1 Introduction

The paper *The Game of Hex and The Brouwer Fixed-Point Theorem* by David Gale mainly goes with two purposes:

- The first one is to show the equivalence between the Hex Theorem with two players and the Brouwer fixed point theorem on a unit square.
- The second part of the paper gives a generalized version of the Hex Theorem into higher dimension, which sheds light upon multiplayer game theory problem in mathematical economics.

The author notes that the equivalence proof, given its simplicity compared with n dimensional case, has other versions, while the generalization has more newness and originality by offering an algorithm for computation of fixed points.
Section 2 gives relevant facts about Hex Theorem, section 3 proves the equivalence, and section 4 generalizes it.

2 Hex

1. The rule of the Hex Game
   Two players are given a board that typically contains 11×11 hexagons. Each player moves alternately by occupying a hexagon with a stone with different color (typically black and white) representing each player. Note that in Gale's paper, stones of different colors are replaced by x and o as in tic-tac-toe game as another way of showing the occupancy of the hexagon (or tile in Gale's paper). When one uses stones to draw a connected path from one side to its opposite side, the game is over and that one wins. The author gives the definition of being connected as:

   Definition 2.1. *A set S of hexagons is said to be **connected** if any two members h and h' of S can be joined by a path $P = (h = h^1, h^2, \cdots, h^m = h')$ where h^i and h^{i+1} are adjacent.*

   Put vividly, a connected set of hexagons in one color (for example, white) means that there exits a lane made up of white stones placed over this set of hexagons allowing one to walk on stones just next to each other from one side to the opposite side of the board. The following shows a real-life example of the hex game board.

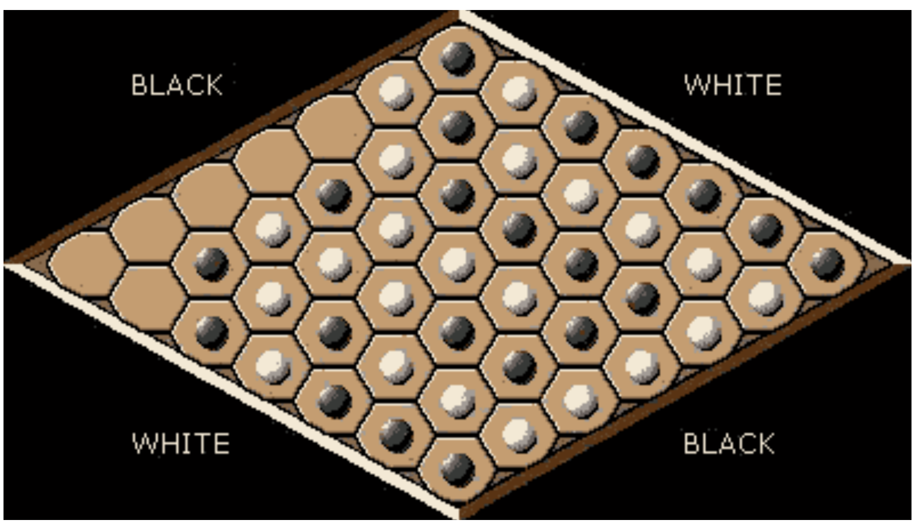

Figure 1: Hex game board (in the above figure, the white player wins the game by creating a connected path starting at one side and reaching the opposite side). Source: https://www.ams.org/publicoutreach/feature-column/fcarc-partizan2

2. Hex Theorem
   To be consistent with mathematical language, we shall continue using object representation described by Gale (x and o instead of the visual way, black and white) to present the following theorem.

   Theorem 2.1 (Hex theorem)**.** *If every hexagon of the Hex board is occupied by either* *x or o, then there is either a black (x-) connected path beginning at one side of the board and ending at the opposite side or a white (o-) connected path behaving similarly on the other pair of sides.*

There is a strengthening of this theorem, by replacing the inclusive "or" with an exclusive one that rules out a scenario of "both", however, this is not of primary purpose of this paper.

3. Geometric intuition behind the proof of Hex Theorem
Back to the analogy of black and white stones. Imagine there is water all over the board, and the winning rubric of constructing a connected path from one side to the opposite is can be naturally translated as putting stones each one just next to the other over the water to let a person walk step by step on the stone bridge to cross the river and reach the other bank. While building this bridge, it is intuitively clear that it prohibits the other player building his bridge to connect the other pair of banks to cross the river.

   Now, we want to prove it in a more informal but rigorous way. Suppose we have a board filled with x and o, like the one in figure 2. We give a notation of each hexagon by numbering them in rows (the letters A to M) and columns (the numbers 1 to 13) (note that it's still a 11×11 board but with a circle of o and x just around it to make it easier to visualize. Another notation is about the six vertices of the hexagon, see the figure 3. The $A1$ hexagon has 6 vertices, upper vertex (u), bottom vertex (d), left upper vertex (l), left bottom vertex (L), right upper vertex (r), and right bottom vertex (R). We also use edges a, b, c, d to separate the banks (see figure 3). Lastly, when we say X-face, we mean either hexagon marked by x or the X region (and its opposite region X'), and the same for O-face. For this purpose, we augmented the 11×11 board to be an 13×13 board with extra 4 banks each filled solely with x and o.
   We give a touring rule for how we will draw edges and then show that the fact that we can always draw a path by edges connecting the ends a, b, c, or d according to this touring rule is equivalent of saying that we can always draw a connected path by hexagons from one bank to the opposite.
   The **touring rule** is given by: starting from starting from the edge a, which separates bank X and O apart, keeping drawing along the edges which is a common boundary of an X-face and an O-face.
   We then have three observations about the path by edges, and we will show the equivalence of existence of the edge-path and the hexagon-path.

   (a) Uniqueness of the edge-path and unchanging orientation
   Suppose one is at some edge and moves from its one endpoint to the other. According to our touring rule, for an edge to be drawn, it must be the common boundary between an X-face and an O-face, which means that it must be between an X-hexagon and an O-hexagon if we use the augmented board (figure 2). Notice

that we will not have any edges drawn within the augmented part (but possibly between the augmented part and the original board, which then serve as the boundary between the bank and the original board), because they are banks with the same type while an edge to be drawn should have different hexagons on its two sides. Continue with the edge we are at. Since two sides of it need to be X-hexagon and O-hexagon, we see X can be on its left and O the right and the other order (X right O left) can run through the same argument. See figure 3 case 1 and case 2. the edge we at are, respectively, B8Rr and H2Ll, sided by B8 & C7 and H2 & G3. the next hexagon one will meet will be either an O-hexagon (B7 in case 1) or an X-hexagon (G2 in case 2). In case 1, we draw the edge B7dL according the touring rule, and we observe that we arrive at a new edge (B7dL) which has X-hexagon of its left and O-hexagon on its right again. In case 2, we draw the edge G2dR according to the touring rule, and again we has a new edge with X left and O right. Thus, in both cases, we will deterministically draw only one edge further and arrives at the exactly same situation with the orientation (defined as X left and O right). Whenever one is at an edge and follows the touring rule, if the three hexagons incident to the edge are fixed, there will be no more than one options to draw a new edge because we have shown that the rule determines the next single on edge, which gives the uniqueness of the path.

(b) Never revisiting the vertex
First, be revisiting, we mean that there is a vertex with three drawn edges attached to it. Note that it is possible to have a loop but without any other edges protruding to connect the loop to some other paths. We will then prove the property (b) in a generic example without loss of generality. See figure 3, where one is now at the edge J5dL and then goes into a loop running counterclockwise to go back to revisit where it starts. We want to ridicule such situation. Let us keep what we used in showing property (a), setting J6 and J5 to be X and O. By the property (a), we see that the loop has to enclose a circle of X to make one runs counterclockwise and revisit. However, this results in I6 and I7 being X and J6 being X too. Again, note that it is possible to form a loop (if one draws the edges J6Ld J6dR, J6Rr), but that's not revisiting by definition.

(c) Always terminate
The hex game board is finite, so there are finitely many edges one can draw. By property (b), one can never revisit a vertex and thus never revisit an edge. By property (a), we see that for each time we arrive at a place sided by two different hexagons, according to the touring rule, we have exactly one option to do, which is to draw an edge between them (we do not draw zero edges or 2 edges). Therefore, keeping following the touring rule and using property (a) and (b) will make these finitely edges one by one decreasing. The path drawn then has to terminate at least when all edges are used (it's not possible to use all the edges, but that suffices for us to show the property).

Now we present a theorem from graph theory, which we want to briefly prove in our case.

Theorem 2.2 (Graph lemma). *A finite graph whose vertices have degree at most two is the union of disjoint subgraphs, each of which is either (i) an isolated vertex, (ii) a simple cycle (loop), (iii) a simple path.*

By property (b), we have shown that the edge graph of the hex board is indeed a finite graph whose vertices have degree at most two. Then we classify these vertices into several cases:

case 1 vertex of degree 0: then just an isolated vertex.

case 2 all vertices are of degree 1: then it will just be a single edge. However, by property (a), we see that's impossible in our 11×11 board if we follow the touring rule.

case 3 all vertices are of degree 2: it is a simple cycle (loop). That's a possible scenario in our edge graph.

case 4 vertices of degree 1 and degree 2 combined: this should a simple path which can be shown by induction. It's easy to see that three vertices with two 1-degree endpoints and a single 2-degree middle point form a simple path connecting two endpoints. Afterwards we append another vertex with one edge to the former path and we will get another simple path with the old endpoint being a middle one with degree of 2 and the new appended vertex as the new endpoint.

So, if we put this last case into our hex board. We observe that there are four edges of degree one: a, b, c, d. Therefore, starting from one of them, say a, one follows the touring rule and end up appending edges of degree of two and finally terminating (by property (c)) with a vertex of degree of one, which has to be one of b, c, d. In other words, if one starts with a, then one will gets a simple path connecting a to one of b, c, d.

Finally, we need to see that such a simple edge-path allows us to find a hexagon-path that makes on win the hex game! The main idea is to use property (a), where we show that for every edge on the simple path, if we start with a's orientation (X left and O right), then will will keep that orientation along the whole path. Then, this path has one side with all O and the other all X. They are connected because the edges of them form a connected simple path (a separation of two hexagons need another hexagon to be added, which result in another edge added to disrupt the connectedness of the simple path). In figure 2, if the player puts the stones on these uniformly O hexagons on the right side of the orange path, he/she will get a connected hexagon-path. Since it starts at a, the first piece of this hexagon-path is O in the 13×13 augmented circle, and the last piece is on the right of b, which is O in the 13×13 augmented circle. We have then finished creating this winning set.

3 Hex $\Rightarrow$ Brouwer

3.1 Preliminary notations

We first introduce Nash's representation of the Hex board. A square taken from the $\mathbb{Z} \times \mathbb{Z}$ lattice where two squares are considered adjacent if they are next to each other horizontally, vertically, or along the diagonal of slope of 1. This is equivalent to the Hex board. Take a hexagon in figure 3, say A1 for an example. Imagine shrinking the side A1br and A1Ld into a single point, which deforms the hexagon to be a parallelogram. Grouping hexagons A1, A2, A3, B1, B2, and C1, and we see in figure 3 that they are deformed into blue patterns in Nash's representation. Before we give a more precise definition of the Hex board, we first define an order for which the elements (vertices) can be compared and the norm we based on.

Definition 3.1. *We define the* ***maximum norm*** $\|\cdot\|$ *in the space* $\mathbb{R}^2$ *as:*

$$\|z\| = \max_{i=1,2} z_i$$

for some $z = (z_1, z_2) \in \mathbb{Z}^2$*. And the induced inner product is* $|\cdot|$

Definition 3.2. $\forall x, y \in \mathbb{Z}^2, s.t. x \neq y$ $(x < y)$ *is defined as* $\forall i = 1, 2$ $x_i \leqslant y_i)$*. Let* $>$ *be similarly defined. The two points* x *and* y *in* $\mathbb{Z}^2$ *are* ***comparable*** *if* $x < y$ *or* $y < x$*.*

We then give the formal definition of Nash's representation of the Hex board and shows that they match.

Definition 3.3. *A* ***Hex board*** B_k *of size* k *is a graph whose vertices are* $\{v \in \mathbb{Z} \times \mathbb{Z} | (0,0) \leqslant v \leqslant (k,k)\}$*.*

Definition 3.4. *Two vertices* z *and* z' *are said to be* ***adjacent*** *in a Hex board* B_k *if:*

- $|z - z'| = 1$
- z *and* z' *are comparable.*

We want to see how this definition of being comparable is equivalent to Nash's setting. We notice that $|z - z'| = 1$ does not use the Euclidean norm one usually encounters to draw the circle. The following picture gives an example of the unit circle $|z - 0| = 1$.

In fact, for a point $z = (z_1, z_2) \in \mathbb{Z}^2$, $|z - z'| = 1$ is the set

$$\{z' = (x, y) | x = z_1 + k_1, y = z_2 + k_2, k_i \in \{-1, 0, 1\}\} \backslash \{z\}$$

The second condition is that the two points are comparable. The definition of being comparable rules out the possibility where the two pair of coordinates have different directions in comparison by simple order in $\mathbb{R}$. Namely, for $x, y \in \mathbb{Z}^2$, x and y are not comparable if $x_1 \geqslant y_1$ but $x_2 \leqslant y_2$ or $x_1 \leqslant y_1$ but $x_2 \geqslant y_2$. Thus, among the eight points on the circle $|z - z'| = 1$, the upper left and lower right points are not comparable to z. The left six points

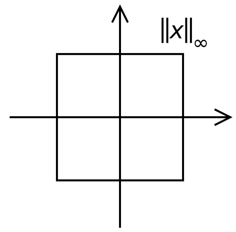

Figure 4: The maximum norm. The figure shows the unit circle in an $\mathbb{R}^2$ space with the maximum norm.

lie on the horizontal, vertical, and positively sloping diagonal near z, which are exactly those comparable to z both by the above definition and by Nash's definition.

Figure 5 continues the deformation we did in Figure 3. The Orange area shaded with black is the same of the grouped six hexagons in Figure 3, and we add three others (orange hexagons shaded with blue) to form a Hex board of size 3. A deformation shows that it is the same as a lattice $([1,4] \times [1,4]) \cap \mathbb{Z}^2$. Figure 6 gives an example of Hex board B_5, and we label each bank with N, S, E, and W on the compass. The horizontal (vertical) player wins by draw a path connecting E and W (N and S).

3.2 Stating Theorems

We now restate Theorem 2.1. Notice that Theorem 2.1 says that if we cover all the hexagons with x and o, then one will have a connected path connecting two opposite sides. By the last part (begins with "finally") in the Section 2, we see that this connected path made up of hexagons is equivalent to a connected path by vertices. Thus, the Hex board B_k is like the edge graph of the original Hex board whose basic elements are regarded as hexagons rather than vertices. We then have

Theorem 3.1 (Hex Theorem*). *Let H and V be two sets partitioning a Hex board B_k, then there will be a connected path joined by edges in H that meets E and W or one in V that meets N and S.*

And we have the classical Brouwer fixed-point theorem in a unit square $I^2 = [0,1] \times [0,1]$:

Theorem 3.2 (Brouwer Fixed-Point Theorem). *For a continuous mapping $f : I^2 \to I^2$, there exists a fixed point $x \in I^2$ such that $f(x) = x$.*

3.3 proof of Hex $\Rightarrow$ Brouwer

We will use theorem from Munkres's (denoted as [M]) to build up our proof.

We first observe some topological properties of the map f and prove a lemma. Since $I^2 = [0,1] \times [0,1]$ is clearlty closed and bounded in the Euclidean space $\mathbb{R}^2$, by [M] Theorem 27.3 we know that I^2 is compact. Besides, by [M] Theorem 28.2, in a metric space like $\mathbb{R}^2$ and its subspace I^2 which is consequently metrizable, compactness is equivalent to sequential compactness.

Lemma 3.3. *The statement that* $\forall \varepsilon > 0, \exists x \in I^2, s.t. |f(x) - x| < \varepsilon$ *implies the existence of a fixed point (i.e.* $\exists a \in I^2 s.t. f(a) = a$*)*

Proof. Since $\forall \varepsilon > 0, \exists x \in I^2, s.t. |f(x) - x| < \varepsilon$, we can then define sets A_i such that they are nonempty. Let $A_i := \{x \in I^2 || f(x) - x| < \varepsilon_i\}$ where we can randomly pick $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0$, $i \in \mathbb{Z}^+$. Clearly, we have $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$.
Since the sets $\{A_i\}_{i\in\mathbb{Z}^+}$ are all nonempty, we can take $x_i \in A_i$ from each of them and obtain the sequence $\{x_i\}_{i\in\mathbb{Z}^+}$. By the sequential compactness, this sequence has a subsequence $\{x_{i_k}\}$ that converges to some $a \in I^2$. By definition of the convergence in a metric space, we have:

$$\begin{aligned}&\forall \epsilon > 0, \exists n \in \mathbb{Z}^+ \text{ s.t. } i_k > n \text{ implies } |x_{i_k} - a| < \epsilon\\ &\text{In particular } |x_l - a| < \epsilon \text{ for some } x_l \in \{x_{i_k}\} \text{ such that } l > n.\end{aligned}$$

By [M] Theorem 21.3 and the continuity of the function f, we see that

$$\{x_{i_k}\} \to a \Rightarrow \{f(x_{i_k})\} \to f(a)$$

Again, by the definition of convergence, we have:

$$\begin{aligned}&\forall \epsilon' > 0, \exists m \in \mathbb{Z}^+ \text{ s.t. } i_k > m \text{ implies } |f(x_{i_k}) - f(a)| < \epsilon';\\ &\text{In particular } |f(x_{l'}) - f(a)| < \epsilon' \text{ for some } x_{l'} \in \{x_{i_k}\} \text{ such that } l' > m.\end{aligned}$$

Let $L = \max l, l'$. Since $x_l, x_{l'} \in \{x_{i_k}\}$, x_L is also in the subsequence.
For the last inequality, we choose $\epsilon' = \frac{\delta}{3}$ for some $\delta > 0$. We then have $|f(a) - f(x_L)| = |f(x_L) - f(a)| < \frac{\delta}{3}$. Since x_L is an element in the subsequence of $\{x_i\}_{i\in\mathbb{Z}^+}$, we have x_L satisfying $|f(x_L) - x_L| < \varepsilon_L$. By the randomness of $\varepsilon_1, \varepsilon_2, \cdots$, we choose ε_L to be $\frac{\delta}{3}$. Besides, we have the inequality $|x_L - a| < \frac{\delta}{3}$ where we choose $\epsilon = \frac{\delta}{3}$.
Lastly, we get

$$\delta = \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} > |f(a) - f(x_L)| + |f(x_L) - x_L| + |x_L - a| \geqslant |f(a) - a|$$

By the randomness of δ, we have $f(a) = a$ (we can choose $\delta = \frac{|f(a)-a|}{2}$ to get a contradiction if not). Also notice that this trick of contradiction is not applicable directly to the condition of our theorem $|f(x) - x| < \varepsilon$ because for each ε there are different x satisfying the inequality. □

According to the lemma, we want to use Hex theorem to show that $\forall \varepsilon > 0, \exists x \in I^2, s.t. |f(x) - x| < \varepsilon$, which then gives a fixed point.

Theorem 3.4 (Hex implies Brouwer)**.**

Proof. By [M] theorem 27.6 and the continuity of f in the compact metric space I^2, we see that f is uniformly continuous. Therefore, by definition, given $\varepsilon > 0$, $\exists \delta > 0$ such that $\delta < \frac{\varepsilon}{\sqrt{2}}$ (this can be done by taking whichever smaller) and $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \frac{\varepsilon}{\sqrt{2}}$. Given δ we find by ε, we can find a k such that $\frac{1}{k} < \delta$ and take a Hex board of this size B_k. We make the Hex theorem get involved by regarding the Hex board a greater version of I^2, or more mathematically by scaling the Hex board to I^2. We define the following scaling function r:

$$\begin{aligned} r : B_k &\to I^2 \\ z &\mapsto \frac{z}{k} \end{aligned}$$

Thus, we want to see how the composite $f \circ r$ behave by the Hex theorem. Testing if the set $\{x \in I^2 | |f(x) - x| < \varepsilon\} \neq \varnothing$ is then converted to be on the set $A = \{z \in B_k | |f \circ r(z) - \frac{z}{k}| < \varepsilon\}$. We define four sets than can cover the complement of set A in B_k to correspond to the horizontal and vertical players in the hex theorem and want to show that the four sets cannot cover the whole B_k, which leave spaces for A to take elements. The fact that some sets even bigger than the complement of A in B_k cannot cover B_k suffices to show that A^c itself cannot cover B_k.

Let the component function of $f : I^2 \to I^2$ be f_1 and f_2 such that $f(x) = f(x_1, x_2) = (f_1(x), f_2(x)) = (f_1(x_1, x_2), f_2(x_1, x_2))$ for $x = (x_1, x_2)$. Notice that the metric used in the set A is a Euclidean metric, and then

$$\begin{aligned} A^c &= \{z \in B_k | |f \circ r(z) - f \circ r(z)| \geqslant \varepsilon\} = \left\{ z \in B_k | |f(\frac{z}{k}) - \frac{z}{k}| \geqslant \varepsilon \right\} \\ &= \left\{ z \in B_k | \left| f_1(\frac{z}{k}) - \frac{z_1}{k} \right|^2 + \left| f_2(\frac{z}{k}) - \frac{z_2}{k} \right|^2 \geqslant \varepsilon^2 \right\} \end{aligned}$$

Let

$$\begin{aligned} H^+ &= \left\{ z | f_1(z/k) - z_1/k \geqslant \frac{\varepsilon}{\sqrt{2}} \right\} \\ H^- &= \left\{ z | z_1/k - f_1(z/k) \geqslant \frac{\varepsilon}{\sqrt{2}} \right\} \\ V^+ &= \left\{ z | f_2(z/k) - z_2/k \geqslant \frac{\varepsilon}{\sqrt{2}} \right\} \\ V^- &= \left\{ z | z_2/k - f_2(z/k) \geqslant \frac{\varepsilon}{\sqrt{2}} \right\} \end{aligned}$$

Let $H = H^+ \cup H^-$ and $V = V^+ \cup V^-$. We see that
$H = \left\{ z | f_1(z/k) - z_1/k > \frac{\varepsilon}{\sqrt{2}} \right\} \cup \left\{ z | f_1(z/k) - z_1/k < -\frac{\varepsilon}{\sqrt{2}} \right\} = \left\{ z || f_1(z/k) - z_1/k| > \frac{\varepsilon}{\sqrt{2}} \right\}$
$V = \left\{ z | f_2(z/k) - z_2/k \geqslant \frac{\varepsilon}{\sqrt{2}} \right\} \cup \left\{ z | f_2(z/k) - z_2/k < -\frac{\varepsilon}{\sqrt{2}} \right\} = \left\{ z || f_2(z/k) - z_2/k| \geqslant \frac{\varepsilon}{\sqrt{2}} \right\}$
Also,

$$A^c = \left\{ z \in B_k | \left| f_1(\frac{z}{k}) - \frac{z_1}{k} \right|^2 + \left| f_2(\frac{z}{k}) - \frac{z_2}{k} \right|^2 \geqslant \varepsilon^2 \right\} \subseteq H \cup V$$

(that's because if $z \notin H$ then one of the two squares is smaller than $\frac{\varepsilon^2}{2}$ then it must be the case $z \in N$ to make the other square greater than $\frac{\varepsilon^2}{2}$ to possibly make the sum of the two square greater than ε^2, namely $z \in V$, and vice versa)
To show that the four sets cannot cover B_k, we need to show that they are not contiguous.

Definition 3.5. *A pair of subsets A and B of a graph are said to be **contiguous** if there exists $a \in A$ and $b \in B$ such that a and b are adjacent.*

Let $z \in H^+$, $z' \in H^-$. By definition 3.4, z and z' being adjacent means $|z - z'| = 1$. WLOG, let $z'_1 - z_1 = 1$. We proceed the proof showing that they should not be adjacent by contradiction. Suppose they are adjacent and hence $z'_1 - z_1 = 1$. Then in the board B_k where we choose $1/k < \delta < \frac{\varepsilon}{\sqrt{2}}$, we have $z'_1/k - z_1/k = 1/k < \frac{\varepsilon}{\sqrt{2}}$. Thus,

$$z_1/k - z'_1/k > -\frac{\varepsilon}{\sqrt{2}} \Rightarrow z_1/k - z'_1/k \geqslant -\frac{\varepsilon}{\sqrt{2}} \tag{1}$$

Since $z \in H^+$, $z' \in H^-$,

$$f_1\left(\frac{z}{k}\right) - \frac{z_1}{k} \geqslant \frac{\varepsilon}{\sqrt{2}}$$
$$\frac{z'_1}{k} - f_1\left(\frac{z'}{k}\right) \geqslant \frac{\varepsilon}{\sqrt{2}}$$

which gives

$$f_1\left(\frac{z}{k}\right) - \frac{z_1}{k} + \frac{z'_1}{k} - f_1\left(\frac{z'}{k}\right) \geqslant 2\frac{\varepsilon}{\sqrt{2}} \tag{2}$$

Adding inequalities (1) and (2) gives

$$f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right) \geqslant \frac{\varepsilon}{\sqrt{2}}$$

However, by the uniform continuity,

$$\begin{aligned} \frac{z'_1}{k} - \frac{z_1}{k} < \delta < \frac{\varepsilon}{\sqrt{2}} \Rightarrow & |f(z/k) - f(z'/k)| \\ = & \sqrt{\left| f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right) \right|^2 + \left| f_2\left(\frac{z}{k}\right) - f_2\left(\frac{z'}{k}\right) \right|^2} < \frac{\varepsilon}{\sqrt{2}} \end{aligned} \tag{3}$$

but

$$f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right) \geqslant \frac{\varepsilon}{\sqrt{2}} \Rightarrow \left|f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right)\right| \geqslant f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right) \geqslant \frac{\varepsilon}{\sqrt{2}}$$
$$\Rightarrow \left|f_1\left(\frac{z}{k}\right) - f_1\left(\frac{z'}{k}\right)\right|^2 \geqslant \frac{\varepsilon^2}{2}$$

which is contradictory to inequality (3). Thus, z and z' cannot be adjacent and thus H^+ and H^- cannot be contiguous. It can be similarly shown that V^+ and V^- are not contiguous too. Suppose Q is a connected set contained in H. Since non-contiguousness clearly implies disconnectedness, we see that Q must lie entirely in H^+ or H^- by [M] Lemma 23.2. Also notice that H^+ does not meet the boundary $E = 0 \times I$ because there is no element in I^2, which is the codomain of f, that has horizontal coordinate greater than 1 by a nonzero $\frac{\varepsilon}{\sqrt{2}}$. Similarly H^- does not meet W. Thus, Q, which is either in H^+ or H^- cannot meet both E and W. It can be similarly argued that V contains no connected set meeting N and S too. Therefore, H and V do not cover B_k, for if they are, the fact that there is no path meeting either pair of banks, which contradicts the Hex Theorem. $\square$

4 Brouwer $\Rightarrow$ Hex

To complete the proof of equivalence between Hex and Brouwer, we need to do this last piece: showing that Brouwer implies Hex.

Theorem 4.1 (Brouwer implies Hex)**.**

We first have some preparations for proving the theorem. Notice the fact that any maps f from B_k into $\mathbb{R}^2$ extends to a continuous simplicial (or piecewise linear) map $\hat{f}$ on I_k^2 where I_k^2 is the $k \times k$ square in $\mathbb{R}^2$. Namely, if

$$x = \sum_{i=1}^{3} \lambda_i z^i \tag{4}$$

where

$$\lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{3} \lambda_i = 1 \tag{5}$$

$$\hat{f}(x) := \sum_{i=1}^{3} \lambda_i f(z^i) \tag{6}$$

We then prove a lemma about this.

Lemma 4.2. *Let z^1, z^2, z^3 be the vertices of a triangle $\triangle$ in $\mathbb{R}^2$, and let $\hat{p}$ be the simplicial extension of a map p defined by $p(z^i) = z^i + v^i$ where v^i $(i = 1, 2, 3)$ are given vectors. Then $\hat{p}$ has a fixed point iff 0 is in the convex hull of $\{v^1, v^2, v^3\}$.*

Proof. Let x and λ_i be those in (4) and (5). By convexity, $x \in \triangle$. By (6), we have

$$\hat{p}(x) = \sum_{i=1}^{3} \lambda_i p(z^i) = \sum_{i=1}^{3} \lambda_i (z^i + v^i) = \sum_{i=1}^{3} \lambda_i z^i + \sum_{i=1}^{3} \lambda_i v^i = x + \sum_{i=1}^{3} \lambda_i v^i \tag{7}$$

Thus, x is fixed in $\hat{p}$ if and only if $\hat{p}(x) = x$. By (7), this means

$$x = x + \sum_{i=1}^{3} \lambda_i v^i \Leftrightarrow \sum_{i=1}^{3} \lambda_i v^i = 0$$

By definition of the convex hull of a set X, which is the set of all convex combinations of points in X, we see that since $\sum_{i=1}^{3} \lambda_i v^i$ is such a combination by (5), 0 is in the convex hull of the set $\{v^1, v^2, v^3\}$. $\square$

Now we shall start proving theorem 4.1.

Proof. Let H and V be a partition of the B_k. Namely, $H \cap V = \varnothing$ and $H \cup V = B_k$. We define $\hat{W}, \hat{E}, \hat{N}$, and $\hat{S}$ as follow (by using the symbol $a \dashrightarrow_h b$ to represent there there is an H-path h connecting a and b):

$$\begin{aligned}
\hat{W} &= \{v \in B_k | \exists w \in W, v \dashrightarrow_H w\} \\
\hat{S} &= \{v \in B_k | \exists s \in S, v \dashrightarrow_V s\} \\
\hat{E} &= H - \hat{W} \\
\hat{N} &= V - \hat{S}
\end{aligned}$$

Intuitively, $\hat{W}$ and $\hat{S}$ are the sets in H and V that is attached to (touched by) the boundary W and S, respectively. The following is a demonstrative example where $\hat{W} = H_1, \hat{S} = V_1, \hat{E} = H - \hat{W} = H_2, \hat{N} = V - \hat{S} = V_2$.

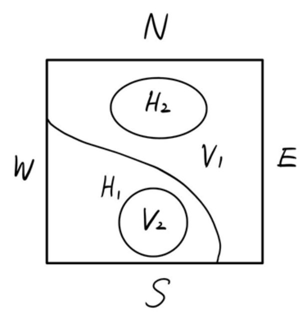

Figure 7: A demonstrative example of the four sets defined

We see that by definition, $\hat{W}$ and $\hat{E}$ (and similarly $\hat{N}$ and $\hat{S}$) are not contiguous because for if they are, then by definition 3.5, there is an edge ab connecting $a \in \hat{W}$ and $b \in \hat{E}$ and $\exists w \in W$ s.t. $a \dashrightarrow_H w$, which then causes a contradiction: the edge ab can be included into the H-path h to form a path h' such that $b \dashrightarrow_H w$, making $b \in \hat{W}$, but $b \in H - \hat{W} = \hat{E}$.

Let $\vec{e_1} = \hat{i} = (1,0)$ and $\vec{e_2} = \hat{j} = (0,1)$ be the standard basis of $\mathbb{R}^2$, and we define $f : B_k \to B_k$ by

$$\begin{aligned}
z = (z_1, z_2) &\mapsto z + \vec{e_1} = (z_1 + 1, z_2), \text{ if } z \in \hat{W}; \\
z = (z_1, z_2) &\mapsto z - \vec{e_1} = (z_1 - 1, z_2), \text{ if } z \in \hat{E}; \\
z = (z_1, z_2) &\mapsto z + \vec{e_2} = (z_1, z_2 + 1), \text{ if } z \in \hat{S}; \\
z = (z_1, z_2) &\mapsto z - \vec{e_1} = (z_1, z_2 - 1), \text{ if } z \in \hat{N}.
\end{aligned}$$

We first need to show that the map f is well-defined for the four operations $\pm\vec{e_i}$ do not make the image out of bound (namely, $f(z) \in B_k$ for $z \in B_k$). For $z \in \hat{W}$, the only possible

scenario that $+\vec{e_1}$ can make $f(z)$ goes outside of B_k is when z is on the right-most boundary E (i.e. $z_1 = k$). However, the assumption we made in this process of contradiction proof is that there is no H-path connecting W and E, rendering it impossible for $\hat{W}$ to include an element on the boundary E (if so, the element can be connected to W by definition of the set $\hat{W}$, contradictory to our assumption). For the operation $-\vec{e_1}$, we need to show that $\hat{E}$ does not include elements on the boundary W to see that the operation does not make the image exceed of the board's left-most boundary. Notice that $\hat{E}$ includes exactly all the vertices in H that can not be connected to W by an H-path. Therefore, if $\hat{W}$ has an element w on W, w cannot be connected to W, which is absurd since w already arrives at W without any path needed.
It can be analogously shown that the other two operations on the vertical direction do no exceed the board too.
We prove a lemma following the non-contiguousness of $\hat{W}$-$\hat{E}$ and $\hat{N}$-$\hat{S}$.

Lemma 4.3. *Given a triangle $\triangle$ in $\mathbb{R}^2$ with vertices z^1, z^2, z^3 that are mutually adjacent to each other in B_k, when computing the image of the triangle under f, there will never be one of the three vertices z^j translated by operation $+\vec{e_i}$ and another z_k translated by $-\vec{e_i}$ ($i = 1, 2$ and $j \neq k \in \{1, 2, 3\}$).*

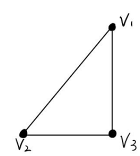

Figure 8: A triangle with vertices z^1, z^2, z^3

Proof. Since each vertex in the triangle $\triangle$ is adjacent to the other two, the whole triangle never lies in both $\hat{W}$ and $\hat{E}$ or in both $\hat{N}$ and $\hat{S}$ due to the non-contiguousness of $\hat{W}$-$\hat{E}$ and $\hat{N}$-$\hat{S}$. Notice that the operations by f is conditioned by where z belongs ($\hat{W}, \hat{E}, \hat{N}, \hat{S}$), and then the conclusion of the lemma is clear. □

An immediate corollary is that the whole triangle will be translated by unit vectors that both lie in one quadrant of $\mathbb{R}^2$ because all other possible combinations of different vectors that can be used in during mapping are

1. first quadrant: $+(1, 0) = (1, 0), +(0, 1) = (0, 1)$
2. second quadrant: $-(1, 0) = (-1, 0), +(0, 1) = (0, 1)$
3. first quadrant: $-(1, 0) = (-1, 0), -(0, 1) = (0, -1)$
4. second quadrant: $+(1, 0) = (1, 0), -(0, 1) = (0, -1)$

We complete the proof of Hex theorem by contradiction. The assumption of the contradiction is that there is no H-path connecting E and W AND there is no V-path connecting N and

S. The fact we will build to contradict Brouwer fixed-point theorem (Theorem 3.2) is that there is no fixed point of the continuous $\hat{f}$ on I_k^2, where $\hat{f}$ is the simplicial extension of f on I_k^2. We will show that by Lemma 4.2 and Lemma 4.3 we proved.

To be more specific, the "given vectors" in lemma 4.2 can be one of the following groups (recall the above list of two vectors in each of the four quadrants we made):

1. group 1: two vectors in one of the quadrants plus one vector duplicating one in these two chosen vectors.
2. group 2: all three vectors operated upon the vertices are the same single vector chosen from $\pm\vec{e_i}$ $(i = 1, 2)$.

Since all of the possible groups of three vector operations belong to one quadrant, the convex hull of the three vectors do not include 0, because the only combination that can make it zero is $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which is not a convex combination as (5). Lemma 4.2 then shows that there is no fixed point for $\hat{f}$ over I_k^2, which contradicts to Brouwer fixed-point theorem. The assumption, which is the negation of the conclusion of Hex theorem, is thus false. We're done. $\square$