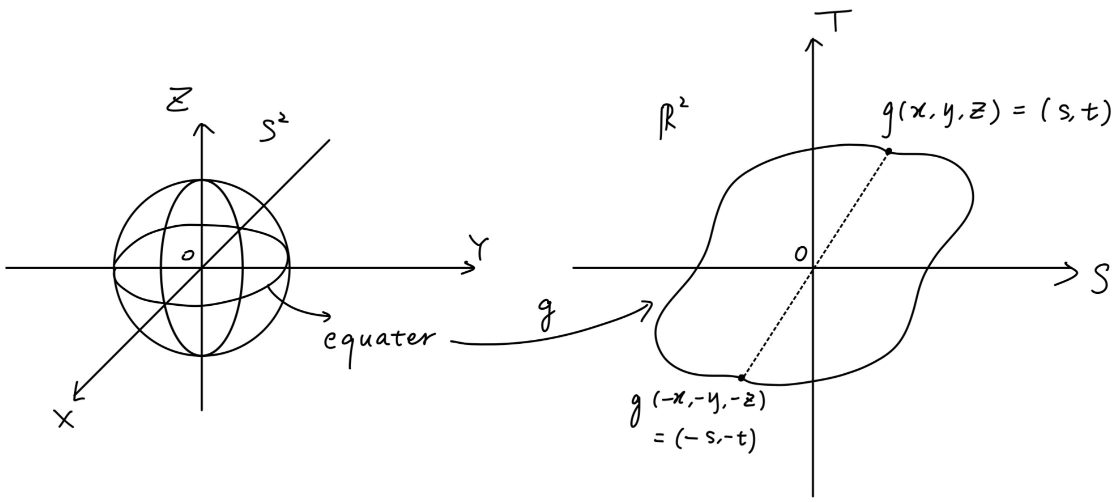

Figure 4: function g maps the equator to a rotationally symmetric closed curve

Suppose g maps the north pole $*$ to a point Q on the plane (again suppose Q is not the origin, for if it is we are done). Recall our notion of continuity before. When we have some input in the domain S^2 and another input that is close enough to the original input, they should be mapped (by continuous function g) to corresponded outputs in the range that are also close enough. Thus, we may pick the equator and another circle very close to it, we will find that there will be a closed curve very close to one mapped from the equator. If we proceed this continuous deformation of circles from equator to the north pole via all the circles of latitude from 0° to 90° (6 of them are picked up for illustrating this process in the diagram next page), we will then have a continuous deformation of the equator-mapped closed curve that surrounds the origin the point Q. Notice that it is the property that the equator is consists of infinitely many antipodal pairs that makes its image under g to be a closed curve that surrounds the origin.

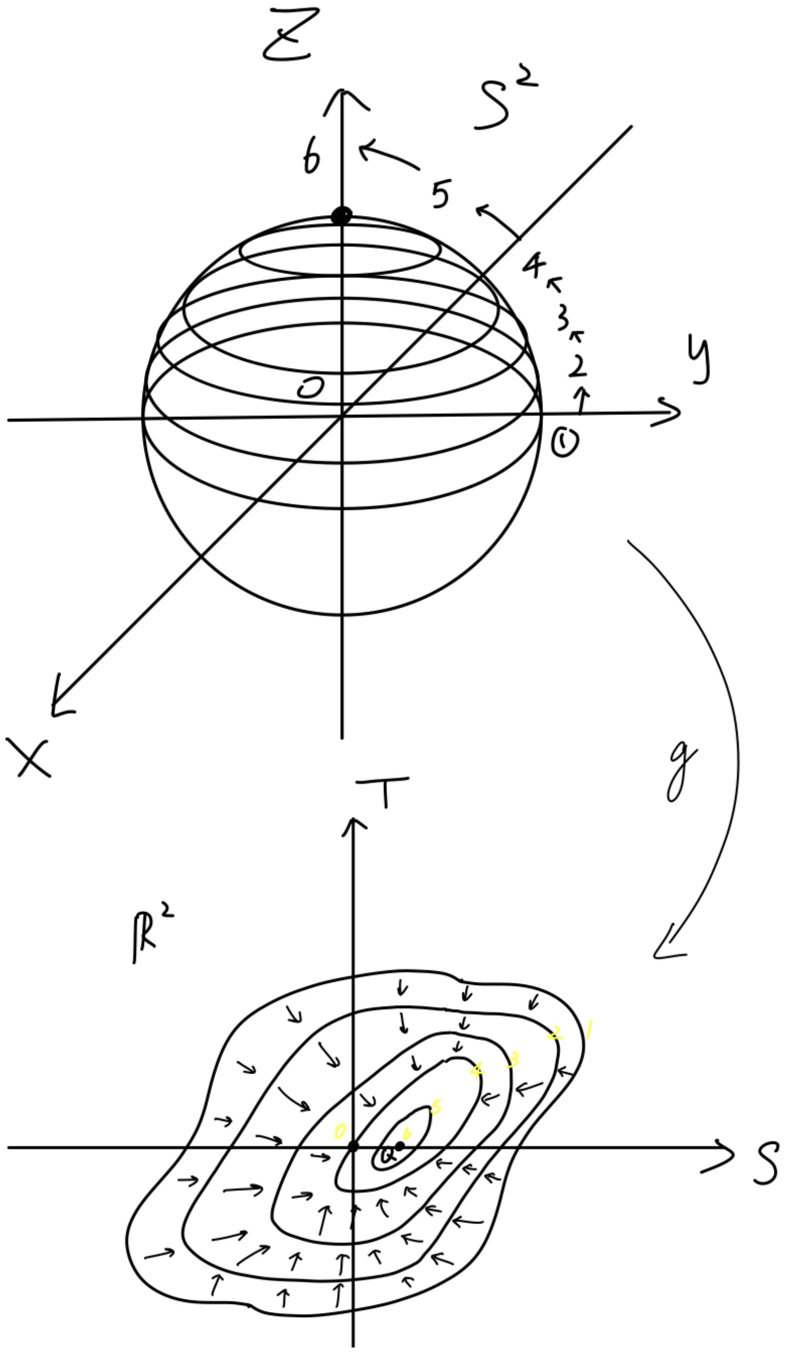

Figure 5: function g maps the equator to a rotationally symmetric closed curve

Therefore, this continuous deformation will at some time pass through the origin! Try yourself. If you are given a rubber band that can shrink or stretch. Surround the origin on the plane with the rubber band and slowly deform it to encircle the point Q and finally reach it. There is no way such you can avoid passing through the origin. The theorem is then proved. One of its interesting implication is the assertion that given the distribution of temperature and pressure for every point on the Earth's surface, one can always find a pair of antipodal points such that they have the same temperature and pressure. The hint for verifying this assertion is the continuity of distribution of temperature and pressure (the remaining is quite natural).

Well, you may find this continuous deformation elusive and indefinite. It is actually a concept related to homotopy you will learn in algebraic topology, but we won't go deep into it here. Go turn back and see how the Borsuk-Ulam theorem can be magically applied to the necklace problem.

4 Split spoils evenly

4.1 From necklace to Borsuk-Ulam

To apply Borsuk-Ulam theorem to the stolen necklace problem, one of the method is to first find an analogous structure fitting the theorem. A key reformulation should be pointed out there.

Both partitions of the necklace and the points on the sphere are represented by three variables with one restriction. As shown in section 2, two cuts divide the necklace into three pieces a, b, c. Let these three letters denote the length of three pieces. Notice that only the relative lengths affect how many beads each thief get because the length of the cord passing through the beads does not matter. What matters is the amount of beads assigned rather than length, quality, material, or other properties of the cord. Thus, let $a + b + c = 1$ to represent the total length of the cord.

Now, compare the following two sentences (where the entries of (s, e) stand for the amount of sapphires and emeralds assigned to a thief respectively):

- For a continuous function $f : (x, y, z) \in S^2 \to (s, t) \in \mathbb{R}^2$, there is a point (x, y, z) such that $x^2+y^2+z^2 = 1$ and $f(x, y, z) = f(-x, -y, -z) = (s, t)$
- For a necklace assignment A to thief 1 (s, e) (which determines the amount given to thief 2 as all the remaining beads belong to the other) under some partition (a, b, c), there is a partition (a, b, c) such that $a + b + c = 1$ and the assignment A for thief 1 is the same as the assignment to thief 2.

The similarity between the structure of two statement may already make you think about how to construct the missing parts to meet the conditions of the Borsuk-Ulam theorem. Before the construction, we first fill up the equivalence proof (they are similar but not yet equivalent).

4.2 From discrete to continuous

One of the key difference between Borsuk-Ulam theorem and the stolen necklace problem is the notion of discreteness and continuity. Notice that the usage of these two terms may change in different context. For example, one can describe random variable type by these two terms in probability theory; and one can also use discrete to describe a whole branch of mathematics with continuity standing for the mapping in analysis and topology.

Some observations of the two statements in the last subsection 4.1 leave the following unsolved questions to correspond the two completely:

1. The assignment A should be the continuous function referred in Borsuk-Ulam theorem, but how to make it continuous? How do length of the three pieces a, b, c tell us the amount of beads to assign for the thieves?
2. Since A eats some partition (a, b, c) and spits out an assignment (s, e), (a, b, c) should have the same traits as (x, y, z) in S^2, so how to do that? Notice that (s, t) in Borsuk-Ulam theorem and (s, e) in necklace problem can be easily corresponded as they both necessarily belong to some subsets of $\mathbb{R}^2$.

Question 1
The continuity requirement of question 1 is achieved in the next subsection. The main part of question 1 is explaining the correspondence between (a, b, c) and the amount of beads. Namely, how does this continuous variable tell information about a discrete one.

Given a necklace of $(k+l)$ beads (see refined version for meaning of these two letters), one divide the necklace into $(k+l)$ segments of equal length $\frac{1}{k+l}$ so that the total length of the necklace is 1. We identify each segment with the bead on it. This identification can be coloring, as in graph theory. Thus, we color $(k+l)$ segments each with color of blue and green to represent sapphires and emeralds on each segment respectively. An illustrative example on raw version necklace can be

The points on the unit interval $I = [0, 1]$ are $\frac{i}{k+l}$, $i = 0, 1, 2, \ldots$. Suppose we by Borsuk-Ulam theorem find a partition (a, b, c) such that each thief gets the same total length of blue and green parts, $\frac{k}{k+l}$ and $\frac{l}{k+l}$. If each entry

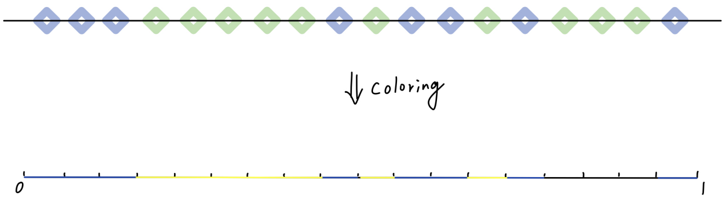

Figure 6: Coloring each equi-length segment with the color of the bead on it

of (a, b, c) are exactly some multiples of $\frac{1}{k+l}$, the two cuts are automatically on some endpoints of the small segments, corresponded to the borderline between two adjacent bead. Thus, we can follow the partition (a, b, c) to have two cuts on the original discrete necklace.

If the entires of (a, b, c) are not all some multiples of $\frac{1}{k+l}$, then we show it can still be followed to give two cuts on the discrete necklace. Since the two cuts on the unit interval are a and $a+b$, with three pieces (intervals) $[0, a]$, $[a, a+b]$, and $[a+b, 1]$. Let $\frac{p_1}{k+l} < a < \frac{p_1+1}{k+l}$ and $\frac{p_2}{k+l} < a+b < \frac{p_2+1}{k+l}$ for some nonnegative integers p_1 and p_2 smaller than $k+l$. By analysis in section 2, we find that there are three way to assign the three pieces to the thieves while the first two ways that both assign adjacent pieces to some same person degenerates the two-cut scheme to a one-cut scheme where only the amounts of bead in both types are equally distributed on both sides of the very middle point of necklace. Thus, one really need to check the third using the partition (a, b, c) from Borsuk-Ulam theorem: assigning the first piece $[0, a]$ and the third piece $[a+b, 1]$ to one thief with the second piece to the other.

The four parts concerning the non-integer cuts are $[\frac{p_1}{k+l}, a]$, $[a, \frac{p_1+1}{k+l}]$, $[\frac{p_2}{k+l}, a+b]$, $[a+b, \frac{p_2+1}{k+l}]$. Since $[\frac{p_1}{k+l}, \frac{p_1+1}{k+l}]$ and $[\frac{p_2}{k+l}, \frac{p_2+1}{k+l}]$ are two colored segments, pieces $[\frac{p_1}{k+l}, a]$ and $[a, \frac{p_1+1}{k+l}]$ are in the same color, and pieces $[\frac{p_2}{k+l}, a+b]$ and $[a+b, \frac{p_2+1}{k+l}]$ are in the same color. By previous assertion, we see that the middle two pieces $[a, \frac{p_1+1}{k+l}]$ and $[\frac{p_2}{k+l}, b]$ belong to thief 1, while the other two pieces belong to thief 2. Notice that the partition (a, b, c) ensures that each thief has the equal total length of both bead types. Besides, all segments except the four pieces we mentioned are intact when being assigned to each thief.

Thus, we see that when $[a, \frac{p_1+1}{k+l}]$ is given to thief 1, there has to be another interval of the same length belonging to one of the pieces a or b. Then, $[\frac{p_1}{k+l}, a]$ or $[b, \frac{p_2+1}{k+l}]$ has to have the same color and same length as $[a, \frac{p_1+1}{k+l}]$. And $[\frac{p_2}{k+l}, b]$'s situation is similar to $[a, \frac{p_1+1}{k+l}]$. In all, one will see that there are only two possibilities:

- segments $[\frac{p_1}{k+l}, \frac{p_1+1}{k+l}]$ and $[\frac{p_2}{k+l}, \frac{p_2+1}{k+l}]$ are of different colors, and the two cuts lie exactly on the middle of the two segments. However, this situation is not possible for other reason:

  Since k and l have to be even integer to equally divided and the segments $[\frac{p_1}{k+l}, \frac{p_1+1}{k+l}]$ and $[\frac{p_2}{k+l}, \frac{p_2+1}{k+l}]$ both take one of the k and l segment, leaving $k-1$ and $l-1$ segments to be equally divided, which is impossible. Indeed, the key observation here is that both the integer part of nominator of a and $a+b$ and the decimal part of the nominator of a and $a+b$ should give a fair division. The next case, however, satisfies this standard.

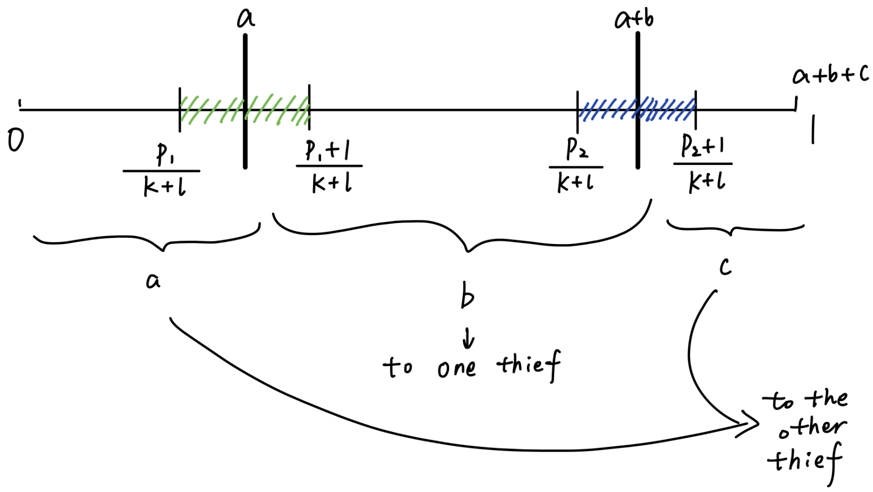

Figure 7: two non-integer cuts under partition (a, b, c) on the unit interval (case 1)

- segments $[\frac{p_1}{k+l}, \frac{p_1+1}{k+l}]$ and $[\frac{p_2}{k+l}, \frac{p_2+1}{k+l}]$ are of the same color, and the position of one cut on a segment is the same as that of the other cut on the other segment. This case is much simpler. One can just move both cuts a

and $a+b$ to the right or left endpoints of their respective segments (e.g. let the two cuts be $\frac{p_1}{k+1}$ and $\frac{p_2}{k+l}$).

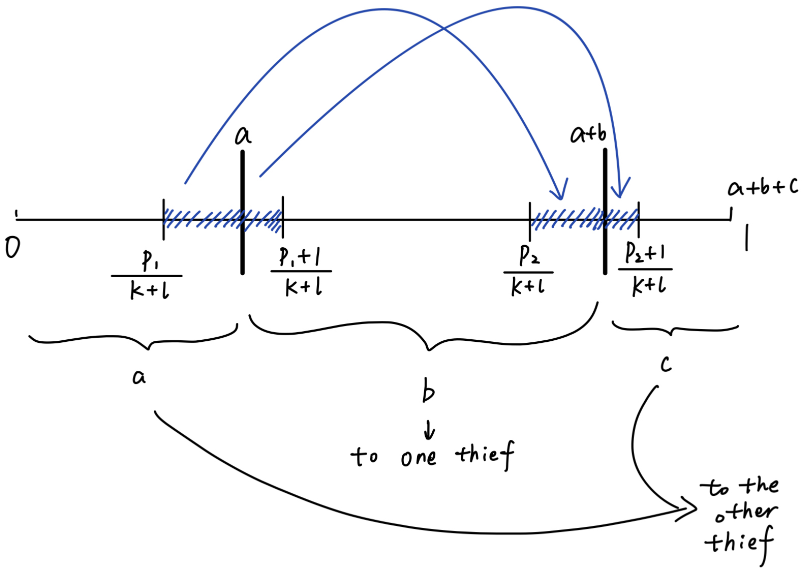

Figure 8: two non-integer cuts under partition (a, b, c) on the unit interval (case 2)

Question 2
We need to correspond (x, y, z) and (a, b, c) in this question. We post the solution first:
The squares x^2, y^2, and z^2 give the length of the division a, b, c while the signs of x, y, and z give a thief 1-or-2 assignment.

For example, the point on the sphere $(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{6}})$ means we have the partition $(a, b, c) = ((\sqrt{\frac{1}{3}})^2, (\sqrt{\frac{1}{2}})^2, (-\sqrt{\frac{1}{6}})^2) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$. Thus, we cut the necklace at point $a = \frac{1}{3}$ and $a + b = \frac{5}{6}$ and assign the first two piece to thief 1 with the last piece to thief 2.

Why does this solution work? Why do we let the sign of entries take that meaning? See the following example.

Suppose the antipodal pair we find by Borsuk-Ulam theorem is $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{6}})$ and $(\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}})$, where they satisfy $(\pm\sqrt{\frac{1}{3}})^2 + (\pm\sqrt{\frac{1}{2}})^2 + (\pm\sqrt{\frac{1}{6}})^2 = 1$.

They both give the division $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$ of the necklace, but the signs of $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{6}})$ says the first piece and the third piece go to thief 2 with the middle piece going to thief 1. The signs of $(\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}})$ is directly the opposite: first and third piece going to thief 2 with the middle piece going to thief 1. The assignment function A then assign the sapphires and emeralds in these pieces to thief 1 and thief 2. The assignment rule is just what we have said regarding the length and sign while A's output is (s, e) that tells us how many sapphires and emeralds thief 1 get due to the partition. In this example, let $A(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{6}}) = (s_1, e_1)$. Since this is the point whose antipode satisfies Borsuk-Ulam theorem, we have $A(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}}) = (s_1, e_1)$ too. This says if we give the middle piece to thief 1, the amount of sapphires and emeralds is the same as that thief 1 gets from assignment that makes the first and third piece going to him. We then solved the necklace problem. We find such an antipodal pair that by Borsuk-Ulam theorem gives the equal assignment to the thief.

Question 3
The last one still concerns with the assignment A. Notice that we need to make A continuous to use Borsuk-Ulam theorem. In Question 2, we only informally use words to say about the assignment and we don't know how does A in specific calculate (s_1, e_1) from $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{6}})$.

Since the notion of continuity we need to use here beyond calculus course in high school, we may just give a more intuitive proof for now.

4.3 The continuity of A

Recall that continuity of a function $f : X \to Y$ is something like:
moving from $x \in X$ a bit to $x + \delta \in X$ only change $f(x) \in Y$ to a nearby point $f(x) + \varepsilon \in Y$, where δ and ε are just letters standing for some very small positive real numbers.

Have a look of the figure in the next page to ease you imagination. There are three situations for moving the cut a to a': (i), (ii), and (iii).

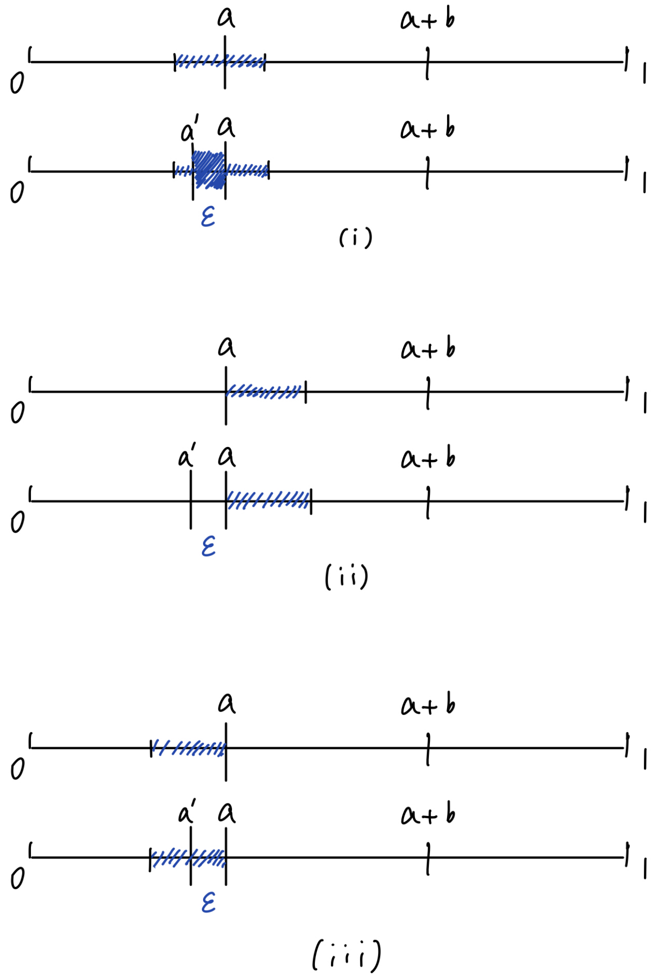

Figure 9: continuity of A. Moving the cuts non-drastically won't change the amount of colored pieces assigned to thieves drastically.

Let the middle piece going to thief 2, with the first and second going to thief 1. In (i), a' is moved a bit rightward by ε by our choice so that it is still in the blue segment. (s, e) will be changed to $(s - \varepsilon, e)$ because blue segment stands for sapphires and A's output (s, e) (notice that $s, e \in \mathbb{R}$, they are transformed to discrete version not by A but by process in question 1). Thus the change of the output is as small as ε. In (ii), nothing is changed for thief 1, so the change of (s, e) is as small as 0. In (iii), the result is the same as (i). You may check that. It is easy to see that the above analysis is completely analogous for green-colored segment and leftward shifting.

5 What's more?

5.1 Generalization of the theorem

You may wonder how to solve the stolen necklace problem if there are more than two thieves and more than two kinds of beads on the necklace.

When there are 2 thieves and n types of beads where each bead has its amount divisible by 2. Still, they want to minimize the cuts to divvy up the necklace. In fact, some analysis similar to what we have done can show that the thieves need n cuts to cut the necklace. In turn, we need n-dimensional Borsuk-Ulam theorem to map the partition of the necklace $(a_1, a_2, \ldots, a_{n+1})$ to the n kinds of beads $(b_1, b_2, \ldots, b_n)$ (the n-dimensional unit ball S^n live in $\mathbb{R}^{n+1}$ space. It is the collection of all points in that space that have a distance 1 from the origin).

Theorem 5.1 (n-dimensional Borsuk-Ulam theorem)**.** *if $f : S^n \to \mathbb{R}^n$ is continuous then there exists an $x \in S^n$ such that $f(-x) = f(x)$.*

How about k thieves and n types of beads where each bead has its amount divisible by k? You may have some trials for cases where the beads of the same type are contiguously distributed, but more complicated cases need deeper math knowledge.

5.2 Larger picture

Hopefully, this article gives a nice introduction to topology in a less cliché way, apart from solving the stolen necklace problem. In fact, two of the branches in topology are point set topology and algebraic topology. The Borsuk-Ulam Theorem in higher dimension is shown by utilizing algebraic topology theorems. Point set topology sets up some of the grammar of topology, and algebraic topology is instead a very powerful tool making sentences on top of grammar. Assuming you are interested in pursuing more in topology at college after reading this article, you will see in the future that a lot of discussions about space in point set topology are closed. That is, they are digested inside the learned knowledge system, and only part of them can be used in analysis. But when it comes to algebraic topology, it was much broader, not only to establish a higher point of view on topology, but also to show the wide range of uses of the tool. During your journey of mathematics, you may find out algebraic topology applicable to many other fields, including complex analysis, combinatorial, manifold, and even computational economics.

What else did you learn? Probably the following methodology.

Solving puzzles by abstracting relationships, classifying cases, simplifying problems, and translating problems.

Each of these steps are used in solving necklace problem. Among them is the most important one, translation. We usually see the word translation as encoding semantic meaning from linguistic symbols, whereas translation is important in math too. Math symbols give a system to abstract and epitomize the logical thinking behind relationships, quantitative or qualitative, between mathematical objects. By our necklace problem, our first translation is from the real-world necklace problem to a math problem. Our second translation, however, exemplifies a whole bunch of works done during communication and collaboration in math society: translation from one field of math to another. We put topology and discrete math together. One of the properties in this transformation is, as what I nicknamed, "discretize and continuitize."

In topology, continuity along with various connectedness, separation axioms, and other topological properties are often assumed in discussion. All these notions contribute to the continuous character of topology. Sometimes these continuous structures are so complex to analyze in whole, we tend to look at their local behaviors, and one of the way to do that is to discretize the spaces we are examining. Triangulation is one way to do that as it breaks down a compact surface to small triangles, which allow us to use Euclidean geometry to analyze them (triangles on a plane is intuitively easier to deal with than an irregular shape).

In our necklace problem, however, we make things back to continuous case. We think about beads as colored intervals to apply Borsuk-Ulam theorem. Another example is regarding the grids (lattices) as a subset of two dimensional plane and further as the domain of a continuous function in some graph problems.

We end the article by a small story about translation of math problems and a collaboration between mathematician Terrace Tao and Allen Knutson, who translates problems "into combinatorics, then sic Terry on them."
Link to this account:
https://www.quora.com/What-is-it-like-to-work-with-the-mathematician-Terence-Tao/answer/Allen-Knutson?ch=10&share=18053581&srid=n7gh